Optimal Control, Guidance and Estimation

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Lecture No. # 34

Constrained Optimal Control – I

Hello everybody, we will continue our lecture series in this Optimal Control, Guidance and Estimation course. And in the last class, we have seen some of I mean some details about other things. But, this particular course, I mean this particular class, we will start this Constrained Optimal Control.

So far, we have not talked about this particular topic. So, next couple of lecture, we will see the, what differences it brings, when you put constraints into action; and when I mean constraints it means inequality constraints. So, let us see first control inequality constraint, and towards the end of this series of lecture, we will talk also about little bit about of state constrained problems as well basically.

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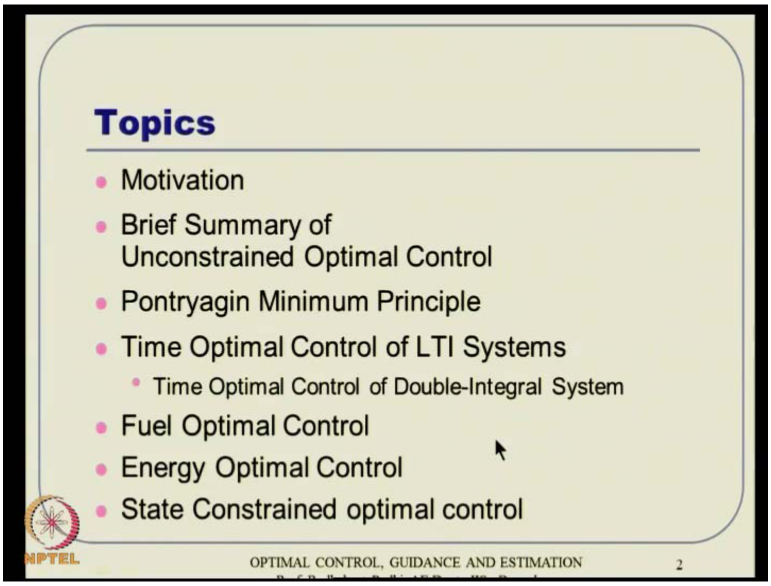

So, this is the outline of the topic; so that, I lined up in this particular lecture. First thing is little bit motivation about, why we want to study again this constraint optimal control for problems. And as I told, lot of practical applications demands that, that we formulate the problem in this framework actually. And little bit brief summary of unconstrained optimal control just to just to recap easily at certain derivation process and thing like that; that will form the basis for constraint optimal control also basically.

Then, what we will study in detail this particular lecture is, Pontryagin's minimum principle in some sought of a generic framework. Then probably this is this will be the topic of this particular lecture. But subsequently, we will also study this time optimal time optimal control of LTI systems in detail.

And especially, this time optimal control of double integral-system, which is a very standard text book sought of a problem is kind of a bench mark problem nowadays actually. We will find this problem in many text books, and we will also cover this in fair detail, which will which will kind of clear our ideas, what we are talking about here basically. That will be followed by fuel optimal control, and as well as this energy optimal control.

So, there are various practical problems, which will demand this kind of control system analysis and synthesis basically. And towards the end of these couple of lectures, we will also talk a little bit on state constraint optimal control actually; and how do we solve or how do we propose solutions for incorporating these kinds of constraints alright. So, let us get started, and this particular topic lecture as I told will contain this first three topics, rest of the things we will study as we go along in next two lectures basically alright. Little bit motivation for why we want to study this topic.

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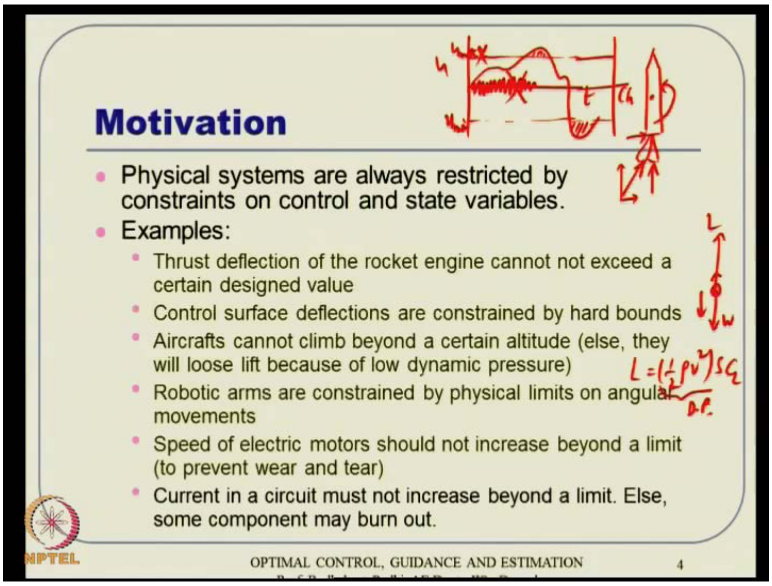

First of all, as we know physical systems are always restricted by constraints on both control as well as state variables by the by that is the reality, we cannot escape from that actually. So, let us see some few examples, how it naturally pops up. So, first of all, suppose you think that ok, rockets are controlled by thrust deflection angle actually typically.

So, either the engine is swiveled, the entire engine will be kind of swiveled in a lateral directions. So that, if you if you have something like let say, if you have some rocket like this (Refer Slide Time: 03:46), and you have something like a thrust coming out, so this is the nozzle part of it; now, the nozzle is deflected like that or that, so then the thrust will be this in initially it was like this. But, if the engine is I mean if the nozzle is deflected, then the thrust will be like that. So, that means, it will still have a component in the vertical direction, but also have a component in the in the horizontal direction, lateral direction.

So, that particular component which will act here hence, because c g is here, it will it will act something like a movement and that is the mechanism for controlling the rockets actually. So, obviously, when you look at it, this engine or this nozzle deflection that we are talking, one of the mechanisms for thrust deflection; cannot be exceed beyond a certain limit and typically the limit is also very small, the limit is roughly about 10 15 degrees actually.

So, if you if you talk about a problem, which demand something like 35 40 degree of nozzle deflection, obviously that is meaningless; because, we simply cannot implement such a control strategy actually. And you can also think that ok well, if is control is constraint, then the system may go unstable well that may not be the case, because we are not talking about something like an instantaneous in effect and all that. I mean the vehicle can still be stable and then we want to have a course correction and think like that.

And also remember, we have a long period of time to kind of take control action. In other words, even if the control saturates for some time, then eventually if it comes out the saturation, then it is as good as a good problem basically. So, that is the whole idea of studying this control, I mean constraint control problems actually. Any way, so that is one example.

The other is let us say, this suppose we talk about aircrafts now not necessarily rockets. Then you also know the aircrafts are having this various control surface deflections, especially like elevator, ailerons, and rudder and thing like that. And we have discussed about that in flight dynamics lectures. So even there, the control surface deflections are typically limited to something like 30 degree, 40 degree and all that.

And just to make your idea little more clear we are talking about control magnitude constraint here. But, in general the control is also constraint in its rate actually. Even the magnitude may be small, the great of rate of change is very high then that also is not acceptable actually. So, these are things that kind of motivates, but we are not talking about rate of rate of control constraints here; they still cannot I mean we are still talking about control bounds.

So, especially if you if you plot it let us say, if you if you plot the control variable u (Refer Slide Time: 06:45), and then the we are just talking about of something about like a u max and then the u min here actually. So, that what I am talking is, in practical really it can still be bounded, but the but the chattering can be there that means the rate can be very high and that is that is also not acceptable.

And if it is not chattering and then something happens like this (Refer Slide Time: 07:08) and thing like that then also it is not acceptable, because of these regions are violating actually, we cannot I mean these are not acceptable controls actually. So, that is what we are interested in. We are we are interested in having something, which will which will talk about some bound of u max and U min; and the control solution will tell will lie between this u max and u min, all the time especially until t equal to t f actually. So, that is what we are worried about, t f can be as good as infinity also basically. Anyway, so this is this is the problem, these are the examples.

Control surface deflections are also constrained by hard bounds. And then coming to the state constraints part of it, let us say aircrafts cannot climb behind a certain altitude; because, if they keep on climb then ultimately they will loose lift, because of low dynamic pressure.

As we know that, lift and drag are typically very strong functions of dynamic pressure, L equal to half rho v square into s into c L right, if you remember the formula. And it turns out this is the lift coefficient is c L, and this surface area is just about this part is something what is called a dynamic pressure. Now, it turns out that rho is not a constant number, rho is a function of height, rather a strong function of height exponential beta s times rho 0 something rho c level let put that. So, if it is like that then obviously, as you keep on increasing height then this quantity, what you see here keeps on decreasing very fast, and ultimately rho will be so small, that your L will be very small.

So then, if you see very first principle that, lift is at least equal to weight or more than weight. So, that it can it can be sustained or taken off in the air, this is w and this is lift. The lift becomes smaller and smaller, then w becomes larger; and then ultimately, what happens? If your lift is very small, w is very high then, this is start coming down actually.

So, it just cannot sustain the lift. That is that is why the state constraints are also important. And if you if it is very low, what happens is your drag is also function of that; that means, drag is again half rho v square into s into c D this time, and it is again a strong function of this dynamic pressure. So, if it is very low, then your rho is very high, obviously you will end up with lot of drag penalty actually.

So, in that is where you your I mean our aircrafts typically have something called climb altitude basically like, so it is not very high. If it is altitude, it should be some where I mean somewhere in the kind of optimum altitude within which the aircraft should fly actually, then it then it alright actually. So, then obviously we need to impose a constraint on the height part of it, height is a state variable in flight dynamics. So, that it is where it is also necessary alright.

So, let us see the next point alright. Next one is let us say robotic arms, they are also constrained by physical limits on angular movements actually. So, that is also I mean various application it is not just aerospace; the first three are aerospace. But, even if you go to non aerospace domain, then also things will pop up naturally actually. Then we can also talk about electrical application, where speed of the electric motor should not increase beyond a limit to prevent, it should not increase beyond a limit to prevent this wear and tear actually. So, that is also necessity.

And also current in the circuit cannot increase beyond a limit. Otherwise, some components may burn out actually. So, we can list out variety of applications, I will list out a little from aerospace, little from robotics and one from something like mechatronics and or may be electric application; and then you have this I mean current circuit applications and thing like that.

So, in almost all engineering applications in real life applications, constraint is satisfaction of constraint is a must actually. In fact, in my opinion constraint is first and then optimization is next actually alright. So, that gives us an enough motivation to study this constrained optimal control. But, unfortunately what happens typically is, when we talk about constrained optimal control, the solution turns out to be in open loop actually. So, that is why things are not very much in order; unless, there is a typical example problem, where we can come up with some sought of a state feedback and constraints, I mean state feedback solution and all actually.

But, in general the solution nature will turn out to be something like in open loop, which is not a very pleasant thing to see actually just keep that in mind. And you can device algorithms and all that for in for generic system, generic non-linear system, which will satisfy all these things; in a state feedback sense is still I think in a big way in a open problem actually.

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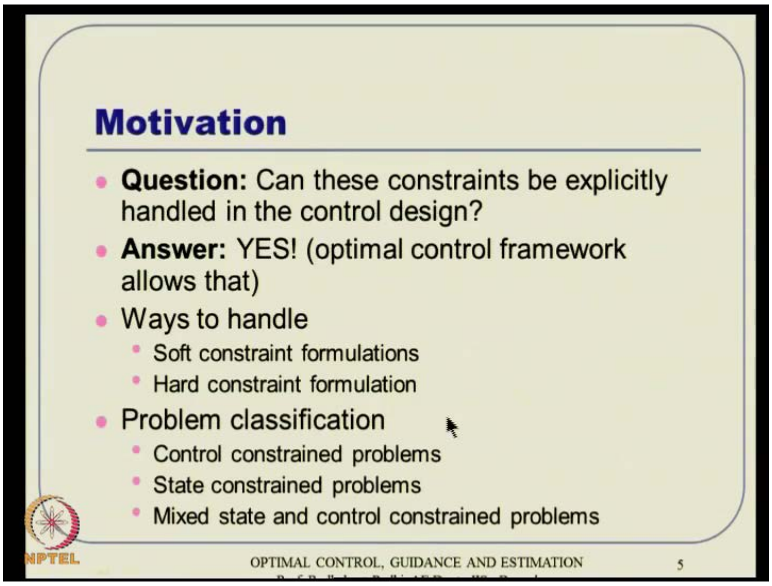

So, what is the summary of it? The question that you are asking is, can these constraints be explicitly handled in the control design itself? I mean in other words, one can also think well these are the constraints and I can also have some sought of a penalty function in the cost function, which will account for that. And that is the usual way of handling that in general also to in many practical applications; the reason being, if you can handle that through cost function let us say the control magnitude is high at some point of time, then you increase the weight associated with the control function; other words, there is a term even, if you if you remember our l q r and things and all, there is a term something like x transpose Q x plus u transpose R u. So, assume I mean this is the term that we handle actually in the cost function, 0 to t f.

Now, here if you talk about this as a scalar quantity or R is a diagonal thing and all that, you can talk about something like u times or q i times u i square actually. So, when u i is approaching to the limit, then you can correspondingly increase q i; so that, it will try to force it down actually. So, that kind of ideas are something called a soft constraint way of handling things or design tuning way of handling things. That is not a very need thing to see.

In a mathematically rigorous sense, we should rather handle it has a state, I mean is a control inequality problem straight away actually. So, that is what we are talking about here. So, the question here is, can these constraints be explicitly handled in the control design? The answer turns out to be is, yes we can do it. And the ways to handle is something like a soft constraint formulation, which I just now described. And then, we can also handle this hard constraint way of problem formulation, which we are interested in mostly.

So, the typical way of classification of this problem is in a explicit way is something like this (Refer Slide Time: 14:56). We can talk about control constrained problems, we can talk about state constrained problems, and we can also talk about mixed state and control mixed state and control constrained problems. So, these three are possible or largely in this couple of lectures will give impasses on control constrained problem, and a little bit on state constrained problem. Mixed state and control are kind of mixed algorithms and all that, we can we can see some references and thing like that, for your own benefit actually.

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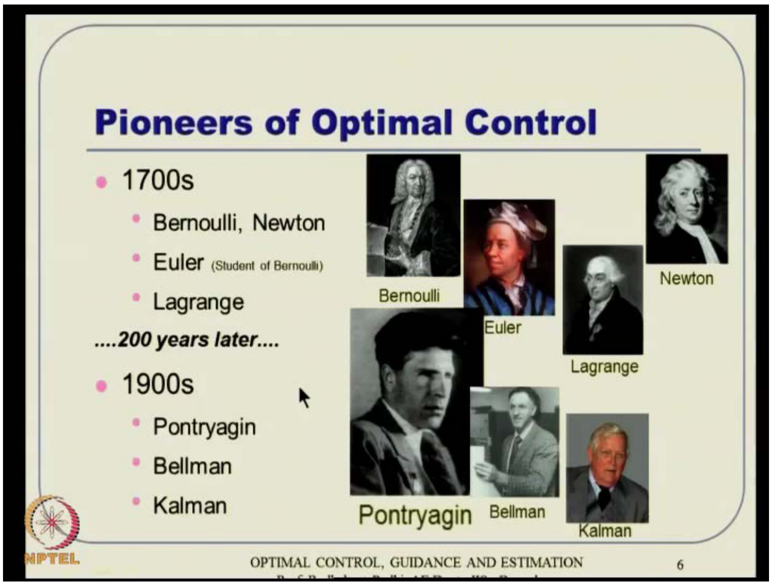

Now, going back to these little bit historical development and all that. As I told in one of the classes before, the very first idea of optimal control theory in general, probably can be credited to Newton all the way back, where you proposed this cheapest recent guidance sought of, I mean cheapest decent methods and all that actually.

Now, he was not alone, but this is a I mean very along go about 200 years back as not alone, they were great pioneers like Bernoulli, Euler, and Lagrange. In fact, we have been talking about E L equation, Euler Lagrange equations and all that actually. So, these are the country reasons from these two pioneers. And there are several others like them, but remember that was an era, where computers were not there and nothing I mean the computational in intensive things were simplified dreams actually, this cannot be done.

So, but something, so it largely remains dormant, lot of people kind of started ignoring and thing like that; but towards mid 1900’s, there was some computers revival or computers were getting developed and that was the time, where the optimal control got a reworth as well. So, something happens 200 years later, and these are the pioneers in the 1900’s, which almost revolutionized this field actually.

So, this Pontryagin, Russian mathematician, and a Bellman American; and Kalman he is a European person, but then he migrated to US, and came back to Zurich town is in Switzerland now. But, any way, so Kalman’s contributions are largely into linear systems theory and Kalman filtering in the linear system framework. Bellman is he now as the Hamilton Jacobi Bellman theorem and all that and dynamic programming.

But, here is this Pontryagin, Russian mathematician who almost single handedly kind of revolutionized this field in around 1950’s. And little bit on Pontryagin is will talk here because, that is his, it is his ideas that we are talking here constrained optimal control and all that actually.

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So, this is a S L Pontryagin he is a he was born in 1908, and then he survived until 1988 so and something like he born in Moscow Russia. And but he the interested thing is, he lost his eyesight, when he was about 14 years old due to some explosion actually. Because of this explosion, even though he lost it website I mean his sorry lost his eyesight; he continued his exploration in mathematics and especially he was very much keen on optimal control ideas and all that.

So, if we try to perceive his person, then his mother came to recue, and then mother helped it in a helped him in a very big way. In other words, she was writing everything and explaining the symbols and all that. She was not a mathematician by the by. So, this in that with that great help, he continued his person and he went in fact, became a very big name in Moscow, I mean in USSR first; then later in the entire world as well.

So, here lot of these significant contributions in topology especially in 1930’s and 1940’s. But, around late 1940’s and 1950’s, he got very much in I mean interested in optimal control and that is where he revolutionized the field actually. Ultimately, he also headed this great institute mathematical institute in USSR currently Russia. And he also focused on this theory of singularly perturbed systems and singular perturbed systems in ODE’s or Ordinary Differential Equations, and t and this is what we are talking here maximum principle in optimal control theory.

Here, we are talking everything in the set of minimization, but when he studied he studied everything on maximization principle actually. So, what on the thing I mean, if you want to minimize a cost function, it is equivalent of maximizing the negative part of it actually or sorry the negative of the same cost function. If you maximize negative j, it as good as minimizes positive j basically. So, this is what it is.

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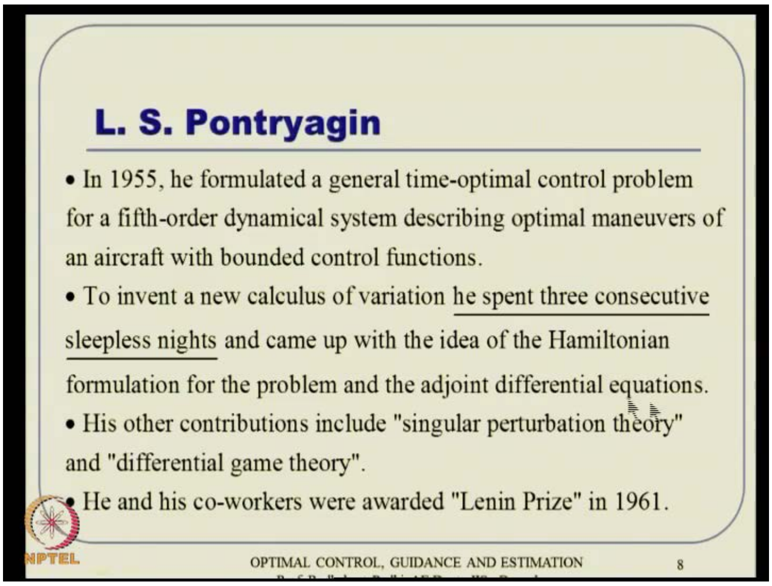

And then in around 1955 that is when he formulated the general time-optimal control problem of a difficult problem, fifth-order dynamical system describing optimal maneuver of aircraft with bonded control functions. And so far, that was never done before in a good analytical sense. And people are almost kind of stun by this development actually.

And to invent this new calculus of variation, which was do not want for a while, he spent three consecutive sleepless nights as well; and he came up with this idea of Hamiltonian formulation from the for the problem formulation; and that is what we are talking about even in unconstrained domain I mean in unconstrained problems. First, we established that from the Hamiltonian formulation and then in this particular lecture, we will see how to exploit that for constrained problems as well actually.

So, here coming back, he spent three consecutive sleepless nights and then, this idea great idea came to him; and then, he proposed these adjoint differential equation methods, these costate equations and all that. This is primarily, because of this Pontryagin’s development actually or Pontryagin’s contributions rather. But as we as I told, he also contributions I mean he also contributed towards this singular perturbation theory; as well as latter on differential game theory, which are not talking about we are not talking anything about that in this particular lecture, I mean course really.

But, essentially differential game theory some sought of an extension of optimal control theory, where essentially we have two classes of control variables; one favorable, and one kind of unfavorable. On the one tries to minimize the cost function, and the other tries to maximize the cost function; we can think of something like a air come back scenario, where somebody wants to attack, the other one tries to kind of get away from that to the maximum possibility; that means one tries to kind of minimize the mid systems, the other tries to kind of maximize the mid systems.

So, both are I mean the problem is in relative dynamics, so both controls play the role actually. So, that kind of problems is called differential game theory. And it has taken a very good impact on the on the modern war games, and war game solutions and all that. And especially for air come back scenarios, if somebody is interested you can you can see that actually some literatures around that you can see; there are books available for differential game theory as well actually.

And this prestigious Lenin Prize, he and his co-workers were awarded in 1961 actually. So, that that is a great Pontryagin is one of the heroes of this modern optimal control theory. And lot of this development around optimal control is happening, because of huge insight into the problem. And his way of giving rather simplified solutions, avoiding this E L equation sought of ideas and all that actually.

In other words, when you follow this state equation costate equation and optimal control equation like that, we do not really talk about Euler Lagrange equation and all that; we do not even though even though we can derive this three conditions from E L equation, we really do not worry so much about E L equations any more actually. So, that is the contribution from Pontryagin’s.

So, let us quickly see, a little bit overview of this unconstrained optimal control that we have studied before and then, we will come back to this constrained optimal control actually (Refer Slide Time: 23:18).

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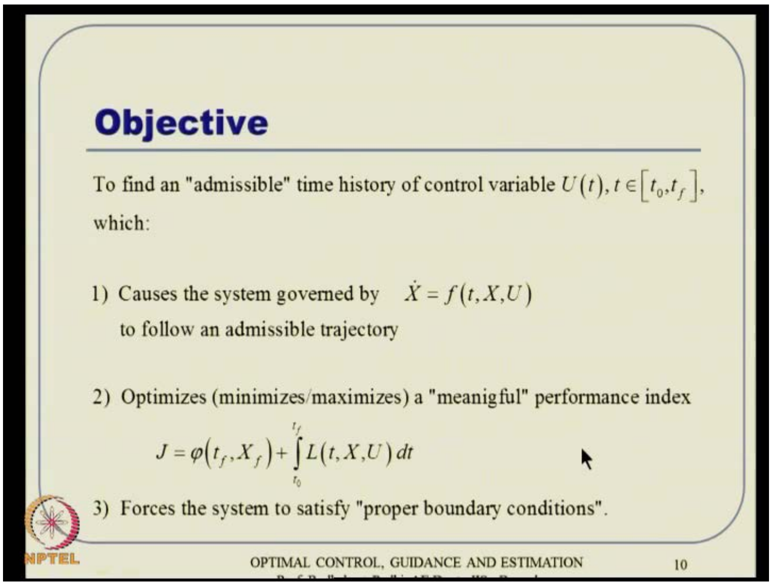

So, objective was, what we have been studying so far, is to find an admissible time history of the control variable from t naught to t f, which causes the system governed by this non-linear system dynamics to follow an admissible trajectory. And at the same time, it should also minimize or maximize that means it should also optimize a meaningful performance index of this form is typically called Bolja class of problems, which is fairly very much generic actually. And also, forces the system to satisfy proper boundary conditions. So, all this things we have seen before.

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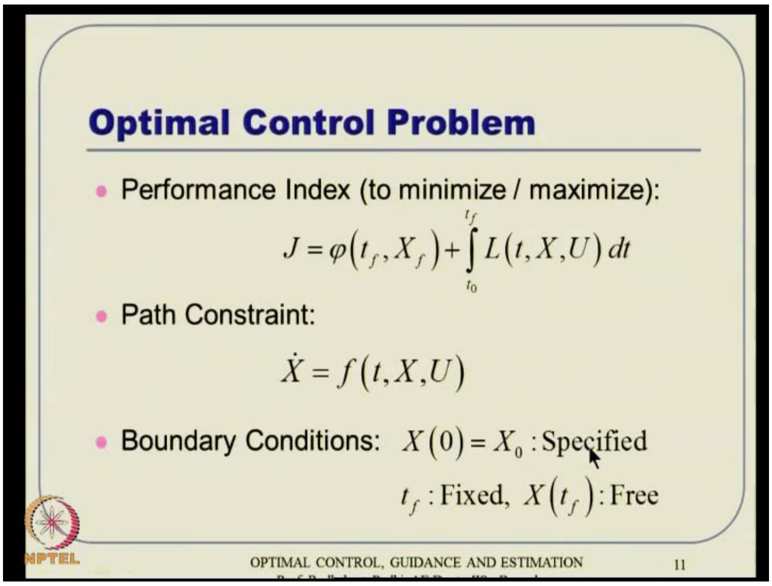

Now, to summarize it has to have, it has to minimize or maximize a cost function of this form. It has to satisfy the path constraint, and it has to satisfy the boundary condition as well basically.

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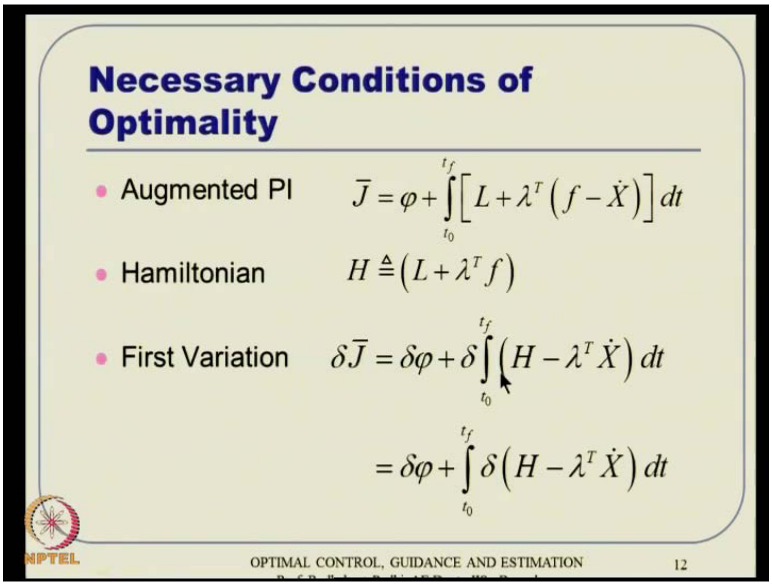

Now, what we followed is, we formulated an augmented cost function of this form, and the dynamics part of it we went we took it inside the integral, the terminal penalty was still outside that is J bar; and then we defined a Hamiltonian, which can does not contain any differential terms in inside the integration. So, this only this part L plus lambda transpose f. And then, we define that as Hamiltonian; then you can continue our analysis for first variation.

In the first variation happens to be the first variation of phi, then first variation of entire integral term, but then integral and I mean the integration and this variation they are commutable.

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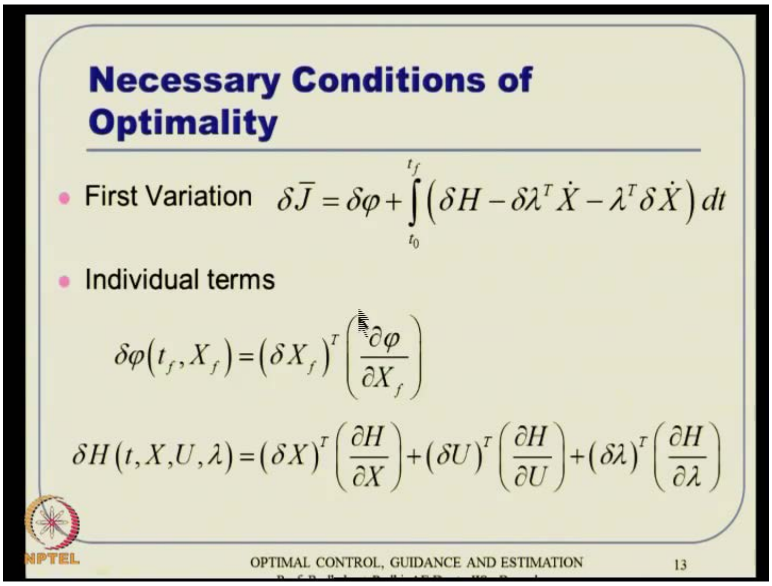

So, the variation went inside and then we expanded that and carry out this algebra of derivation, and then this term by term, we analyze.

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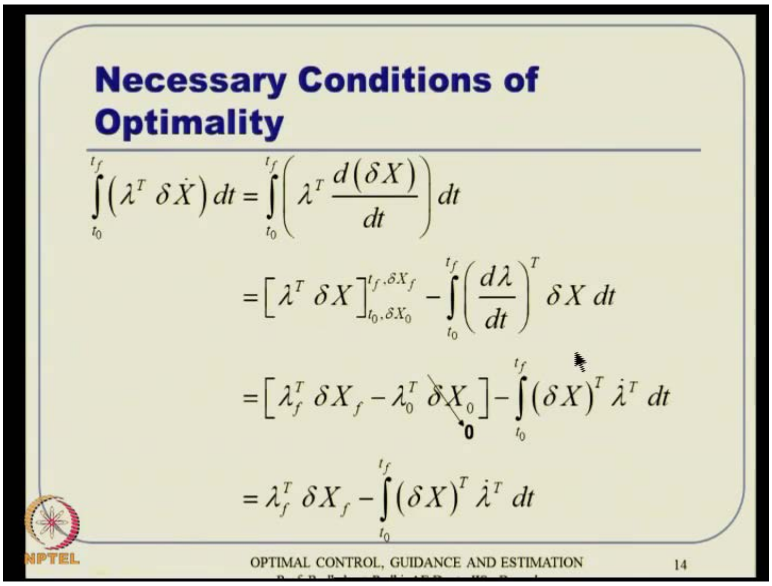

And finally we kind of analyze this as well through this sorry this is the term here this particular term (Refer Slide Time: 25:18), we analyzed in integral sense; and then put some no variation in the initial condition, that condition gives us that variation of X naught is 0 and we continued this analysis.

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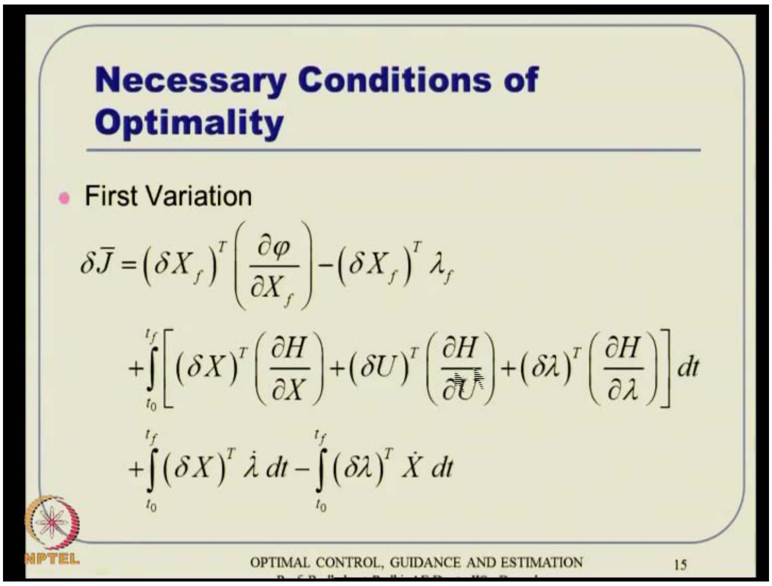

And then finally, we combined all this things, all the first variation terms.

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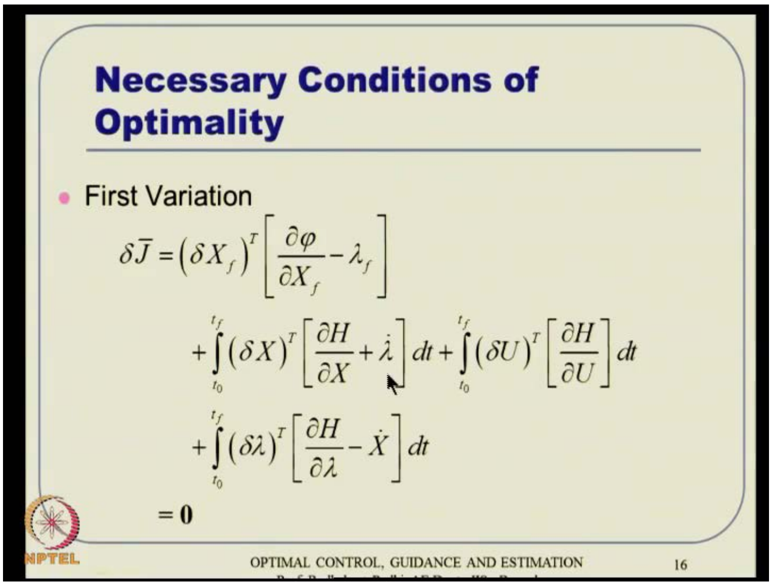

And then we will end up with something like that. Then we excited a theorem, which tells us that, if this equations are true, this equation is true for all sought of possible variations. Then the only way it can happen is, when the coefficients are 0.

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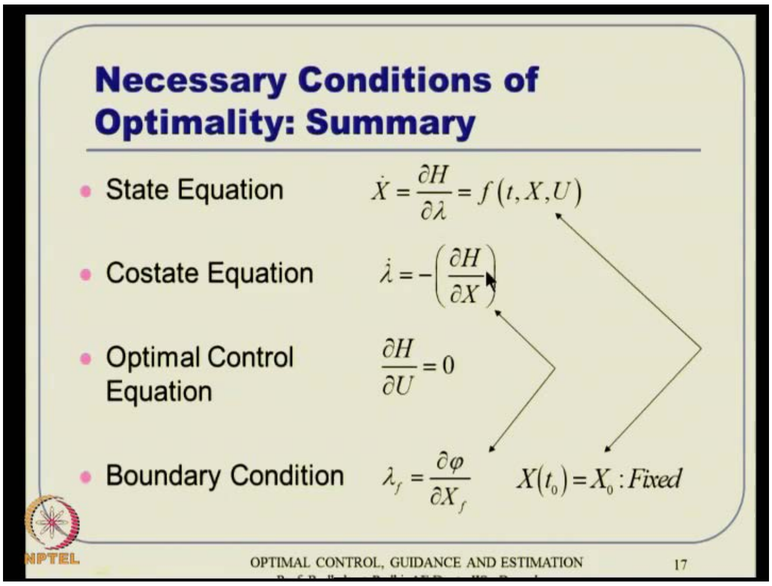

And when we invoke that condition, we will ended up this state equation, because if you see this part (Refer Slide Time: 25:55), this cannot be 0, so this has to be 0, but X dot is del H by del lambda; but lambda, if you go I mean see definition of lambda this is what it is. So, del H by del lambda is nothing but f; and then, f is I mean what we ended up with is same state equation that we started with actually.

Then, similarly we will end up this costate equation from this coefficient being 0 (Refer Slide Time: 26:19). So, this lambda dot is negative of del H by del X; then you have this optimal control equation coming out of here; and then the boundary condition coming out of here actually. So, this is where we observe that these two equations are compatible (Refer Slide Time: 26:36).

This is a state equation with initial condition, and there is a costate equation with final condition; and hence, it has this split boundary conditions. And then this lead to this so called two point boundary value problem; and that is what we call as curse of dimen curse of complexity, because of this complex nature of the problem formulation actually. So, not only the boundary conditions are split, but the differential equation nature itself is opposite actually in a way. The state equation is stable, we will end up with costate equation being unstable that is a drawback actually.

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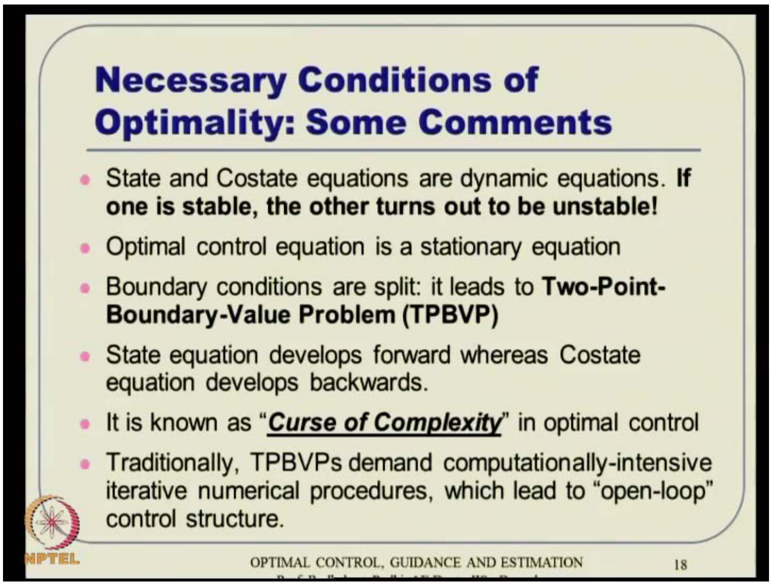

So, these things also we summarize there that point of time. And we told state and costate equations are dynamic equations. And if one is stable, the other one turns out to be unstable. And then, the optimal control this particular out of these three equations, optimal control equation is a stationary equation, algebraic equation; whereas (Refer Slide Time: 27:31), these two equations are dynamic equations actually.

And state equation develops forward, because of the boundary condition being given at initial condition, and the costate equation develops backward. So, this is known as curse of complexity. And traditionally, this two point boundary value problems demand computationally-intensive iterative numerical procedures, which leads to this open-loop control structure. All these things, we summarized a long before, almost at the beginning of the course actually.

So, this is what will form the basis of our derivation, you can see some of this first variation we made it equal to 0 (Refer Slide Time: 28:06). And that is where we will see how this particular constrained optimal control problem defers from unconstrained problems actually alright.

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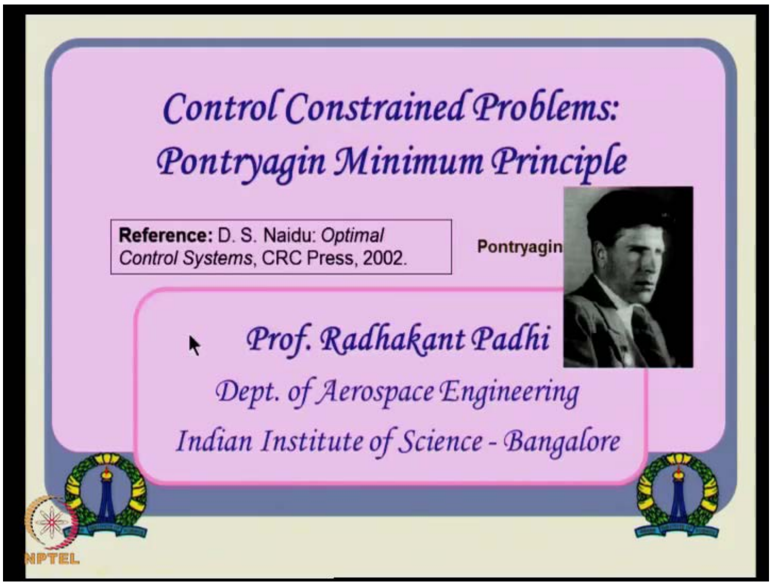

So, let us start studying that. So, what we are talking now is control constrained problems; essentially we are talking Pontryagin's minimum principle. And you can also see here, that these I mean he is blind his eyes are not there. And the last of, I mean lastly this material I have it from this reference; but, I will also list out 1 or 2 very earlier reference, including Pontryagin's own reference, which is now available in English as well actually. We will see that towards the end of this lecture basically.

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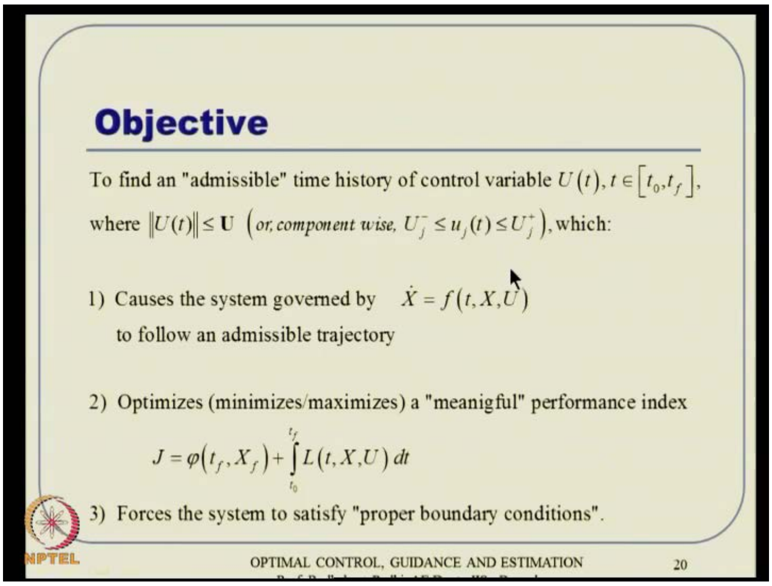

So, what is the difference here, now the difference is, so what we studied I mean before is something like this (Refer Slide Time: 29:00), to find an admissible time history of the control variable U I mean U of t, where t belongs to t naught to t f, which satisfy certain conditions here. So, we did not impose any condition on this U of t at all actually, it was unconstrained and it can take very large value, if it is so required actually.

So, that part we are changing here and we are telling that, the objective remains same for all the other things, but there is additional condition out here for the for the control variable U of t; that means to find an admissible time history of the control variable in this segment U of t, but t belongs to t naught to t f, where the norm of U of t is bounded by certain value or in component sense the each of that, each of the component of control variable that is u j is bounded between its maximum and minimum value; this is the maximum value of U j, and this the minimum value of U j actually.

So, with that condition that additional conditions in action, all are things has to be satisfied actually. That is a they is a different class of challenge actually. And again let me also admit here, that we are not going to follow these is very regress of a topological way of dealing things, that the way Pontryagin's did. What we are going to do is very engineering intuition and then some if somebody is interested in regress things and mathematical way, you can always see the reference that I will give at the end actually.

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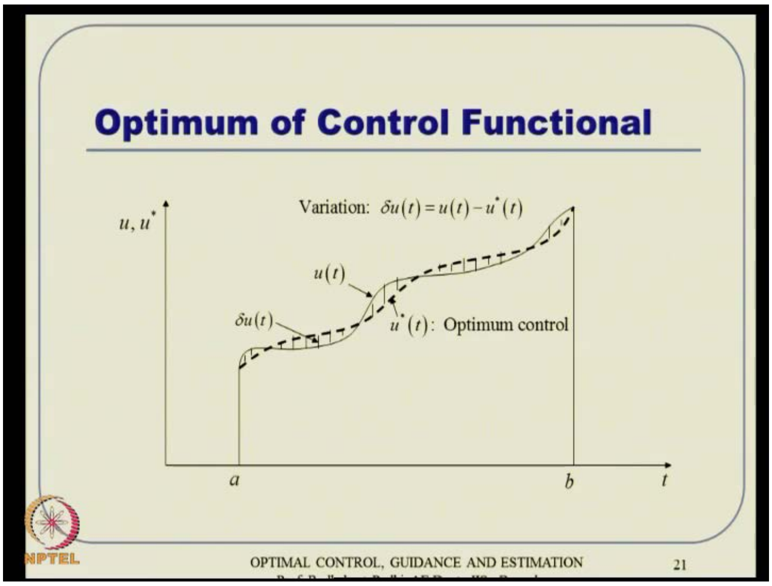

Any way, so little again going back to what we discussing before, what we mean by variations and all that. And then, when you talk about variation and control variable, let us say we have an optimum control or optimal control, this thick solid, thick dotted line that is the optimum control that we have already found. But, what you mean by variation? The variation is something that should happen around that actually; that means delta u can be u minus u star. So, what you are talking here is this derivation beings this, if you take it together collected it all point of time from t naught to t f, there is nothing but the various actually.

So, this is the u star is an optimum control path or optimum control trajectory, where as u of t is something that is closed to optimum; it is varying around optimum, but it is not really optimum actually. Some how the idea is, if you tell something is optimal that means, if I take any variation around that, that is going to give me non optimal. So, that is the concept of local optimum things actually. So, that is the whole thing that we studied before.

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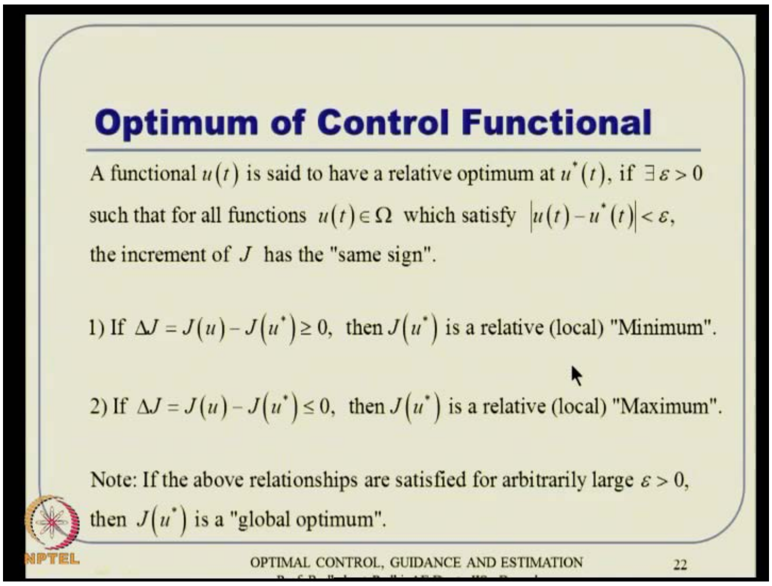

And what we you can summaries as well. That means, if u of t minus u star of t magnitude wise is less than epsilon, epsilon being a small positive quantity. And then, although it should happen, if I evaluate J at u and then J of u star, then J of u minus J of u star should always be positive 0 I mean greater than equal to 0; no matter, what kind of variation we talk here. Remember, u is nothing but, u star plus delta a.

So, irrespective whatever is the delta u, then this is a condition has to be satisfied, then only you get this local minimum; if it happens otherwise, you get local maximum. And also remember, if epsilon happens to be arbitrarily large that means there is no bound on epsilon and all that; obviously, the solution that we are talking here actually global optimum actually; we have discussed all that in one of the earlier lectures in this course.

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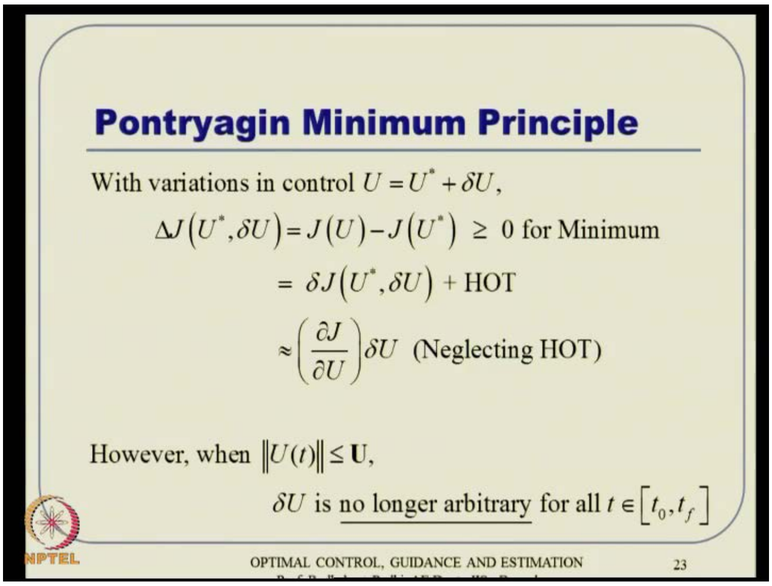

Now, here is a problem actually. So, what is happening is something like U is U is U star plus delta U. However, so this delta J it can always interpreted something like J of U minus J of U star that has to be greater than equal to 0. And here remember the U can be general vector variable sought of thing actually. So, if I write this delta J, which is nothing but, the first variation plus higher order terms. So, if I neglect higher order terms, then I will end up with this first various, which is written something like this (Refer Slide Time: 33:13). So, del J by del U into delta U basically.

However, when norm of U is is constrained, let ne norm of U is less than equal to U, the problem the core issue here is, this delta U is no longer be arbitrary actually; that means we cannot take arbitrary actually that means we can take arbitrary variation around the boundary values actually. That is were this fundamental philosophy that we assumed in the derivation process (Refer Slide Time: 33:50), while deriving this condition; this particular del H by del U is equal to 0, we assumed that this can be arbitrary, and hence this has to be 0.

Now, this is no more arbitrary; so, obviously, we cannot talk del H by del U is equal to 0; that is the (( )) fundamental drawback of or rather challenge of this constrained optimal control. So, we cannot tell del H by del U equal to 0.

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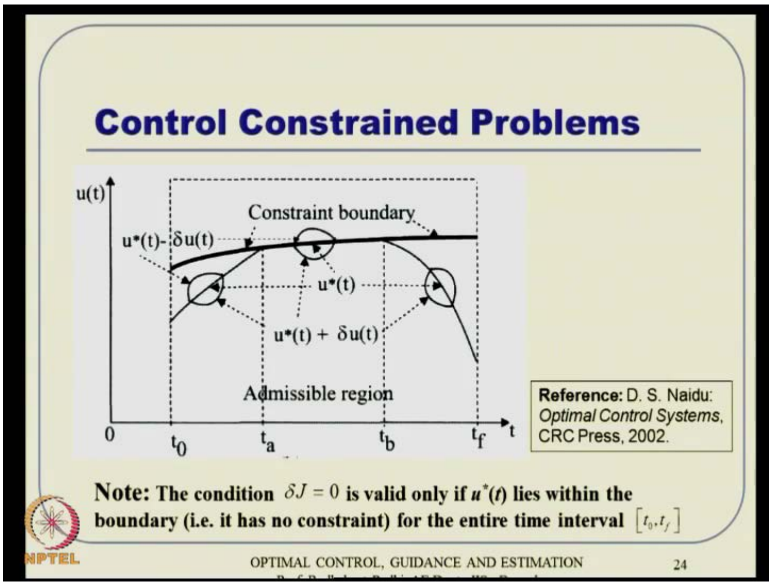

Now, let us see little bit pictorially, what is happening here? Let us say, this is the constraint boundary, and this is what we have already found something like u star. Now, if I take a perturbation, delta u over u star and then here, I have no problem, because I can go both ways, here I have no problems I can go both ways, at least there is some perturbation that is allowed in both ways. But, on the boundary, this cannot happen, because this perturbation on the top side is not allowed, whereas on the bottom side is still allowed actually.

So, delta J but remember delta J is 0 valid only if u star lies within the boundary actually. Here, we have just generate this delta J we are talking as a function of it is an generic control variable only; it is not yet tied up with optimal control variable actually, we will see that in a second. The problem I repeat it is I mean it is allowed both side variation are allowed here, both side various are allowed here, but here it is not allowed. So, only one side variation is allowed actually. And that is where the coefficient cannot be 0 we have to do something else basically.

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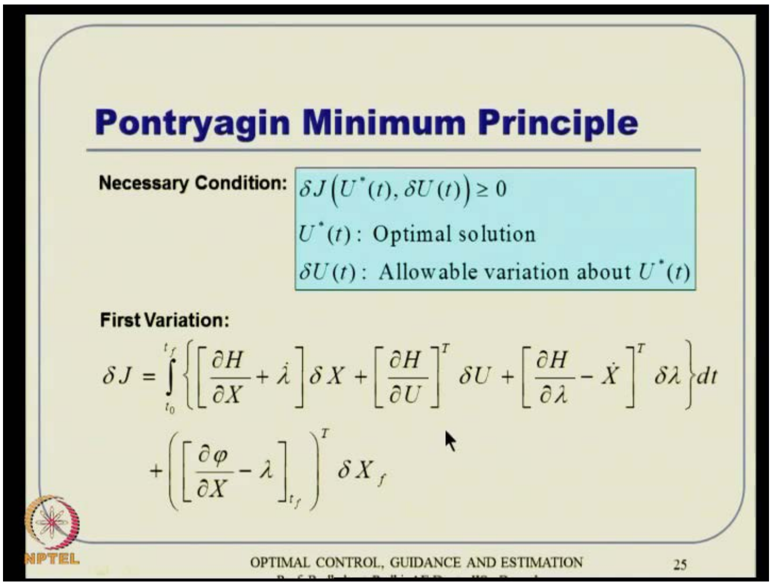

So, if we but this conditions is still there (Refer Slide Time: 35:34), this has to be satisfied; because, if we rather really talk about U star is the optimal solution; then, if you talk variation of delta J around this U star in allowable variation actually, it is not just arbitrary variation. But, if you talk about allowable variation, again this bottom part of it and all that actually (Refer Slide Time: 35:52).

So, even if we take allowable variation, then this delta J with respect to till that has to be positive actually; then only, we can talk that U star is actually an optimum value or thus the value which leads to minimization of U minimization of J really alright. Let us go back to our derivation of unconstrained problems; so, here we have something like this delta J is given and this big expression. And now we are telling that ok wait a second.

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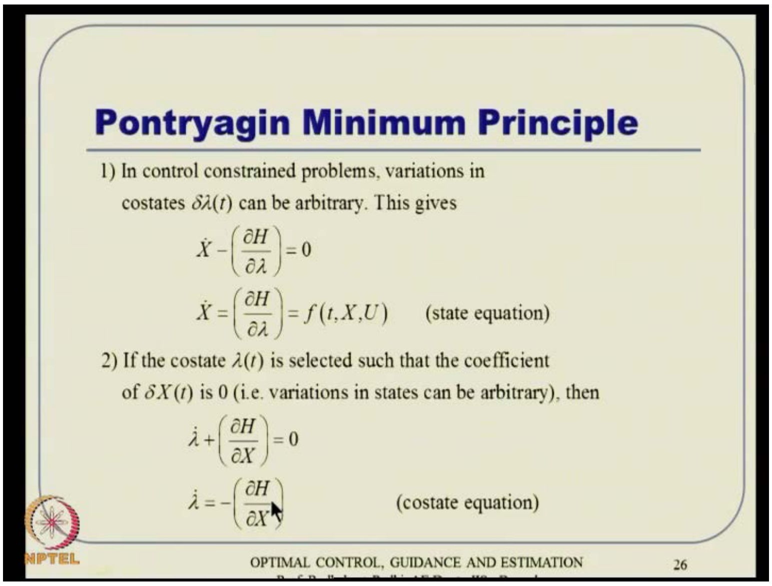

We can actually think of this like this actually; and control constrained problems, variations and costates can be arbitrary. We are not bothered about costate variations, we are not put any constraints on that. So, let the costate be arbitrary, and arbitrary variation can be allowed. That means we can still talk about this (Refer Slide Time: 36:40), coefficient of that delta lambda, whatever wherever it is appearances that can be 0 actually. So, that is why we getting X dot is H by del lambda and that is nothing but, f of (t, X, U) that means the state equation we got it back. That is what it is also part of the formulation.

Now, if the coefficient as the costate is also selected in such a way that the coefficient of delta X is 0; now, delta X may or may not be arbitrary, because delta U is not no more arbitrary. But, remember the lambda is something like a different dimensional all together, it happens in a differences space altogether.

So, if the idea is, if it is the costate lambda of t is selected in such any selected in such a way that the coefficient of delta X is 0. We are not telling the delta X is arbitrary and hence it is 0, hence the coefficient is 0, we are not talking about like that. What we are telling is the delta lambda I mean lambda is selected in such a way that is equation is evaluator. So, then we will end up with same costate equation, lambda dot is minus del H by del X.

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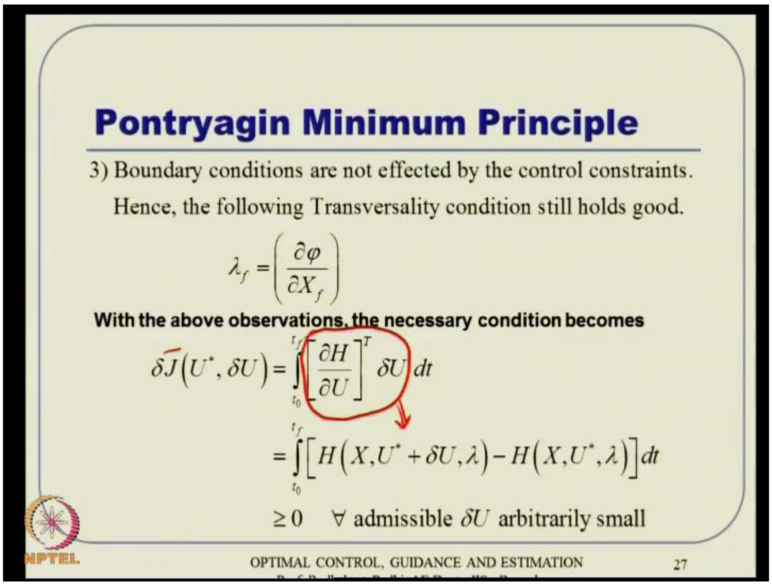

Now, even you can also talk about this boundary condition are not affected by the control constraints, because ultimately the problem objective will met and all that. And hence, the Transversality condition still holds good. So, lambda f is still there actually in this form. So, what you left out really, if you talk about this del J bar ultimately, I think this is little bit del J bar; then, we will end up with only control terms actually right it generate del J bar turns out to be something like this (Refer Slide Time: 38:15), I think is also. So, delta J bar turns out to be like this.

So, what you are telling here is this coefficient is 0, we got the costate equation; this coefficient is 0, we got the state equation; this coefficient is 0, we got the boundary condition. So, what is left out? We left out with this one, so let us keep it actually. So, this is were del J bar is these quantity actually. Now, here we cannot tell let the del H by del U is really equal to 0 that is not possible.

So, what you are telling here is, it is a we do not want to talk independently individually this term and all, but will talk combinely; that means we know that, this particular term that we are talking about (Refer Slide Time: 39:01), I want to see that as it actually together. And then, as we know the we can all actually talk something like, when delta U is small, you can talk about something like this also actually, I mean this is actually this is more appropriate; the second line is more appropriate, the first one is the approximation of that actually first sought of sense.

So, delta here small you can that this two were equivalent. So, we can still interoperate that the entire term is something like this (Refer Slide Time: 39:30). So, as long as we can make sure that, the entire term is greater than equal to 0 we are done actually; and here also a small thing to worry about. Remember, U and U star and all that what you are talking, there are not point variable actually there are also functions of time concepts actually.

So, here is the trick that you if you are in, if you are serious to see rigorous things probably we can say some references or in fact, the original work of entrancing. But, here imagine something like this actually.

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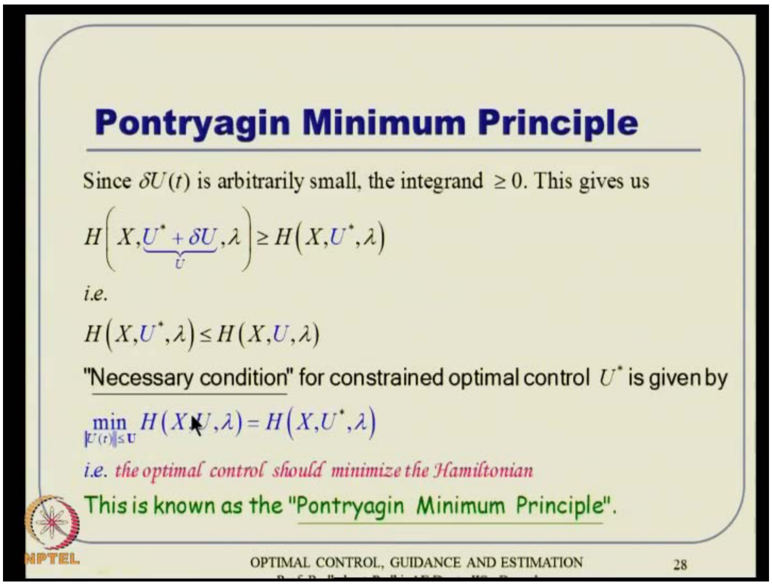

Now, assume that this delta U t is arbitrary small, but this integral value has to be positive (Refer Slide Time: 40:07), and there is no choice for that, know only then we can talk about delta J being something like delta J bar has to be greater than equal to 0 actually; then only we can say that, we are end up with optimal solution basically.

So, if it is arbitrary small that means everywhere else let us say it as gone to 0 everywhere else. There is a this variation (Refer Slide Time: 40:35), every else it is gone to 0, only would this region it has happen some little bit of deviation let us say something like that; everywhere else, there is no variation, that is the variation that we are talking about actually. So, it is a very small variation sought of thing; and if they even in that small variation, if this condition is has to happen then the way it can happen is, if the integrant value is positive all the time, irrespective of I mean whatever delta U it is, but you want the integral value after integration has to be positive.

So, it is a path obviously; and what you are telling is, if we cannot exclude very small variation, we have to include that also obviously, in the allowable sense actually. So, if that condition has to be happen, then this integral will be greater than equal to 0, provided this quantity is greater than equal I mean is greater than equal to 0. And that is were we will end up this condition that this Hamiltonian with some perturbation, we get greater than equal to the Hamiltonian without control perturbation we get greater than equal to the Hamiltonian without control perturbation in the optimal path basically.

By the way, in the entire lecture I will not talk about X star, lambda star and all that; some, text books including I do will talk about X star, lambda star to denote that those are actually optimal values associated with U star; Just to minimize our notational complexity I thought I will avert that, but when you talk about U that is the primary important here. So, we will talk about U when you talk about U that is non optimal U; and when it is U star that is optimal U actually.

And when is X and lambda by default, the mean that by applying U star and generating X and lambda. That means is not just X and lambda anywhere actually. So, I just I thought you just keep that in mind I thought let put a comment about that alright. So, ultimately, what you are telling here is, if you have had this positive quantity satisfied; so alternatively, I can take this one on the left hand side and write this equation. That means my Hamiltonian has to be minimized with respect to the U variable. So, then whatever U, it will ends up with that is my optimal control that is bottom line actually.

So, essentially the necessary condition for control in our control constrained optimal control problems is to find a U star, which satisfy this equation I mean this minimization condition really. We have to minimize this Hamiltonian with respect to this constraints phase, and then whatever control turns out, it turns out to be optimal control. As you remember, all other conditions are already satisfied, state equation satisfied, costate equation satisfied, boundary condition also satisfied.

The only thing that is not satisfied is this del H by del U equal to 0 (Refer Slide Time: 43:31); but the optimal condition is satisfied. Optimal control turns out that instead of making it equal to 0, we have to just do minimization of Hamiltonian within the constraints phase. Now, the constraints phase appears to be large enough and this is almost invalid, and obviously, this minimization tell us static minimization condition tell us that, the first derivative of gradient of H with respect to U has to be 0, del H by del U equal to 0.

So, that means, if it is unconstraint problem, we still use del H by del U equal to 0; but, if it is a constraint problem, we will go ahead and minimize the Hamiltonian within the constraints phase actually. So, that is the difference between the two approaches actually.

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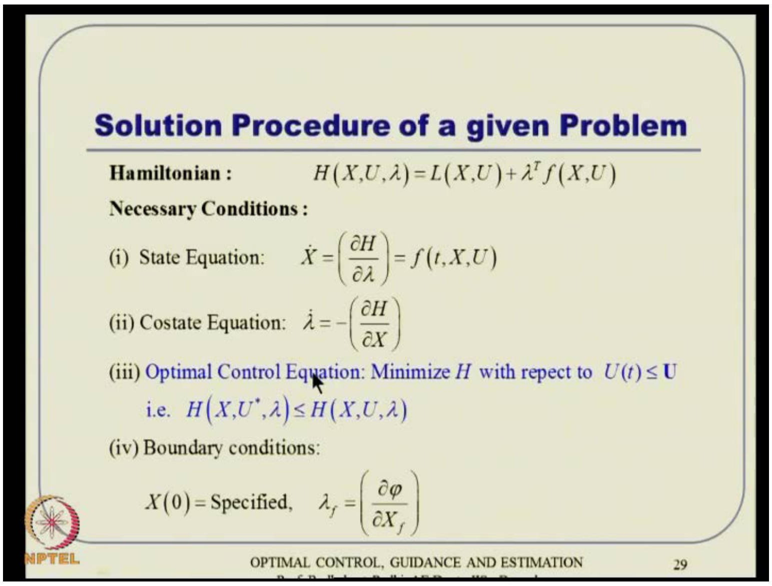

So, that is what I summarized here. It tells how do, we solve that given problem. First you go ahead and define the very similar Hamiltonian that we have defined before L plus lambda transpose f. Then we still use this state equation, costate equation, and boundary condition as it is; whereas, the optimal control equation we change a little bit, and tell it has to the all that we need to do is to minimize the Hamiltonian H with respect to the constraint control that means with respect to the U. But U has to be constant within the space that is allowable actually; that means, Hamiltonian with respect to U star evaluated with respect to U star has to be less than or equal to Hamiltonian evaluated without U I mean without U star any other U basically.

Even there is a small notational difficulty here sought of thing; just to simplify things and all, I have not used X star, lambda star in this notations actually. In Naidu book, you will see this star values everywhere and all; some books prefer to follow that, but I thought I will ignore it here; anyway, so these are the boundary conditions. This is the procedure everything else remains same, only the optimal control equation takes a little bit different and that different results in a huge different later in the solution part of it. But, it tells us that, del H by del U is not equal to 0; however, the Hamiltonian is to be minimized with respect to the control variable and the allowable spaces.

And you can now see that this framework, what we discussed before in Hamiltonian method of what Pontryagin give, state equation, costate equation, optimal control, boundary condition like that; then constraint optimal control becomes so much easier to handle actually; so, much easier to see that. So, what we need to do?

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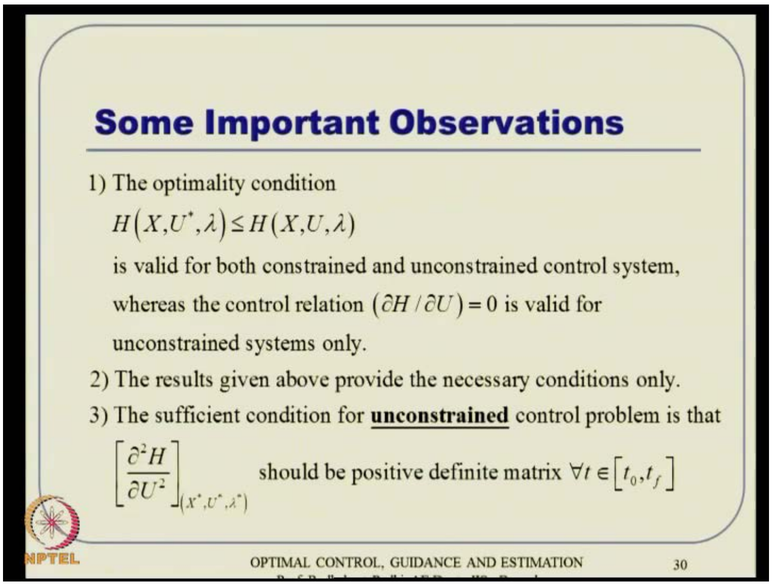

Some little bit important observations is the optimality condition, this one what we just talked about is valid for both constrained equation and unconstrained equation, I mean both constrained and unconstrained controls problem; whereas, this particular equation del H by del U equal to 0 is valid for unconstrained systems only. So, what I mean is again this del H by del U equal to 0 that becomes a special case condition, where the control is not really constraint.

Second point is the results given are only necessary conditions, what you see the sufficiency condition is the different ball game altogether, we are not even talking about that. However, if it is an unconstrained problem, then one of the sufficiency, I mean the sufficiency condition turns out to be something like this (Refer Slide Time: 46:56), del square H by del U square should be positive definite matrix. But, if the constrained problem, this is not valid you have to do several other things; and again, I am not going to discuss anything here actually.

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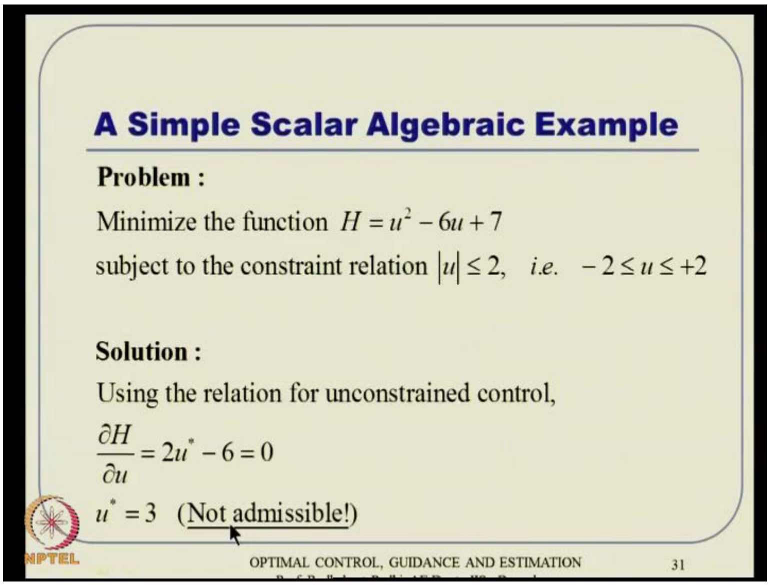

A simple scalar algebraic example just too kind of demonstrate your ideas a little bit clarity sense. So, let us say we have satisfied everything else, and ended up with this Hamiltonian with respect to control variable, when I see it turns out to be something like this. And in general in this H, I mean this particular example H can be anything, I mean H need not be Hamiltonian; but, let us not loose the focus, you can still talk about Hamiltonian is a function something like this; assuming that, 6 and 7 this coefficients and this 1 here, they are all coming from this after satisfying the other conditions actually.

So, our task is to find a particular u, which will be satisfying the constraint. The constraint is magnitude of u has to be less than or equal to 0 that means the control has to be bounded between minus 2 and plus 2 actually. Then using the relationship between for this unconstrained control, what happens is del H by del u equal to 0. So, essentially, del H by del u is, if you do that, u from here minus 6 from there, put u star minus 6 is 0 that means u star is 3.

Unfortunately, this u star equal to 3 is outside this domain, it is not possible to take accept the solution. So, we cannot talk about that as the solution for the variable.

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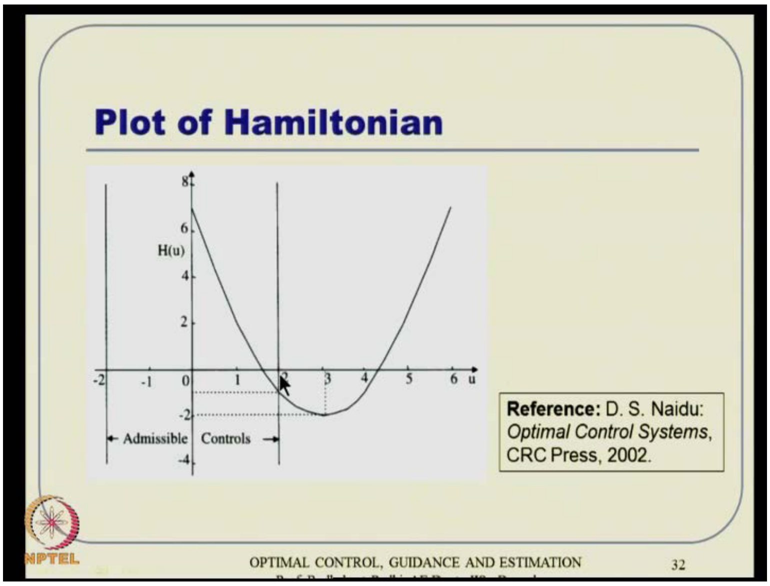

So, pictorially this is something like this, if you picture this H of u that quadratic function, then this takes something like this shape, in the positive side of it and then negative, this is the admissible control actually. So, what turns out to be is obviously, this point, if you do not put any constraint, like the unconstrained solution u star is 3, which is something here (Refer Slide Time: 48:59), which is pictorially clear as well. But, unfortunately, it turns out to be outside this domain, only this domain is admissible actually.

So, what you find out this, then all that you tell is with in this constraint phase, what that particular value which leads to minimization of my Hamiltonian. So, this is the Hamiltonian function (Refer Slide Time: 49:20). So, within the constraint phase, this is the point where it is minimized, the Hamiltonian value takes a very minimum value at this point of time within the constraint; and hence u star equal to 2 not 3 basically.

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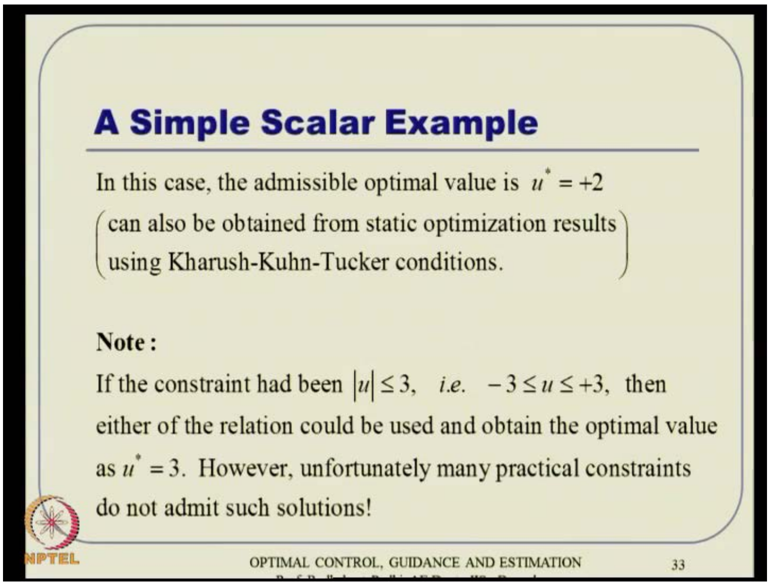

And you can also see that, in this case, that the admissible optimal control u star that can also be obtained through the static optimization results using this K K T condition, that Kharush-Kuhn-Tucker condition; if you apply very rigorously, we will ultimately end up with the same value. And we have discussed about static optimization problem at the very beginning of this course, you can verify that, if you want to.

If the constraint has been something like this rather let us say, norm of u less than equal to 3, then you are lucky; because, u star is 3 is still allowable actually. Unfortunately, many practical constraints are not large enough to incorporate or to pose the problem as something like unconstraint problem in general, because the bounds are too far away.

Having said that, there are also many problems, which will satisfy that; and there are also many problems, where you can actually do it in a soft constraint way; that means increase the waiting, when control approaches the boundary that is the method. And once you do that, there are certain nice thing that happens to the cost function; cost function still retains the quadratic nature for example and hence, convexity and thing like that. And also that becomes much more easier to get this state feedback solutions and all that actually.

However, I also remember that, unfortunately many problems will demand that, you somehow explicitly address this issue. So, that we do not turn into too much of tuning exercise; and the tuning exercise, unfortunately happens to be case by case. That means, if you start from this one particular result (( )) and somewhere down the line, you may exceed the bound actually. Now, if you increase the waiting at that point in time, if you try with a different initial condition, the value can be higher at a different point of time.

So, you will be puzzled to where to use up and where to come down, and all that actually. So, those are the difficulties of tuning in that other approach; whereas, if you just incorporate this out constraint, the inequality constraint then that requirement is not there actually.

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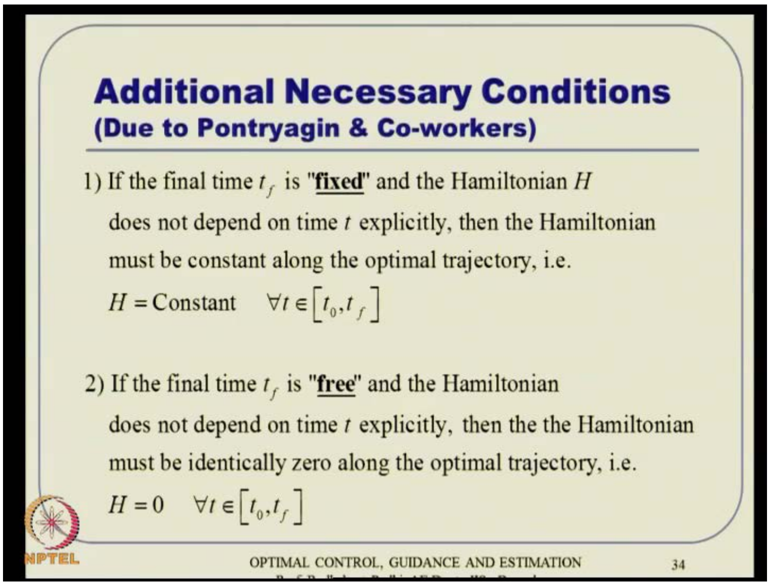

And the additional conditions are, if the final time t f is fixed and the Hamiltonian does not depend on time t explicitly, then the Hamiltonian must be constant along the optimal trajectory; and that is true for control constraint problem as well, if it is not control constraint that is also very most true basically. I mean essentially because, constraint problems are subset of the unconstraint problems actually.

And as a corollary of that, if the final time is free, then what happens? If you I mean this constant becomes 0 as t to t f, typically if you have a quadratic function like that. So, that is generalization of the result is, if the final time t f is free and the Hamiltonian does not depend on time t explicitly, then the Hamiltonian must be identically zero along the optimal trajectory. So, that is the way basically. So, we have proven that in fact before, I think probably about lecture number (( )) you remember that.

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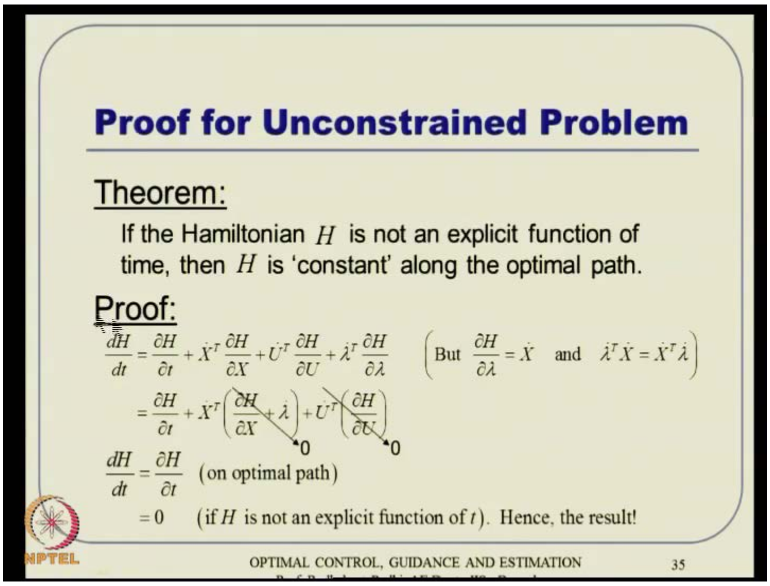

So, this theorem at that time I talked something like that, remember still it is unconstrained problem, I thought I will recap it a little bit before going further. So, d H by d t is del H by del t plus all these terms X dot transpose of del H by del X, U dot transpose del H by del U plus lambda dot transpose of del H by del lambda. Now, if you combine these two terms using this types that del H by del lambda is nothing but, X dot and this is this is scalar ultimately, so I can rework the things.

So, then I do that here, and then combine this two. What I see here is, this term is 0, because optimal control equation is 0 remember it is a unconstrained problem. So, that is 0 and this is also 0, because lambda dot is minus del H by del X; so, that this term also cancelled out. So, d H by d t essentially turns out to be del H by del t on the optimal path. And that is the reason both actually; like, if del H by del t d H by d t is del H by del t, and then what you are telling again is Hamiltonian is not an explicit function of time that means, del H by del t has to be 0 basically; so, if del H by del t is 0, then d h by d t is also 0; and hence, the result actually.

That is a very simple proof for unconstrained thing; for constrained thing, similar things do exist you can try for arguing out yourself or you can see some reference actually.

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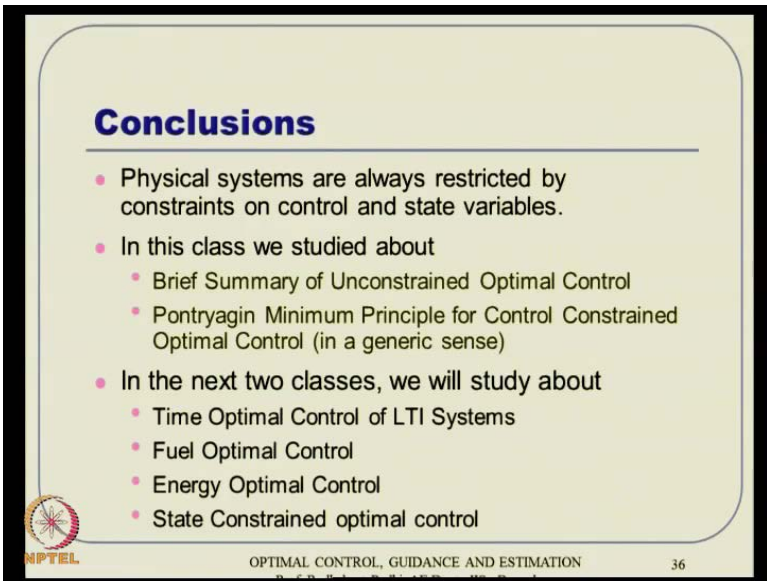

So, in conclusions about this particular lecture, just remember that physical systems are always restricted by constraints on control and state variables both. And in this class, we studied about something like a little bit brief summary about constraint of unconstrained optimal control just to recapitulate the ideas that we talked long time before, remember the early lectures.

And then, we try to kind of correlate and try to derive this Pontryagin minimum principle and some sought of an intuitive argument actually; and in that actually is a very generic sense, what you are talking here instead of del H by del U equal to 0, we are talking about minimizing the Hamiltonian within the available constraints phase.

And also remember that nowadays, this static optimization and numerical procedures are very strong; that means, even if you put a constraint like that, there are going to solve it very fast actually. So, I mean I am not telling that it can be solved in real time and all that is a different ball game. And In fact, this pseudo spectral method of solution and all, they are actually in fact talking about it can be handled online also it can be that fast basically.

So, do not get worried that, just because it is constrained problems we cannot solve it fast and all that actually, and it can be done probably. On the next two classes, we will study about various application problems; and towards the end of the lecture, we will talk about state constrained optimal control as well. And this application problem will typically three classes, time optimal control including a double integrative problem, then fuel optimal control problem, as well as energy integral control problem. And especially, in the framework of linear systems actually; because, that is where state feedback ideas can be brought in and then you will see it may result in discontinuous controller, but it will satisfy the bounds actually that is more important actually.

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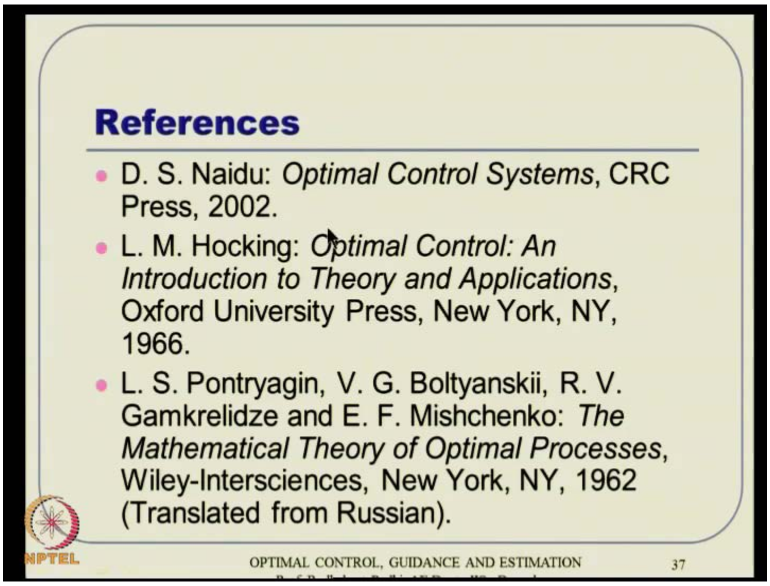

So, with that comments I will stop here, but before that we have some references as I promised, most of my material will be taken from this particular book for this 2, 3 lectures, D S Naidu for this 2, 3 lectures. And then I suggest that, many of you can actually buy this book it is a western print and somewhat economic print and all that are available now, probably you can buy this book as well.

And you have to very early references and this is where the Pontryagin original work translated from Russian to English, which is available in 1962 editions sort of thing, I do not know, whether it is really available to buy or it is simply available online free or something that I have not check. But, you can go back and check it yourself, whether it is really available.

And also remember that, the readings will not be in engineering sense, if you are very much math oriented and you have sufficient mathematic background and topology and all that; you can probably think about studying that and understanding that in a good way. And somewhat a simplified treatment is done in this particular book, this is also not a very recent book, this is also like something published in 1966. So, but I still think it means some form or rather it may be available over the internet or may be some (( )) work place or something, if they come up with some old classical books in the cheap price versions and all that.

So, maybe it is available or not I do not know, I have not checked it, but you are welcome to check it whether it is available somewhere, and you can buy it actually or you can borrow it or you can download and print out or something you can do that actually. We will we will study more in the next two classes about these problems, but for this particular lecture I will stop here, thank you.