Optimal Control, Guidance and Estimation
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Module No. # 10
Lecture No. # 20
Dynamic Programming

All right everybody, we will start with our lecture series on lecture number twenty on this optimal control, guidance and estimation course.

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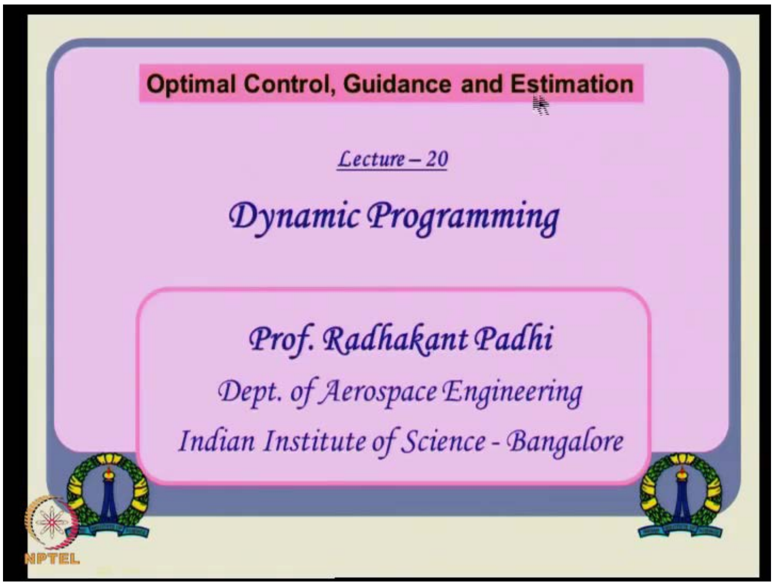

So far, you have seen this optimal control formulation from calculus of variations point of view. But, for here we will see a very different perspective. So, let us get started.

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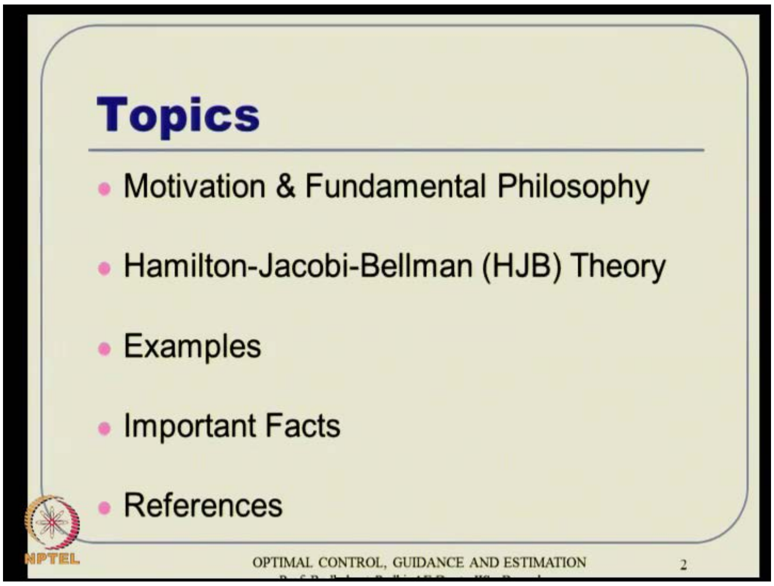

The first thing is motivation and fundamental philosophy, what is there in that. Then, we will continue with this Hamilton-Jacobi-Bellman theory. This is the backbone of this dynamic programming. Then, we will obviously give some examples and list out certain important facts. And then, I will also give some references for everybody to follow.

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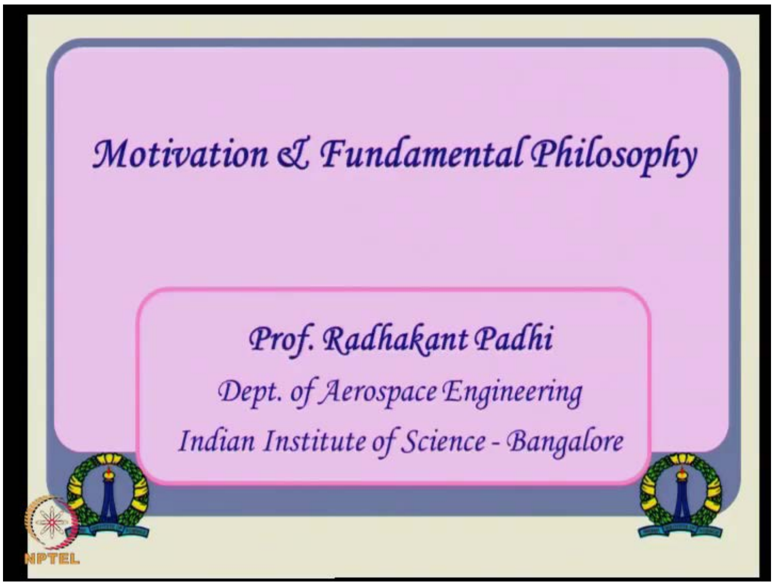

This motivation and fundamental philosophy is like this.

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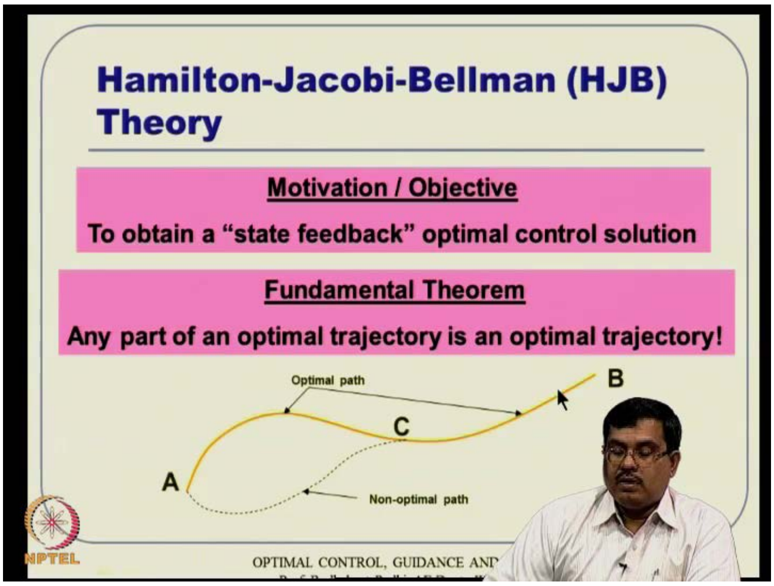

The first motivation is to obtain a state feedback optimal control solution. So far, what you have seen in calculus of variation and the necessary conditions and then two point boundary value problem and things like that. If you remember, it is ultimately to get a trajectory from a given initial condition. So, what you are interested in here is to obtain some sort of a tunnel of trajectories. In other words, given any initial condition, which we will be able to find a solution for the control – now, what does that mean? That means there is something like a state feedback optimal control. Given a state, you want to find out what is the control. And, remember, this control need not be a linear function; it can be any function of state. However, given the state, we will be able to compute the control; that is more important.

What is the fundamental theorem associated with that? It relies on this very intuitive piece of argument that any part of an optimal trajectory is an optimal trajectory; feels obvious, but it is little bit involved to kind of see that. This we are… Intuitively, we will show that here. Let us see, we have found out something like an optimal path from point A to point B via point C. So, this is somehow we have already found out the optimal path. Now, what this theorem tells us is any part, even if you take a small piece of this trajectory, is certainly going to be optimal by itself. And remember, that is a different problem altogether. Problem starts from A and stopping at B, is different from problem starting at A and stopping at C; think like that. So, what this theorem tells us again is if I take any slice of this trajectory, it is going to be optimal by itself.

How does the argument goes? Let us see that we find some other path from point A to C; and, from C to B, we will follow the same; already found an optimal path. Then, what does it tell? Now, let us say that this is optimal (Refer Slide Time: 03:11). If this optimal, then this cost function associated from point A to C, is lesser than this path. So, in principle, you could have taken this path from A to C, this dotted line; and, C to B you could have followed. And then, this path from this dotted line A to C and then C to B – that would have been our optimal path. But, this is not the optimal path, because we started with this A to C via this top line; and, that we claim that is the optimal path that we have already found. And hence, this cannot be the optimal path from point A to C. So, that is the very fundamental intuitive argument sort of thing, which tells us that any part of an optimal trajectory is an optimal trajectory itself. So, let us see how do we use this and to derive this HJB equation.

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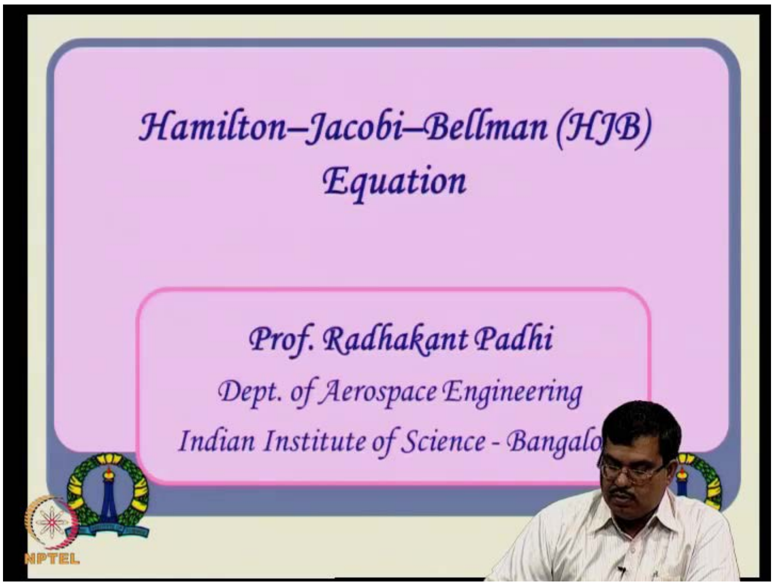

This is famously called Hamilton-Jacobi equation, but later on this name Bellman was also added to it, because of its own country using. So, now, it is famously called as HJB equation or Hamilton-Jacobi-Bellman equation.

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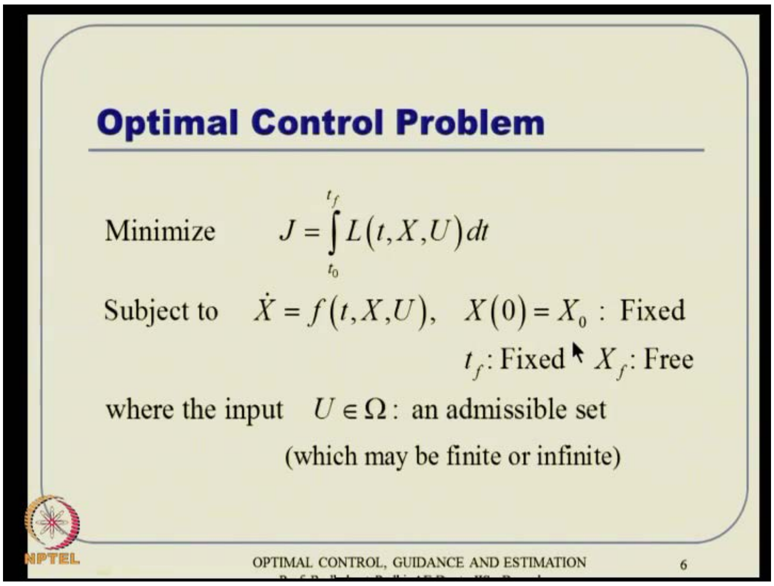

The problem is like this. Again, we go back to this fundamental problem. We want to minimize this cost function subject to this state equation. We have an initial condition and we have final time fixed and we also consider X f being free. Very standard thing, now, what we have been considering so far; the only difference here is outside the cross function, there is no term. That means there is no terminal penalty for say here. And, this is purposefully done, while introduced this terminal penalty slightly later. It does not have the derivation to begin with. And later, we will try to introduce that also.

And also, remember the control here is bounded. That means this omega is an admissible set (Refer Slide Time: 05:03); and, any solution that lies outside this omega, we are not interested in. However, this omega can be infinite as well; that means if you really want to formulate a problem, where the control variable is unbounded, then this is also feasible. But, if it is bounded, for most of the practical application we are interested in bounded set only, then also, this formulation is capable of handling that.

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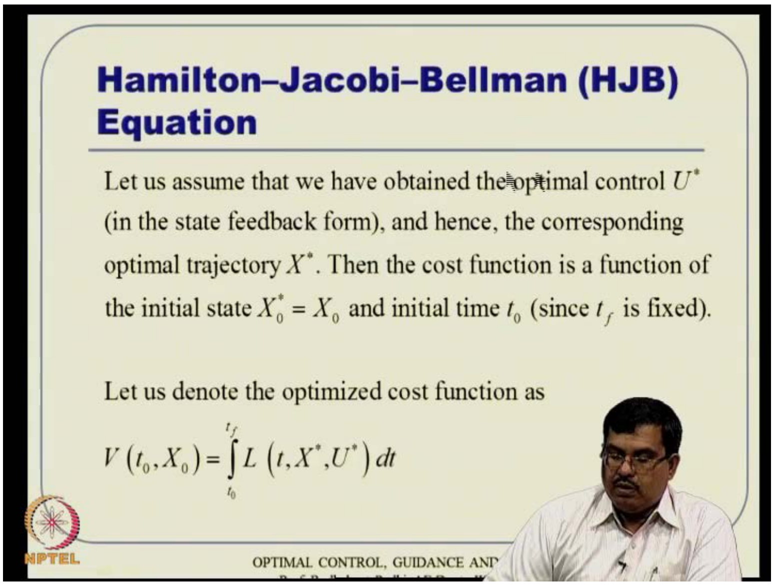

Let us see how things proceed. Now, let us assume that we have already obtained an optimal control solution U star that is already available to us in the state feedback form. And hence, the corresponding optimal trajectory X star is also available. Just for a second we assume that this control has been found out, which is optimal control already; and associated with that, this X star is also available. Then, the cost function is a function of the initial state only and obviously initial time as well. Remember, this cost function – if you see that a little bit, this cost function is here (Refer Slide Time: 06:15). We have found an optimal trajectory and an optimal control associated with that. And, remember, t f is fixed. So, there is no variation on that. So, what remains is initial condition and initial time. So, this needs to be optimized.

And, let us denote the optimized cost function as something like this (Refer Slide Time: 06:35). Now, remember, if I evaluate this cost function now, it becomes a function of only t naught and x naught here (Refer Slide Time: 06:42). So, this cost function what I am getting here, I can denote it something like V, is a function of t naught X naught. Why we are specifically interested in this? If somebody wants to know, ultimately, this X naught is nothing but the current state and t naught is nothing but the starting time; wherever I am, that is the t naught and whatever X I am having, that is X naught. So, I want to interpret this entire problem as something like initial time and initial state. So, this V is defined as something like this. Everything else is evaluated such that it remains as a function of t naught and X naught.

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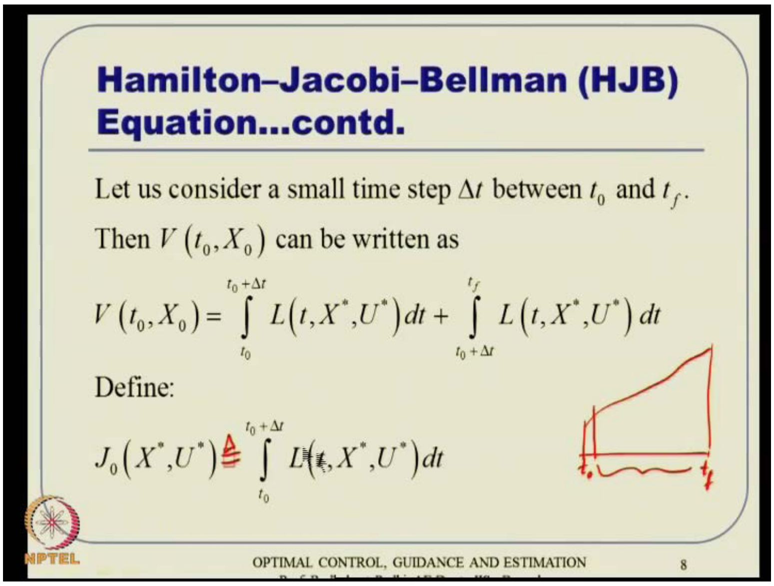

Now, let us consider a small time step delta t between t naught and t f. So, what we are telling is, there is a t naught there; it starts with something like this; there is a t naught and there is a t f; the trajectory is there, t naught to t f. But, what I am interested is t naught to some delta t. And then, this one is rest of the time. This is what I am interested in. Then, what I am interested, what needs to be done here is this is V of t naught, X naught, which is nothing but an integral. Remember, V of t naught is something like this (Refer Slide Time: 08:11) – t naught to t f. So, what I have done is, this entire segment of t naught to t f, I have divided in two parts: one is t naught to t naught plus delta t and the other one is t naught plus delta t to t f. So, obviously, this integration formula that we know can be split into something like this (Refer Slide Time: 08:27). First is t naught to t naught plus delta t; and, the second part is t naught plus delta t to t f.

Next, we define something like this (Refer Slide Time: 08:37). So, this is a definition symbol, because of this power point version, there is some message here. So, essentially, you can define it something like this. This j naught – I define it as something like this (Refer Slide Time: 08:55). So, what happens here is, I want to analyze this particular thing little more carefully; and, rest of the things is the optimal cost anyway. Remember, this is the optimal cost already (Refer Slide Time: 09:10); this is (Refer Slide Time: 09:15) optimized cost. So, if I have a path from t naught plus delta t to t f, that becomes an optimized cost from that point of time; that will also use that. Now, this particular time, I mean this (Refer Slide Time: 09:28) t naught to t naught plus delta t is the immediate effect of the control. And hence, I want to give you a little bit more emphasis for that.

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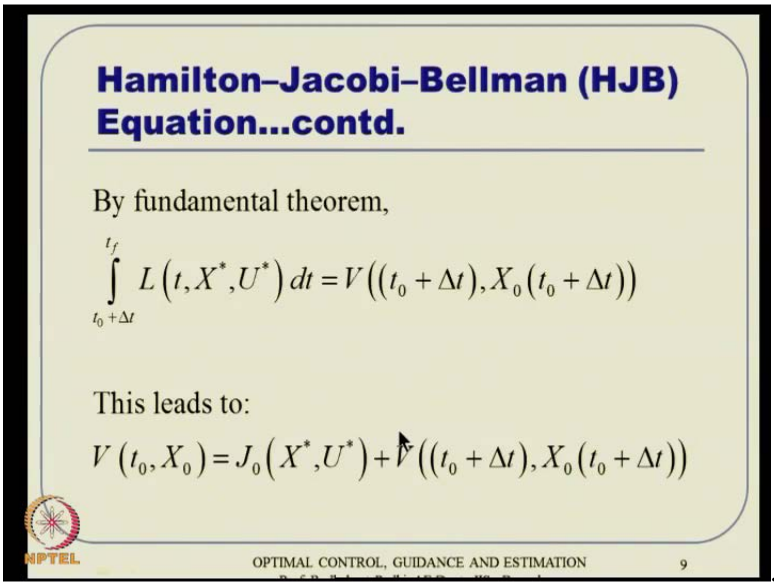

Then, what happens here? Then by using this fundamental theorem as I told, that means, any part of the optimal trajectory is also optimal. What you can claim first is t naught plus delta t to t f, that part of the cost function is same as this one. That is also an optimal path. So, this leads to this kind of identity that V of t naught, X naught is nothing but j naught of X star U star. This is coming from (Refer Slide Time: 10:05) the first part of it; and, the second part is coming from this side here. So, some people call this as cost to go and something some people call it a utility function and things like that. So, this is also possible to define. In the next class, we will see in approximate dynamic programming. Those are the things terminologies used there. Just to repeat, this is the utility function and this is the cost to go from t naught plus delta t.

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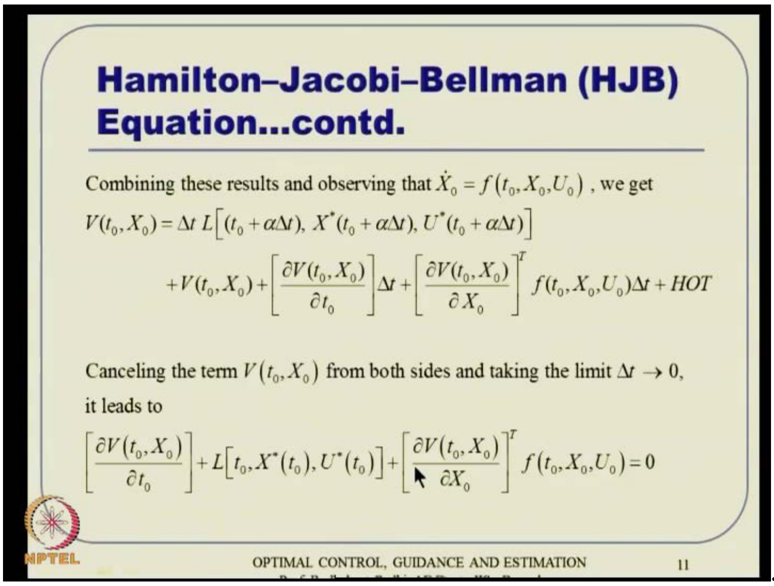

Now, what? Let us try to analyze this little more.

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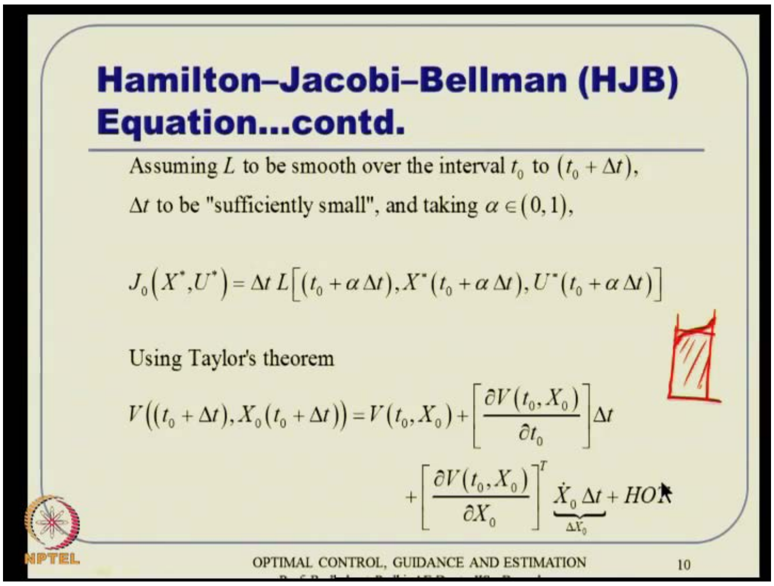

Now, first thing is this portion. Now, this portion is something like this (Refer Slide Time: 10:52). This portion can be written as… This is nothing but an integral (( )) This j naught of X star, U star is nothing but that integral there – this integral (Refer Slide Time: 11:07). So, this integral can be approximated or using this calculus ideas, what you can do is, we can write it this way. We assume that this L to be smooth over the interval; obviously, L is part of a cost function what we select. So, obviously, it has to be smooth.

And also, delta t needs to be sufficiently small. And then, there is a theorem in calculus, which tells us that if I select an alpha properly between 0 and 1, then I can exactly write this equation that way. It is almost like trapezoidal rule, but it is something like… Let me pictorially try to put it for the benefit of you over there. What you telling here is, there is a function here; I want to integrate this; what it tells is, there is some value, about which if I take the area of this particular rectangle, then this is same as area under the curve. But, it is not of a three volume thing; there exit an alpha between 0 and 1, so that this is identically equal.

Now, what about the next one? If you remember, there are two parts (Refer Slide Time: 12:28). One is this part also. That part – you can take this Taylor's theorem now (Refer Slide Time: 12:39). What does this Taylor's theorem tells us? That if there is an increment of a function something like this, I can always use the Taylor's series and write it this way. First is V of t naught, X naught; second term is for this first order term with respect to delta t; and, the first order term with respect to delta x. For this, delta X naught, I can always write it in this way (Refer Slide Time: 13:00); X naught dot times delta t also. So, this is what using Taylor's theorem and taking up to the first order terms and keeping the higher order terms this way.

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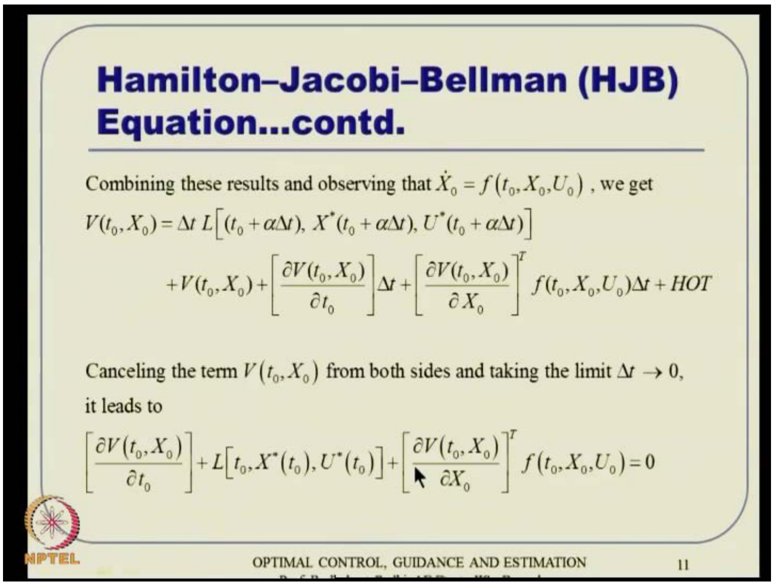

Next, what happens? Now, we combine both the things; then, what happens is V of t naught, x naught; this is the first part, what we get from that calculus area equality; and, this is what the second part, what we got from the Taylor's series expansion. Now, remember, this part – this (Refer Slide Time: 13:34) X naught dot is nothing but f of t naught, X naught, U naught; that is from the system dynamics. So, now, I get some form of something like this. So, what can I do now? You can very clearly see that there is a V naught t naught X here and there is a V naught t naught X here. So, I can actually get rid of these two. If I do that, then what I do is I take delta t tends to 0. So, I can divide everywhere by delta t and things like that.

Remember, the higher order terms will have delta t square and things like that; so, delta t square, delta t cube and all higher order terms. So, what happens here, even if I divide it by delta t and then delta t tends 0, then all these terms will go to 0. So, what remains here? First term is this term (Refer Slide Time: 14:24). Remember, first order delta t's are gone. So, this is the first term; coming from here is the first term. And, coming from here, this term is here; and, coming from here, remember, delta t's are all tends to 0; that means all delta t's are gone here. What is remaining is this one. So, this now you can recognize that as I told you sometime back that what you are interested is solution from current time and current state.

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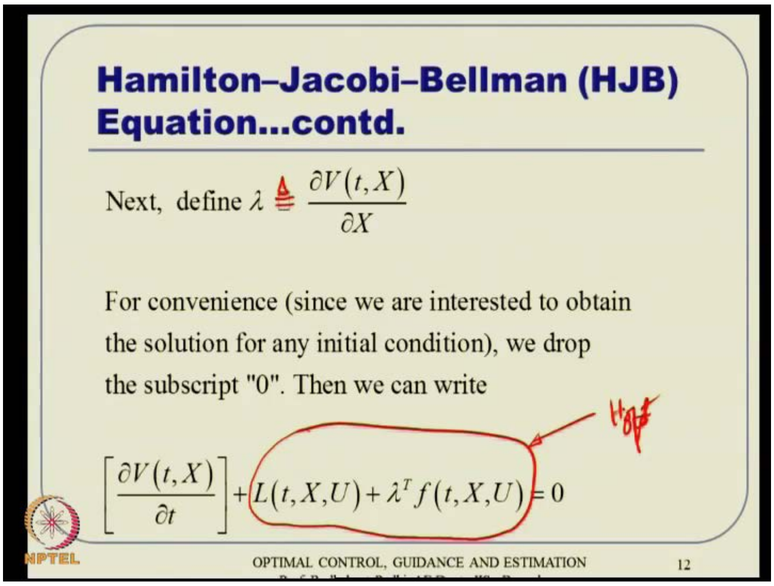

So, all these, what you see here, these arguments can be dropped; this t naught, X naught and all that, you can interrupt with that as current time and current state. So, these arguments I can drop out and write these expressions little bit later. But, before that, I can also define… Again, this is the definition symbol; I can define del V by del X as nothing but lambda. And, in fact, just to comment, from this definition, del V by del X happens to be the same lambda coming from calculus of variation (( )) So, this gives us some different interpretations for this co-state variable lambda, which tells us, lambda is nothing but the sensitivity vector of the optimized cost function with respect to state.

Anyway, coming back to that, we just define lambda like that. So, what you see here (Refer Slide Time: 15:52) is nothing but lambda transpose. The only difference is this V remember is not that cost function; it is the optimized cost function. After you find the control solution, after you put it by and evaluate the cost function, that is the optimized value of the cost function. Anyways, using this definition of lambda (Refer Slide Time: 16:14) and dropping the arguments t naught, X naught to t and X, what you get here is this one (Refer Slide Time: 16:19). Then, this equation is famously called as HJB equation (Hamilton-Jacobi-Bellman equation). But, also remember, what is L plus lambda transpose f. If you remember this part, this L plus lambda transpose f – if you remember, this is nothing but Hamiltonian. But, this is not any Hamiltonian, because remember, this lambda involves V and all that; V is an optimal curve. So, this is optimized Hamiltonian.

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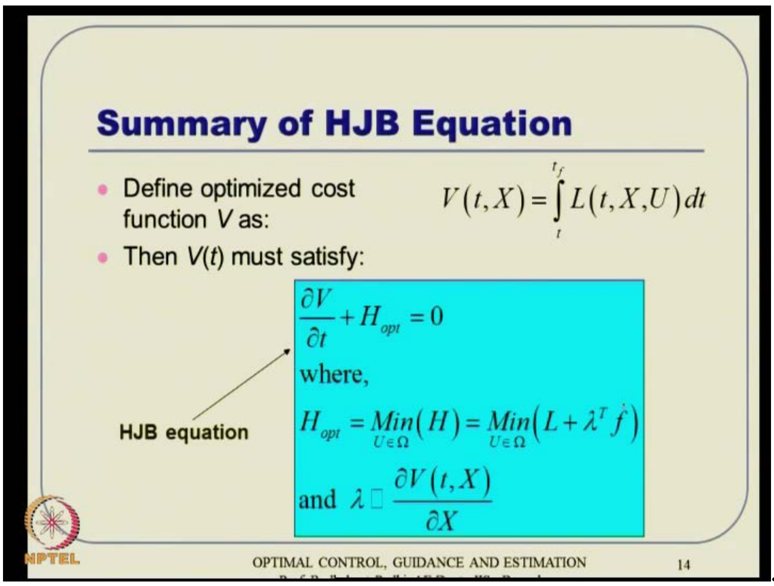

So, this is what is defined.

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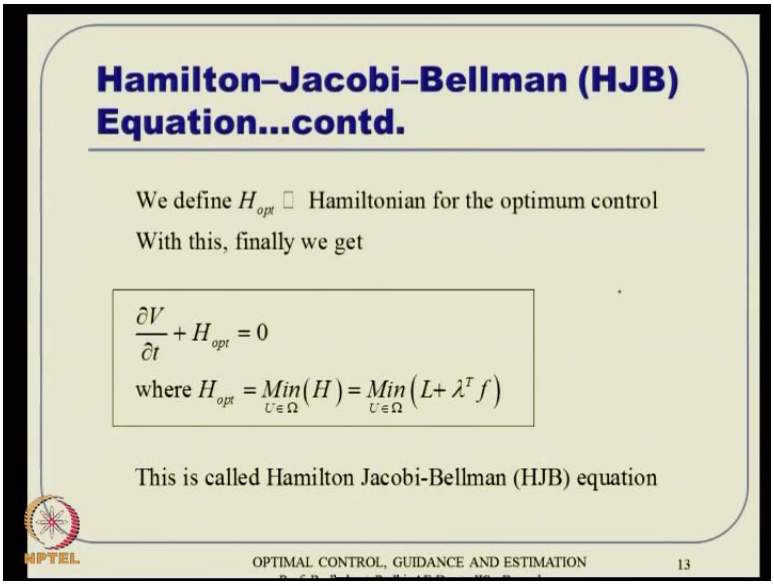

We define this H opt to be Hamiltonian for the optimal control. And, with this definition, this del V by del t plus H opt is equal to 0; that is that what you can write. So, that is what we can write here. I can tell that del V by del t plus H opt is equal to 0; that is very famously called HJB equations, where this H opt is minimum Hamiltonian over this constraint set omega. So, this has to be found out first. As I told, this optimizes Hamiltonian is to be found out. And then, you put it back here and whatever equation you get is called HJB equations. We will see examples and how to do that.

What is the summary of HJB (Refer Slide Time: 17:59) equation? First, you define this cost function; V of t, X is equal to that way. And then, this V must satisfy this equation, del V by del t plus H opt equal to 0. It is very easy to remember way the. All that is to remember very quickly, del V by del t plus H opt; that is equal to 0. But, also remember this H opt is optimized Hamiltonian; that means you have to put the definition of Hamiltonian first, l plus lambda transpose f, where lambda is defined to be del V by del X. And then, you can substitute the expression (( )) here and solve it further to get control in all that. How do you do that? I will give an example to demonstrate that. Remember, L contains a control and f contains a control. So, those control variables need to be solved as a function of lambda and X. But, lambda – remember, is also a function of X, del V by del X sort of things. It will all come in some sort of a state feedback form.

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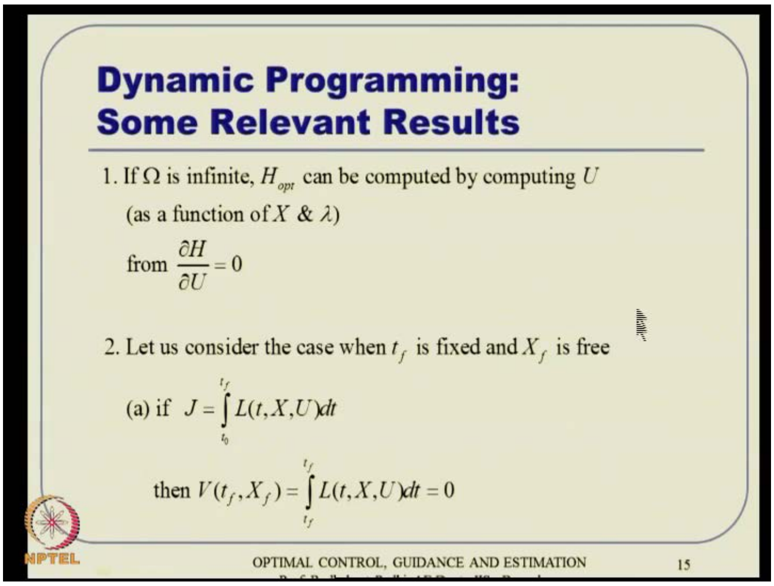

If I move further, there are certain relevant results. Let us first see that. And, the first result tells us that if omega is infinite, that means we talk about something like an unconstrained control problem. Control variable is not confined to a particular finite set sort of thing. Then, this H opt can be computed by computing this equation, del H by del U equal to 0. That is what we have seen from the other side of the story, calculus of variations. So, if you put this equation, del H by del U equal to 0 and solve for whatever control expression as a function of state and co-state, X and lambda, then that is the way to proceed.

You put this equation, del H by del U equal to 0; get a control as a function of state and lambda; and then, wherever control appears here (Refer Slide Time: 20:01) in this equation, substitute it there and try to solve for… Some people try to solve for V directly, because this equation is interpreted as a function of V finally. So, you solve it for V. And, once you solve for V, then lambda is available. And, once lambda is available, remember, all these are available as mathematical expressions. So, once lambda is available, you can get the control back from this equation (Refer Slide Time: 20:25).

Now, what is the second comment? Second result tells us that when t f is fixed and X f is free, if j is like (Refer Slide Time: 20:38) this, then the final boundary conditions tells us that if you put V of t and X; instead of t 0, you put t f. So, V of t naught, X naught – if you remember, that is the definition starting from t naught. Now, what I put is V of t f, X f; first, the boundary conditions sort of thing. Then, this t naught, and I have to put t f; and hence, it does not matter; t f to t f, an integration of any integrand integrating over the same interval happens to be 0. So, what you are getting here is if t f is fixed and if X f is free, that is what we started with, then this is the boundary condition that we need to use.

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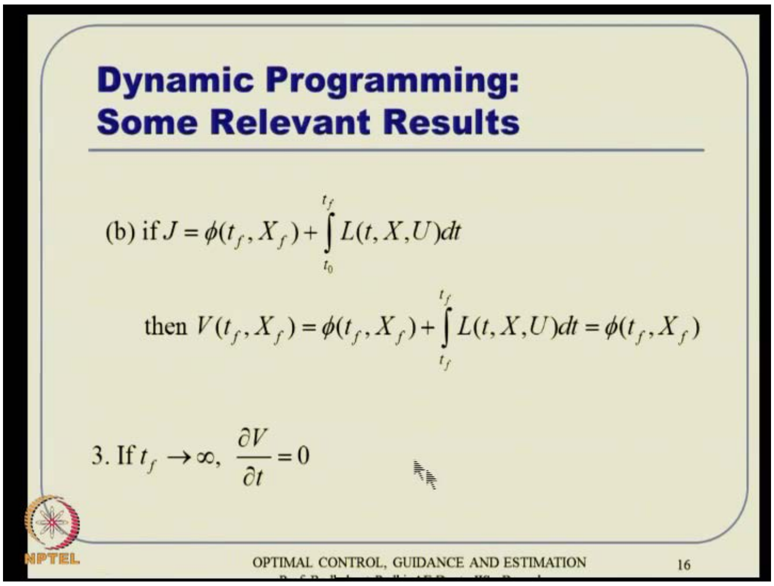

Now, what about this terminal penalty? That is what I told you. I will talk about that. If you have a terminal penalty also, then what happen is, if you put t f, X f, then this part goes to 0 and you are left out with this part. So, what it tells us is just the boundary condition that will be differing, boundary conditions for V. If you do not have any terminal penalty, this happens to be 0. If you have some terminal penalty, then V of t f, X f happens to be that function.

Now, the (( )) tells us that if t f goes to infinity, then what? Remember, many times we talked about infinite term related problems and things like that. So, then, t f is fixed, but due to fixed infinity. So, if t f goes to infinity, then some relevant results tell us that del V by del t has to be 0. This is another useful kind of results. And, this if you analyze a little bit (Refer Slide Time: 22:20) here, del V by del t equal to 0. And, let us imagine a case, where you have something like scalar problem; that means X is a single variable sort of thing (Refer Slide Time: 22:29). In that case, if del V by del t goes to 0, then what we have is an ODE equation.

This del V by del x is nothing but only one state variable. And hence, this del V by del X is nothing but d V by d X in that case. So, you put it back; you get this PDE, results in ODE. So, that remains with that. This HJB equation, if you analyze little bit, this is del V by del t here and then del V by del x here. So, this is a non-linear coupled partial differential equation. So, this non-linear coupled partial differential equation is not easy to solve, but it results in that way. But, if t f goes to infinity, then this part is 0. And, if you have a similar problem, then it results in some set of ODE basically. What you also remember that ODE is not a linear ODE; that means that is typically a non-linear ODE for a non-linear problem. Anyway, we will see examples for that. So, this is the result (Refer Slide Time: 23:38). But, if t f goes to infinity, then del V by del t is equal to 0.

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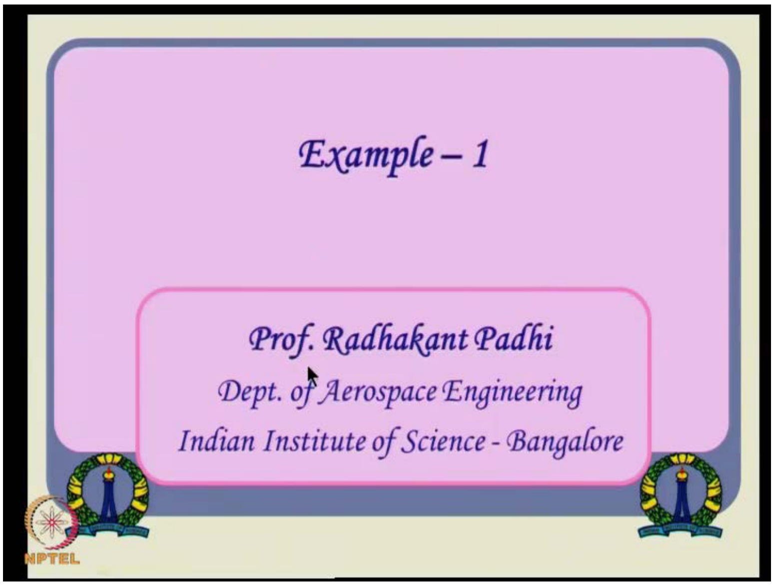

Let us see an example, which will clarify our ideas a little bit.

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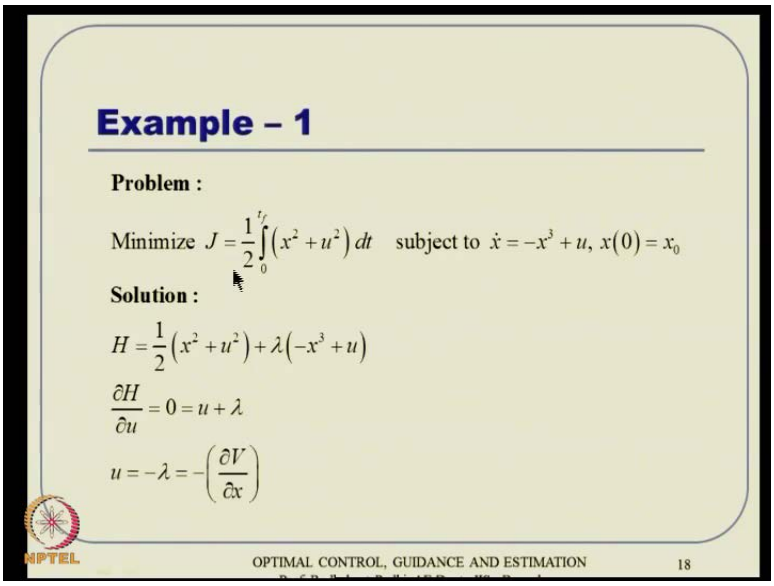

It is a very simple example sort of thing. You have a standard quadratic cost function starting from 0 to t f x square plus u square dt, subject to this non-linear system equation x dot equal to minus x cube plus u; initial condition is known. It is an unconstrained problem; that means control early is not confined to a particular set. So, let us start solving this. First thing is definition of Hamiltonian. Hamiltonian means l plus lambda transpose f. l is half of this fellow; half goes inside. This is l plus lambda times f; f is nothing but this – minus x cube plus u. So, this one you put it back here; this l plus lambda times f; there is a thing called lambda transpose here; lambda is also scalar. And, the second result is this – because it is unconstraint, del H by del u has to be equal to 0; that means…

Now, what is del H by del u? First term is nothing but u; and, second term in nothing but lambda. So, this 0 is equal to u plus lambda. So, what I get as solution is u equal to minus lambda; that is nothing minus del V by del x. So, the point here is if I give a solution for V as a function of state somehow, then I can do this partial derivative; I will get lambda after that. And, once I put it back, negative of that is nothing but the control, equal to minus lambda; remember that. So, all that it matters is getting a solution for V. Let us see how to do that.

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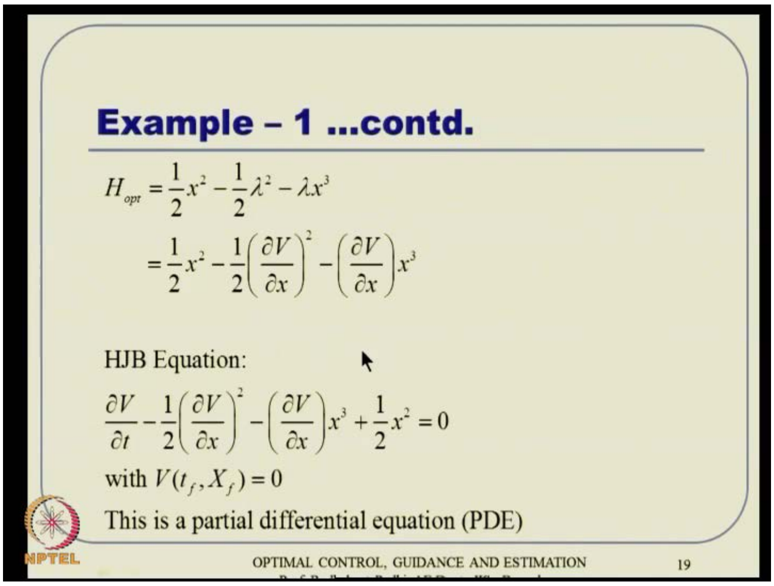

Now, if I do that, you need to evaluate this H opt. H opt is likely… This is the Hamiltonian definition (Refer Slide Time: 25:42). But, remember, u is now like this; u is minus lambda and things like that. So, if you put it back, that is what I get as optimum Hamiltonian. When lambda by definition is nothing but del V by del x. So, wherever lambda is there, I will put del V by del x. So, the H opt is like this. So, in this particular example, the HJB equation tends out to be this. So, if I see this equation in V, there is a partial derivative with respect to time and there is a partial derivative with respect to x, which is also squared already here. So, essentially, this is it; this results in something like a non-linear partial differential equation, becomes difficult to solve.

And, also remember, on the way, this V of t f, X f is equal to 0, because there is no terminal penalty here. There is nothing here. So, that is 0. That is why this result comes (Refer Slide Time: 26:44) – V of t f, X f is equal to 0. Now, this is a PDE and it is difficult to solve. Here we introduce one more assumption; but, what about t f going to infinity? If t f goes to infinity, then obviously one of the results implies that del V by del v goes to 0. Now, this goes to 0. As I told you some time back, this (Refer Slide Time: 27:08) entire thing is nothing but an ODE.

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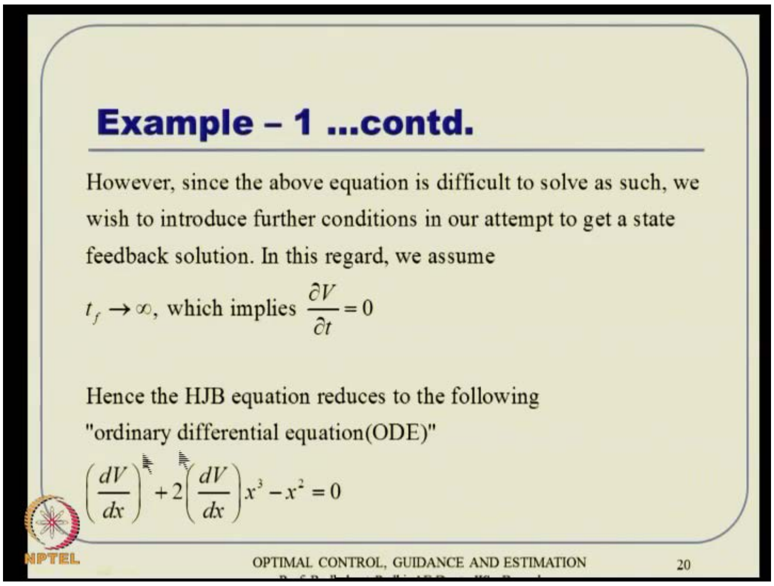

This particular ODE or ordinary differential equation can be written something like this. Still this is a nonlinear ordinary differential equation (nonlinear ODE). What you get is much simpler than this PDE (Refer Slide Time: 27:29); or, still it is a nonlinear ODE. So, how do you solve it? Solving it in closed form is also a trick situation for nonlinear ODEs. And, one of the ways to solve is something called cover series approximation and all that. So, let us in (( )) that.

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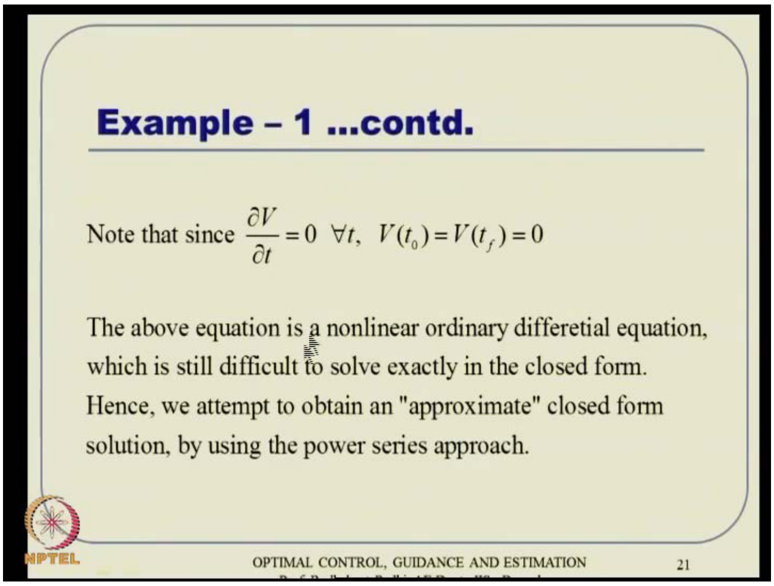

Seriously, this is what I have already told (( )) This is an ODE and still difficult to solve. And, what you are interested in is something like an approximate closed from solution. So, one of the way to explore is power series approach. So, let us explore that.

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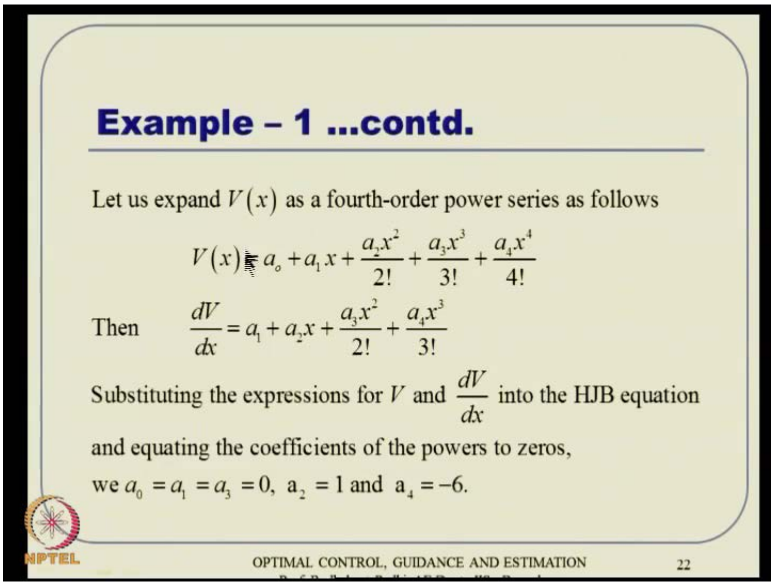

What you are telling here is let me write this V of x as something like five terms here. And, writing up to five terms, you can think that is not the way to get. What I am taking here is zeroth power; starting from zeroth power constant all the way up to fourth-order power. So, what I am expecting here is some sort of a good solution even though theoretically it is an approximate solution.

Now, what you need? If you see this (Refer Slide Time: 28:36) equation, we need an expression for del V by del x here as well as here. So, you put that expression del V by del x; it is very easy; this term will be nothing, zero; this is a 1; this is a 2 x and all that. So, this d V by d x results in this term (Refer Slide Time: 28:52). Now, what you do is, we will not do the entire thing, but what is necessary here is – dV by dx is available now. Now, the HJB is equation is like this – this (Refer Slide Time: 29:07) equation. So, we put d V by d x whole square here, whatever expressions we are getting. And then, we try to kind of equate the various powers and then get a set of coefficient equations sort of thing (Refer Slide Time: 29:21). That I will leave it as some sort of no more exercise sort of thing.

What is needed is you got the expression from V and d V especially for d V by d x. That expression you take it and substitute it in the ODE, whatever ODE is here. Collect various powers of x; and, remember, right side is 0. So, that means all coefficients have to be 0 (Refer Slide Time: 29:47). So, all coefficients will become functions of these constants, whatever constants you are having (Refer Slide Time: 29:52). And then, you will see that this will result in a set of equations in the parameters. These coefficients are parameters. And, if you solve that set of equations, this is what you will get. What you will get is a 0, a 1 and a 3. a 0, a 1 and a 3 – all these things are 0. a 2 is 1 and a 4 is minus 6. This exercise you can do it yourself and see. Now, once you do that, once you have it, then you got an expression for V and expression for del V by del x.

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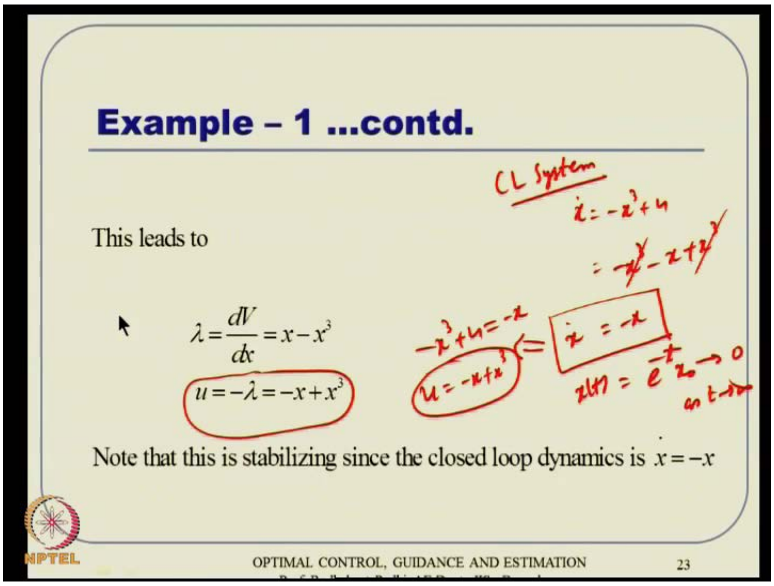

So, let us go and see that. Ultimately, what it leads to if I substitute it here (Refer Slide Time: 30:34). Let me do that probably. I have got V of x; this is equal to… First one remember, a 0, a 1 and a 3 are 0. So, that I am not taking. a 2 is 1. So, this is x square – a 2 is 1, remember that – by 2 factorial, is two; and, a 4 is minus 6. So, what I get is minus 6 x 4 divided by 4 factorial; so, that means 4 into 3 into 2 into 1 obviously (( )) So, this 3, 2 – 6; this one is gone; this – what I am left out is something like x square by 2 minus x fourth divided by 4. Now, if I take del V… This is V of x. If I take del V by del x, obviously, from this part of the story, if I take this expression, this turns out to be nothing but x minus – this 4 x (( )) will go – so I am left out with x cube. This is what we will get. Then, d V by d x ultimately is x minus x cube. That is the (( )) what we are getting here.

Now, u turns out to be minus lambda (Refer Slide Time: 31:52). So, what happens? u is nothing but minus x plus x cube. What is happening finally? The beautiful thing is we have got a state figured control even though it is a nonlinear optimal control problem to start with. And, we had no terminal full empty; and, t f we assume to be infinity and things like that. Ultimately, it resulted in something likely a closed form expression for the controller. So, given a state, I can evaluate this control and I am done. That is the beautiful thing about the HJB equation in general dynamic programming.

Now, what? There is some interesting analysis that follows here. Let us see that also. We started with something like x dot is nothing but minus x cube plus u. Now, u is nothing but minus x. So, if I substitute this… This is the closed loop system. If I substitute this, which is minus x cube plus u and then that u is this one – so, I will substitute that, this is minus x plus x cube. This x cube gets cancelled out. So, what it results is x dot equal to minus x. And obviously, this is a linear equation essentially. And, with minus x, that means eigenvalue is minus 1. And hence, the system is stable. You can also very quickly realize that this is the solution, e to the power minus t into x naught. So, this goes to 0 as t goes to infinity. So, that means you got a solution, which is also stabilizing.

Now, there is another interesting argument here. Let me find it out all also. What you get here is (Refer Slide Time: 33:46) something like stabilized linear equation sort of thing. Somebody can also argue that while going through all these exercise, you go to this power series expression (Refer Slide Time: 33:59); here up to fourth-order term; write this partial derivatives. Put all the coefficients like in other equations, equate the coefficient; that exercises are not shown; equate all the coefficients to 0; solve the equations, find the coefficients and then do this analysis to get it there (Refer Slide Time: 34:13). Ultimately, what it is giving? It is giving something like a stable linear aerodynamic sort of thing.

There if you (( )) know to derivate an equation; and, those who do not know, you can see the other course. There are lectures on that. They also very well argue that I can derive this equation (Refer Slide Time: 34:33) coming from dynamic inversion sort of ideas by enforcing this equation to begin with. How do I do that? Let me see that; can I not solve like control (( )); that means I start from this equation. And, this equation will give me that this is the derived x dot. But, the x dot is nothing but minus x cube plus u; that is the x dot. That is something like minus x. That is what this equation is giving. So, u is nothing but minus x plus x cube; exactly, what I got here. Whatever I got here (Refer Slide Time: 35:20) is same as what I got here. And, here it is just a two line expression. I start with this expression x dot equal to minus x. And, this is the derived x dot; but, the actual x dot is up to minus x u plus. You have to substitute that; and, that solves the control into line.

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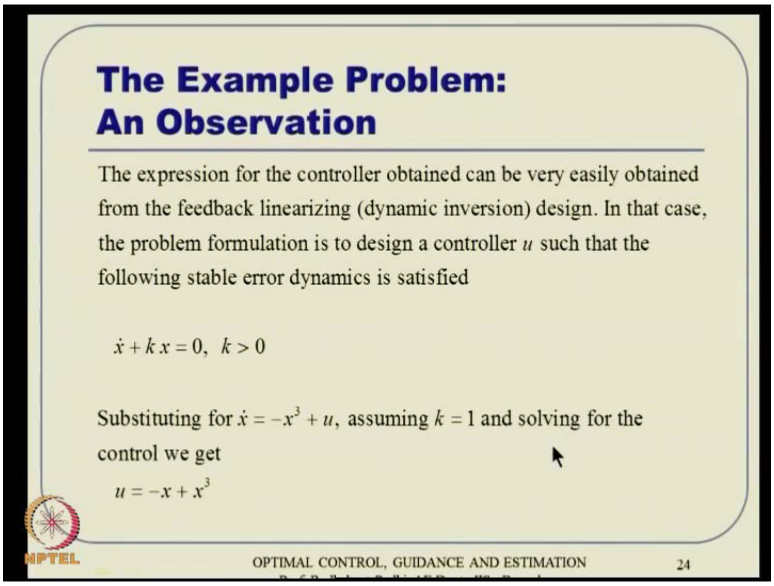

That is what I have written here. You start with something like a stable… desired dynamics sort of thing. You substitute k equal to 1 and then substitute (( )) and get it very quickly; results in the same thing.

Now where is the problem here? The problems are somewhat like this. This (Refer Slide Time: 35:58) particular solution that you are looking here is really not a good solution. In other words, it is a stabilizing solution; it will do the job, but if you are a little bit clever or something, you can also see that this control expression what you are getting here is cancelling out the beneficial nonlinearity. We will see this in this theory lecture also. So, minus x cube is beneficial nonlinearity, because any terms, x dot equal to negative of x to the power n as long as n is odd – this is the stable dynamic, because if I take Lyapunov function, V of x equal to half x square and I get V dot, it is nothing but x into x dot; x into x dot is nothing but minus of x to the power n plus 1. We can write it clearly. Whatever I am getting here is minus of x to the power n plus 1. If n is odd, then n plus 1 is even; and then, minus of that is certainly going to be something like a negative definite function.

If you see this analysis, it tells us that minus x cube what was here was stabilizing term. And, by doing all these, we have cancelled out that beneficial nonlinearity. Why it happened is because of what we thought that fourth-order power series approximation is going to do that job is not. So, if you really want to have a very good solution, probably, you could have continued up to something like twentieth order or things like that. Even people do sometimes more than that. Now, if you do that, then all the elegancy is lost. In other words, even if you take away five coefficients, the algebra involved, which I did not show here, is quite a bit. Now, if you have more and more number of coefficients, the algebra involved will be more and more. So, even for a simple scalar problem, you can realize what kind of algebra difficulties we get into.

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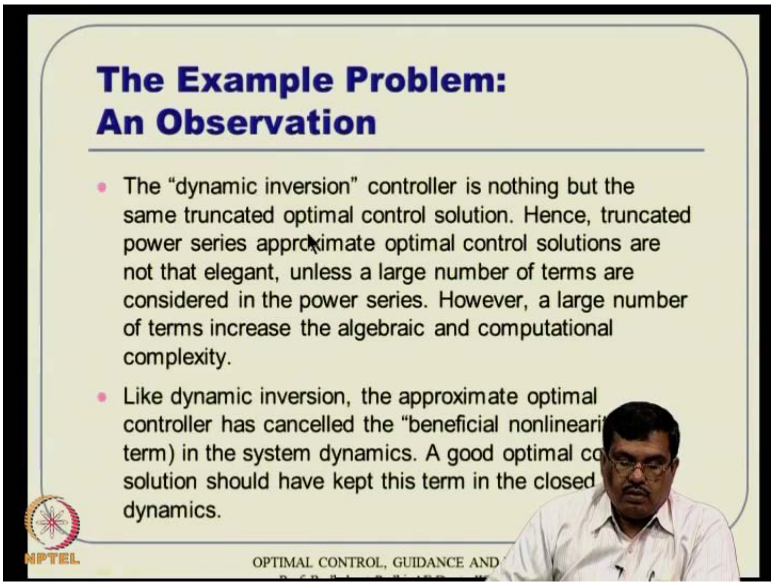

We have some comments; I can think about. This is the observation comments and things like that. So, what I told already is summarized here. The dynamic inversion control is nothing but the same truncated optimal solution. And hence, the truncated power series approximate optimal control solutions are not that elegant, unless a large number of terms are considered in the power series. That is what I told you. Instead of fourth order term, probably, you could have taken tenth order, fifteenth order, like that, twentieth order, things would have been better. But, a large number of terms increase the algebraic and computational complexity. So, just blindly having more and more terms in the power series, is better for the solution property. But, it will also give us lot of ethics for getting the solution.

And, as I also told you, like dynamic inversion, the approximate optimal controller has cancelled the beneficial nonlinearity – minus x cube term. And, a good optimal control solution should have kept this term in the closed loop dynamics. We will also see that how it is possible and all that, how is possible to do that.

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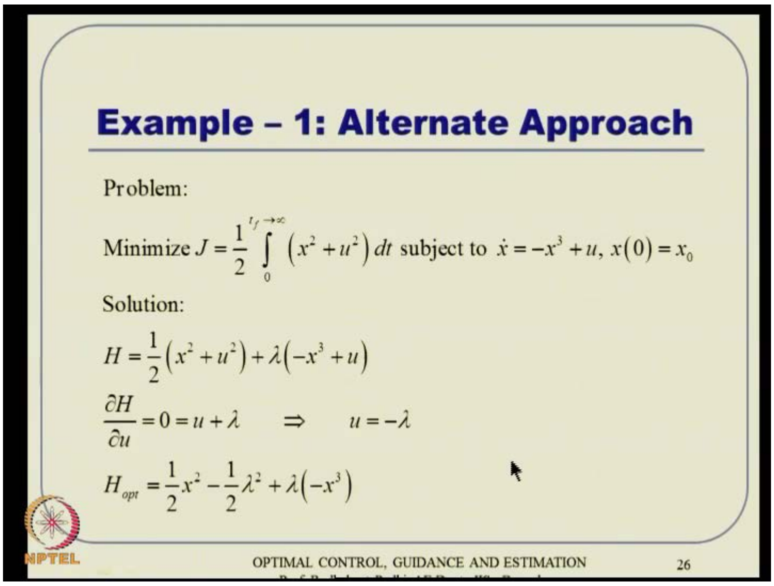

This region – this takes me to different approach. Different approach in the sense, this is my own observation; why the difficulty happened in my own observation is, because we wanted to solve for V. Remember, this (Refer Slide Time: 40:07) HJB equation; whatever you are telling is ultimately in equation 1 in terms of V. So, we attempted to solve for the variable V, which is optimal cost. However, if you just see a little bit more, all other control is a function of lambda, either x or lambda. And, lambda is nothing but del V by del x. So, why unnecessarily going ahead in solving for V, instead you can just simply solve for del V by del x; and, we have done, because our control is ultimately function of del V by del x as you know that; it is not x and V. So, we stop at the equation, where you just simply solve for del V by del x and then we will see what happens. I will give you one example; and, the extension of that example also let us see. In the same example that we have been talking about, this is exactly the same equation and things like that; same example. Here u is again minus lambda; and, H opt is this.

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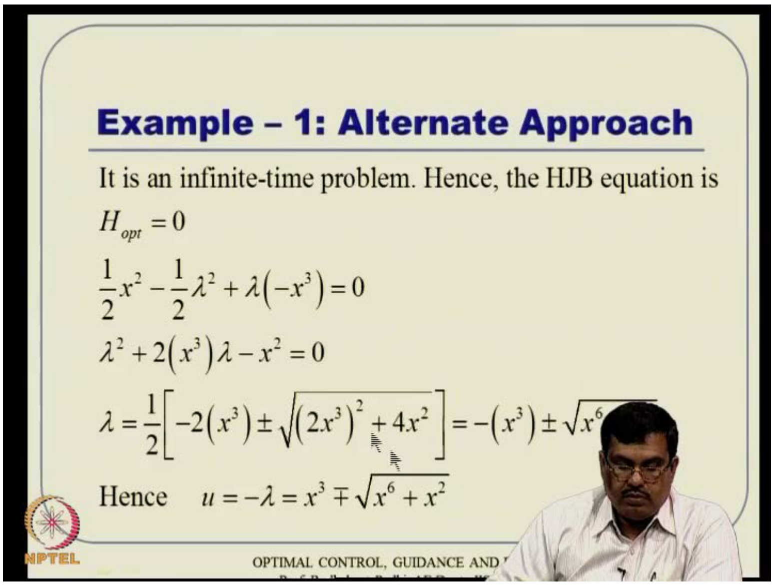

We have H opt equal to 0. And, get this equation. The problem here is… The previous approach was we substituted for lambda, that is, del V by del x and attempted to solve for V as a power series in fact. Instead of that, we want to solve for del V by del x, which is nothing but lambda. So, what I will do is I will interrupt you at this equation, what I am getting here, is something like a quadratic expression for lambda. And, this is nothing but a quadratic equation. So, I get the solution, lambda is nothing but this minus V plus or minus V square minus 4 ac, all that by 2. If I put it there, what I am getting is something like this. So, the control is ultimately minus lambda; remember that the control is minus lambda (Refer Slide Time: 41:56). And, lambda is (( )) is a function of state. There is a little bit ambiguity here, which we need to resolve. So, what we are having here is u equal to minus lambda, is nothing but… lambda is this; so, minus lambda is x cube – this is plus or minus or minus or plus – either way. So, what it tells us? We have to select now. We have two possibilities here and we select. So, what you may claim here is we just select the negative one.

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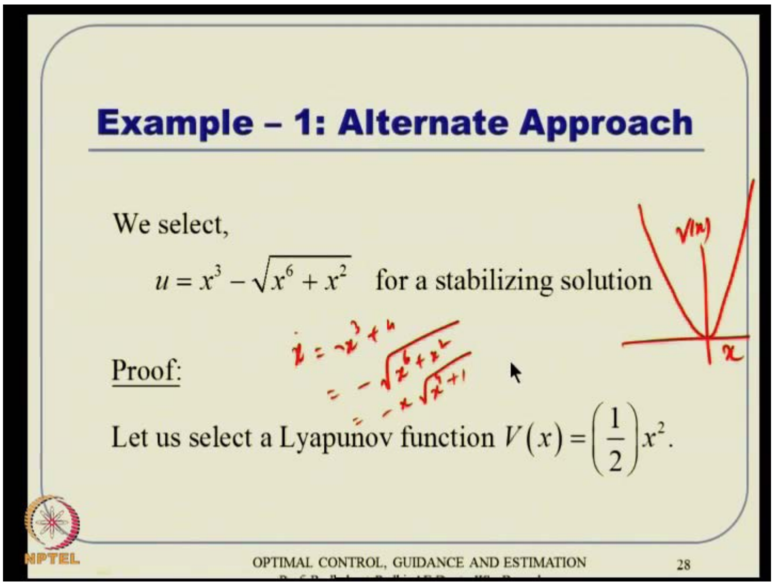

We select this negative term out of this plus or minus or minus or plus; I claim. And, if you select the minus term, it is good; it is also a stabilizing solution. This expression you can never get it as power series expression. This expression is coming out naturally from this equation here. So, this is what it is for a stabilizing solution. And, let us select the Lyapunov function. How do you prove that again that this is stabilizing solution? So, we select this standard Lyapunov equation. And, all that we need to show is V dot is negative definite.

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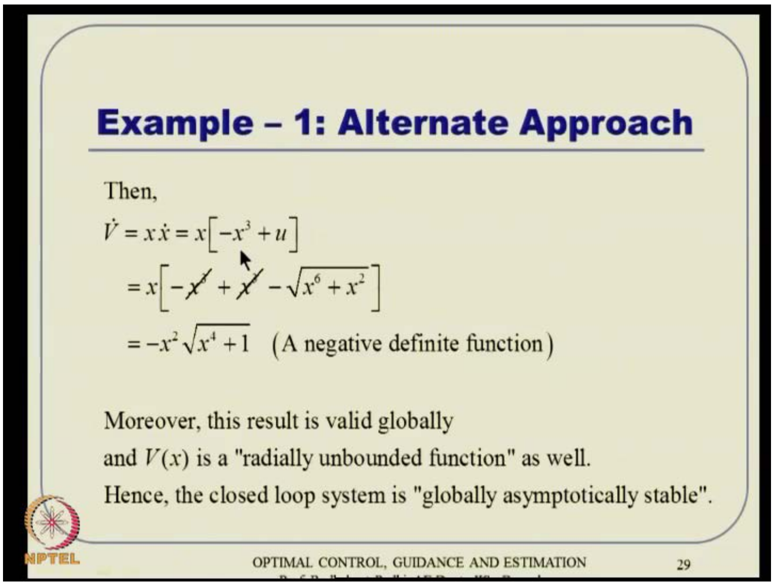

If V of x is half x square, then V dot is nothing but x times x dot; x dot is minus x cube plus u. I remember u, you have selected it as something like this expression (Refer Slide Time: 43:17). So, I can substitute u as this expression here (Refer Slide Time: 43:19). And then, minus x cube plus x cube gets cancelled out. So, we are left out with this one. And obviously, there is a positive square root of this; this is positive term always; this is always positive term and there is a negative sign here. So, V dot happens to be negative a definite function.

By Lyapunov theory, it tells us that this (Refer Slide Time: 43:44) is a positive definite function and V dot is a negative definite function. And hence, what I get is a stabilizing solution, rather asymptotically stable solution sort of thing. And again, those of you know a little bit of Lyapunov theory… And, if you do not know, you can see my other course here. There are I think 2-3 lectures on Lyapunov theory itself. So, what it tells you is V of x is positive definite and V dot is negative definite function. Then, it is asymptotically stable. But, it goes a little bit more than that. Then, V of x is radially unbounded, because as you go radially away, the function has to be away and away; this kind of thing (Refer Slide Time: 44:34). So, x and V of x. There is no saturation sort of thing. So, it is something like radially unbounded.

And, this function (Refer Slide Time: 44:45) is globally defined. V of x is globally valid – half of x square. And, V dot what we are getting it is also globally valid. This negative definite is not confined to a particular region and all that. So, V is globally positive definite; V dot is globally negative definite. And, on top of that, this V of x is radially unbounded. So, this Lyapunov region tells us that whatever results we are getting is global; that means this closed loop system is guaranteed to be globally asymptotically stable. So, this is the power of this HJB theory and all that.

If you are able to get a solution, there are certain very need properties as you said we got. And also, remember, this particular solution that you are getting here will not cancel the beneficial nonlinearity. This we put it here (Refer Slide Time: 45:42) or else, closed loop solution – it is not there; but, I can derive it; that is not the problem. u is like this; so, x dot is nothing but minus x cube plus u. So, what it results in is minus… one x cube and minus x cube will go; and, this results in something like minus x sixth plus x square; that means if I take out this one – one x, minus x into something like x fourth plus one and all that. So, this closed loop system has this property of this not cancelling the beneficial nonlinearity property. It results in a nonlinear closed loop system, which is nonlinear, but globally asymptotically stable. That is of the more important thing – this term (Refer Slide Time: 46:39).

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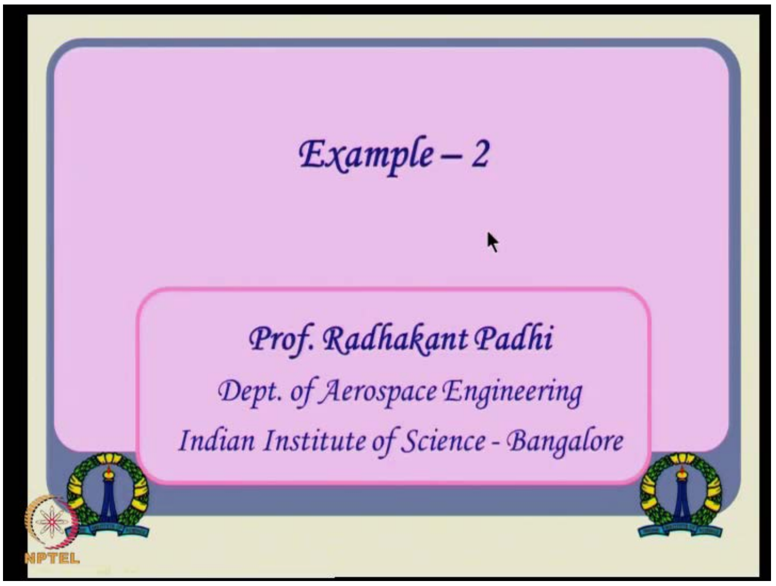

This example 2 is now the extension of example 1. And, I want to say that… somebody should not think that this has become only minus x cube there and hence it is happening around that.

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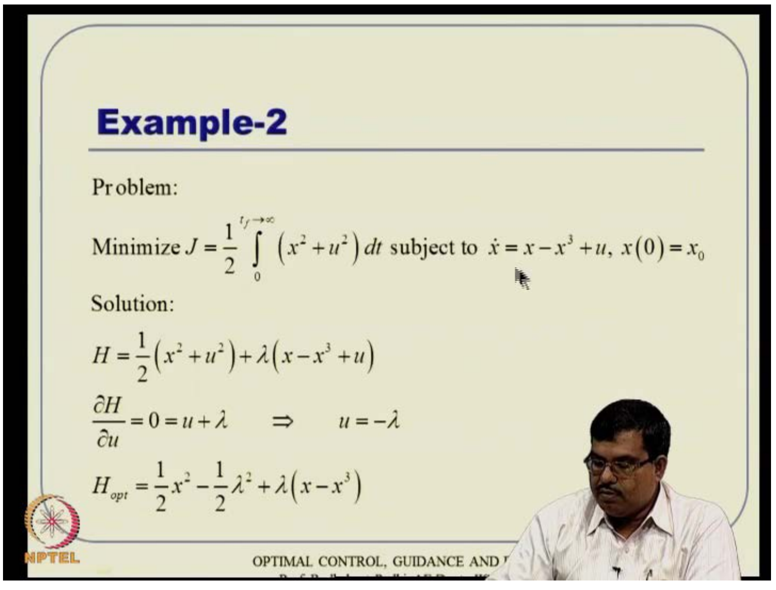

And, also remember in my (( )) lecture, I have already pointed out that point of time that this is a one solution that we like to derive ourselves. So, this is the problem; same problem. We have a quadratic cross function subject to this nonlinear dynamics. Now, you should have simply minus x cube; we have x minus x cube. And also remember, this particular problem, this infinite time quadratic regulator with this non-linear system dynamics, where x is a scalar has become a benchmark example in literature; that means if you have any new technique that you claim that you can solve some nonlinear control problem, optimal control problems and things like that, then probably, this is one problem that you would like to try out first. And, we will see that in many cases including (( )) optic critic and all that, going back this equation again and again.

The same approach we will follow. We will first construct the Hamiltonian. Hamiltonian is l plus lambda transpose f. So, l is this part (Refer Slide Time: 47:58) – half of this term; and, lambda times f; f is this; it is not only minus x cube, but plus x cube is also there. So, as I told you in the yesterday's lecture, there is a nice interesting thing that goes on here. Remember, both are odd powers x to the power 1 – odd; minus x to the power 3, which is odd and so. So, this term is a positive coefficient; this term is a negative coefficient; that means this term is destabilizing; whereas, this term is stabilizing. And, if the modulus of x is less than 1, that means x is bounded between minus 1 and plus 1. Then, x to the power cube is very small; that means within that domain, minus 1 to plus 1, this term is less dominating compared to this; that means this will try to destabilize the system.

However, when modulus of x is greater than 1, this x cube is (Refer Slide Time: 48:55) getting amplified there; that means our system dynamics, this stabilizing part becomes dominating; that means beyond x equal to that modulus of x equal to 1, that means… I will leave that; it is self explanatory anyway. So, beyond this bound between minus 1 to plus 1, this term is dominating, which will pull the system backwards, try to kind of lead the trajectory towards 0. But, as soon as the x value starts, it becomes lesser than 1; it falls between that zone of minus 1 to plus 1. This term becomes powerful and this is destabilizing (( )) So, this funny dynamics goes on here.

Once the longitude is larger, say if the system property is stable… I am talking about homogeneous system without control x n obviously. But, once it is close to 0, it is destabilizing. Again, it poses (( )) So, unless the control is good, we will never go to 0;

the regulator problem will never be happening here, because close to 0, there is a destabilizing term. And, they are the reasons why it is also a kind of a benchmark problem. First, you will notice by the (( )) Anyway, coming back to this, we have a Hamiltonian definition. So, del H by del u is again… That term is not part of… So, we have 0 equal u minus u plus lambda; so, u equal to minus lambda. What we are having here is H opt, is optimal Hamiltonian is something like this. Once you substitute u equal to minus lambda, you get something like that. So, again, t f goes to infinity and there is no terminal penalty here.

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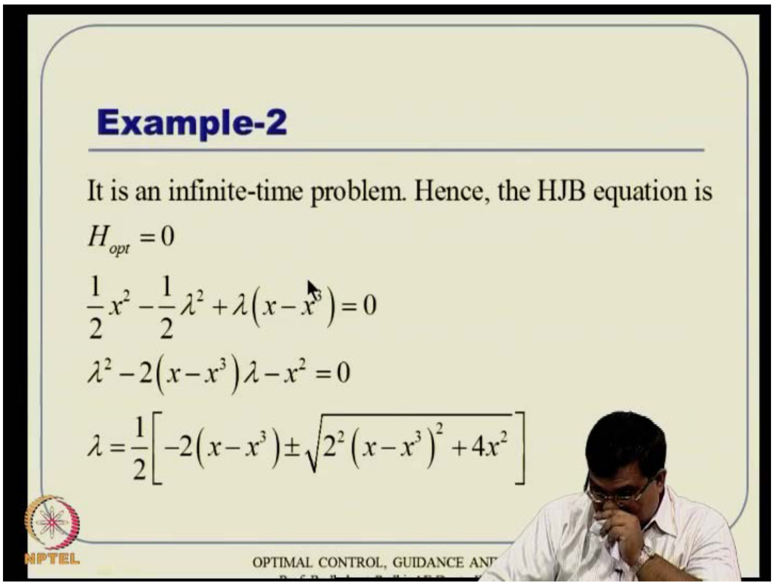

Del V by del t is 0. And then, what we will land up with is something this – H opt is equal to 0; and, H opt – already, we have got it in this form. So, this is what it is. Again, this happens to be a quadratic equation in terms of lambda. Again, we solve for lambda directly like the previous example. All that we are having is this additional term here.

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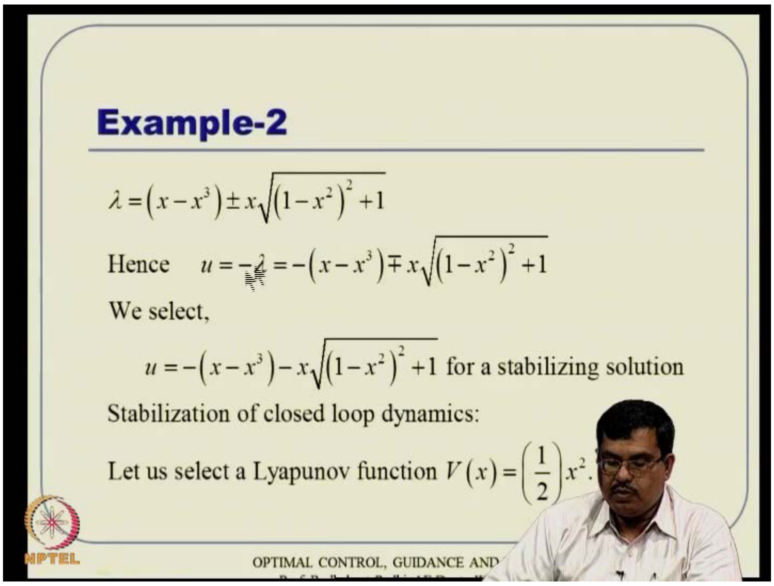

So, what we are getting here is lambda equal to that. And hence, u equal to minus lambda. This is complicated looking expression sort of thing. But, again, if I take a negative sign instead of positive sign, that will result in a stabilizing solution. And again, the similar set of proof can be followed. Let us see very quickly, V of x is again that way. So, V dot of x is (Refer Slide Time: 51:41) x times x dot; and, x dot is this expression (Refer Slide Time: 51:48) – x minus x cube plus u; so, I can substitute it back there. So, I substitute back. Remember, e equal to this expression already (Refer Slide Time: 51:58). So, whatever u we are getting, we can substitute it there.

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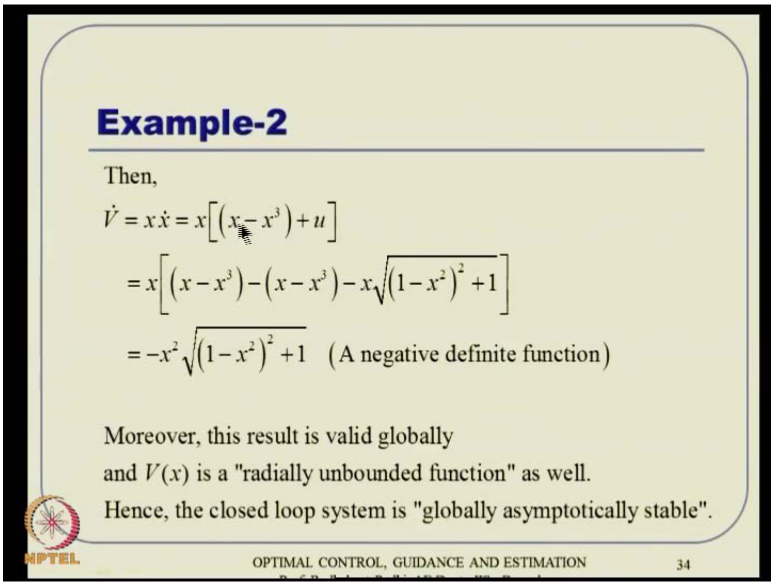

So, x into x dot; x dot contains this term plus u. Again, u is that term already; so, this term (Refer Slide Time: 52:08) with a negative sign here. So, we substitute all that; and, we can rewrite this expression V dot as something like this. Again, this term is a quadratic function. So, this is always positive for x not equal to 0. And, this is also with a minus into x square sort of thing. So, V dot is a negative definite function. And, again, this approach to be globally valid; and again, this V of x is radially unbounded; that means once again, we will land up with the situation, where this control that we are telling is this expression. And hence, it is a closed form expression as this particular state feedback optimal control solution will result in globally asymptotically closed loop system – globally asymptotically stable closed loop system. So, that is the power of this HJB equation and all that.

This small example and an extension of (Refer Slide Time: 53:10) this, where you go (( )) this to small example demonstrate the usefulness of HJB equation. However, in general, it may not be a very good technique for getting the solutions this way. My suggestion is globally, if you can do it, there is a (( )) term should be towards this. But, if you cannot do it, there are many techniques getting double opt and also double opt already, which will try to exploit the HJB equation to get something like an approximate solution and all that. We have seen one solution for theta d approach and all that in the last class.

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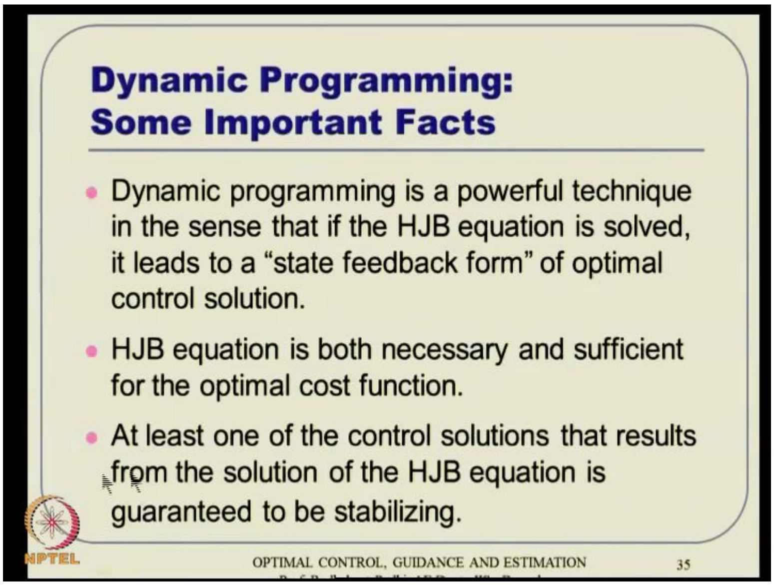

Some important facts are something like this. In general, there are generic facts. First thing is like this. The dynamic programming is a very powerful technique in the sense that if HJB equation is solved, it leads to a state feedback form of optimal control solution. Another technique gives that. And, what you are looking at is something like a tunnel of solutions together; that means (( )) t naught and x naught can fall in from a domain of values. And, I am ready with the control solutions for that. So, that is the power of HJB equation.

HJB equation also tends out to be both necessary and sufficient; that means so far we are not talked out sufficiency conditions and all that in calculus a variation approach; we always talked about necessary conditions only. But, HJB equation tells us, there it is both necessary as well as sufficient. The next one is – what it tells us, even though this HJB equation is a non-linear PDE in general and hence it can throw multiple solutions and all that, it tell us that at least one of the control solutions that results from the solution of HJB equation is guaranteed to be stabilizing. We have seen that example like we took from Lyapunov function, positive definite, so that V dot is negative definite and all that. But, it tells us that these results are quite valid in general while it tells us at least one of the control solution that results from the solution of HJB equation guaranteed to be stabilized; that means we do not want to keep on searching and all that. Within the finite set of solutions that we have, we explore which is the solution that is stabilizing and we are done.

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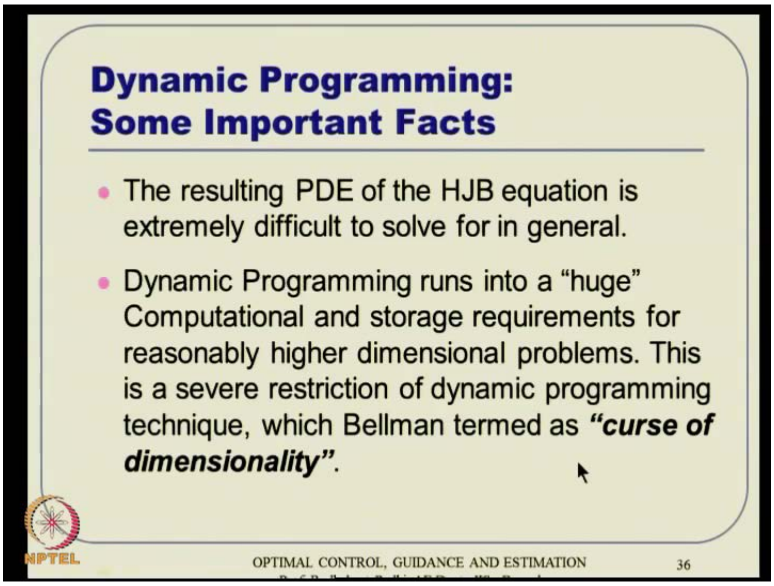

Further things, the difficulty part is rather like this. The resulting PDE from the HJB equation is extremely difficult solve in general. There are that I have been pointing out consistently. And, suppose you want to solve it numerically, there are numerical attempts also to solve this HJB equation numerically. It runs into the huge computational and storage requirements for reasonably good practical problems. And, this tends out to be a severe restriction of the dynamic programming in general, which Bellman himself termed as something like curse of dimensionality; that means if the number of states are increased for the duration of the control application t naught to t f, that is, more, then it runs into this huge computational and storage requirements. In other words, the computational complexities in the form of exponential. That is what Bellman termed it as some curse of dimensionality. Remember, the calculus of variation, we had this curse of complexity. The (( )) problem and all that lead to curse of complexity. Here it will leads to curse of dimensionality. Both are equally varied in a way.

For next classes and all, we will see how to kind of address this curse of dimensionality in a limited sense at least or in a reasonably good way, so that we will get the benefits of this HJB equation without running into these difficulties. That will result in this approximate dynamic programming, new network based adopted critic solutions and all that. We will see that in the next class anyway.

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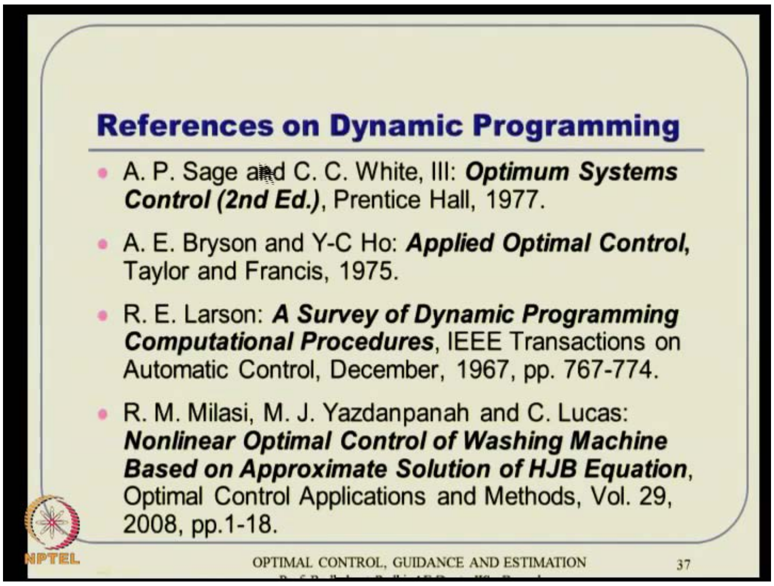

With that, I will list out a few references here. Most of my material here is taken from this first book. And, some of the examples I have solved it myself – this benchmark example especially. And then, always, you can refer to Bryson and Ho; this is the very standard book; there is also a small chapter on dynamic programming. But, my derivation is taken from Sage. Then, all these things are available. These two things are available as a very good sort of papers, which are survey papers in a way. This is survey of dynamic programming and computational procedures, very old paper, 1967 sort of thing.

And, of late, recently, this is also another one, which talks about nonlinear optimal control of washing machine based on approximate solution of HJB equation. It results in a closed form expression. But, the closed form expression is a very high order – fortieth term, fiftieth term – fiftieth order and things like that; very interesting. What I suggest that you can read and probably try to kind of experiment yourself (( )) previous (( )) With that, I will stop. Thanks for the attention.