Optimal Control, Guidance and Estimation

Prof. Radhakant Padhi
Department of Aerospace Engineering
Indian Institute of Science, Bangalore

Module No. # 05
Lecture No. # 13
Optimal Control, Guidance and Estimation
Linear Quadratic Regulator (LQR) – III

Hello everyone, we will continue with our lecture series in this optimal control guidance and estimation course. So, far you have seen many concepts including linear quadratic regulator for about two lectures and we will continue the discussion on L Q R and this lecture as well actually.

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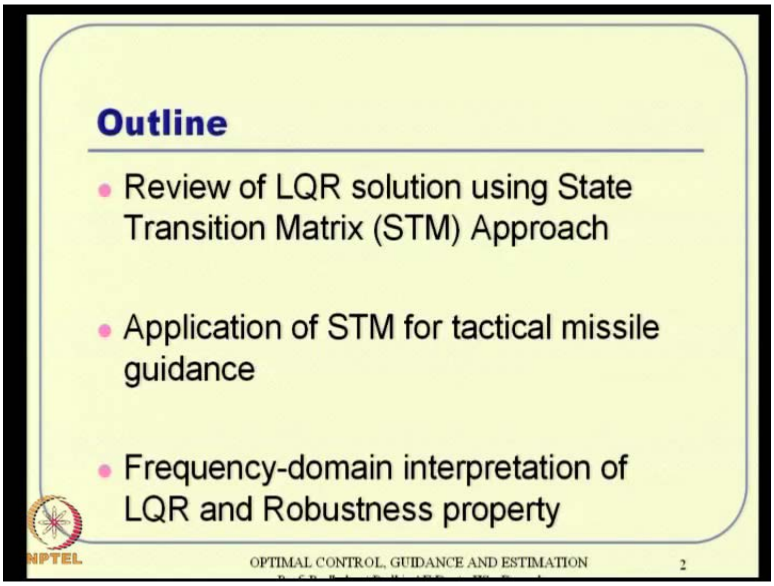

So, this is a what is coming up today I mean this lecture. This is a outline of the outline of the lecture first is review of L Q R using state transition matrix approach; that means, whatever we discuss last lecture we will have a quick review of that. Because we need to use that next for tactical missile guidance application as well just an example of how optimal control whom gives us a platform to design a advance guidance laws and all that more on that we will see little later. But one approach using state transition matrix we will discuss in this lecture actually. Then towards the end of this lecture we will also talk about frequency domain interpretation for LQR as well as some robustness property results.

So, how frequency domain interpretation people can do that but typically when you talk about linear systems then this laplace transformers and other things will come into picture there. So, we will have some idea about what people talk about a frequency domain representation and how it leads to some sort of a equation using which you can also design the (( )) matrix actually we will also give example on that alright.

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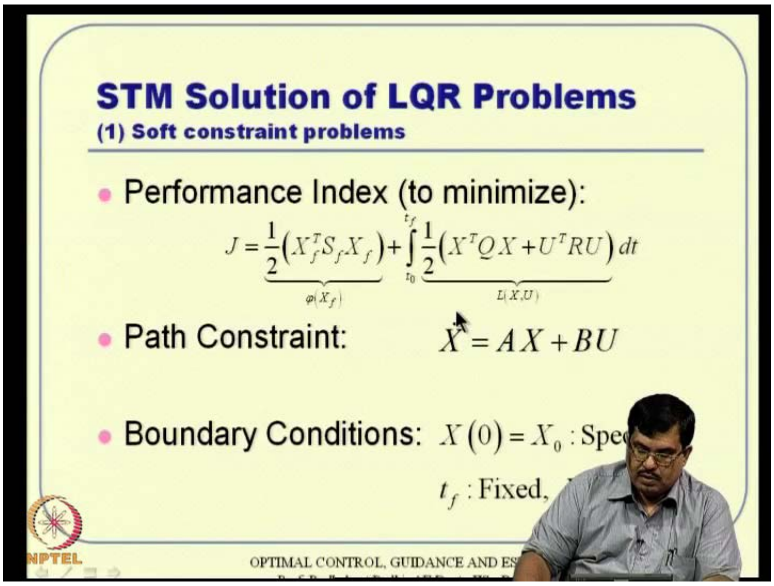

So, first is the summary of what we discussed last time this is what we discussed must revise standard L Q R problems. We have a quadratic cross function, penalty function at the end and cross function for the path actually and the path constraint represents to be static equation boundary condition are given like this. This is the standard form, standard equation that we have been discussing under the frame work of L Q R. And here we will talk about soft constraint problem; that means, X of t f is actually free it is close to 0 by this minimization this type minimization. So, this is a this the problem that we are talking under the frame work of a soft constraint.

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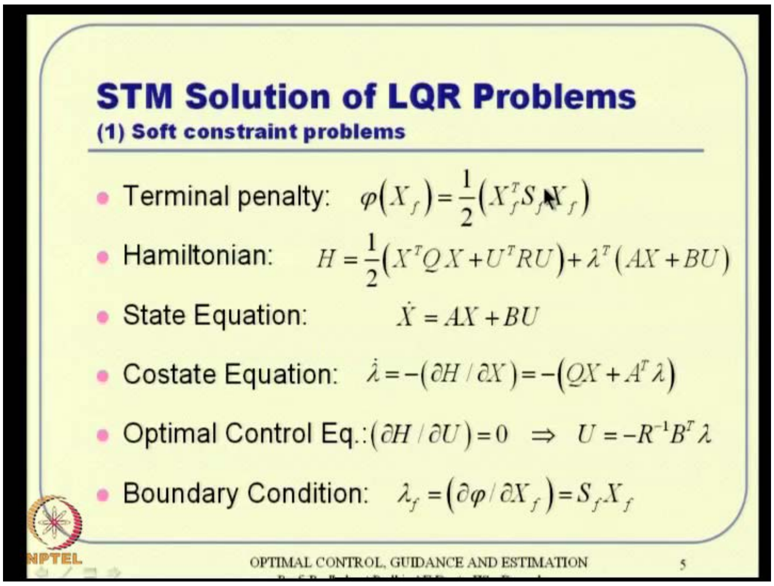

So, we went ahead and derived all these things first of all we noticed that this is phi of X and l of X and all that. And using this Hamiltonian and like that this necessary conditions state constraint optimal that cannot be like this. And then the boundary conditions happens to be lamda f equal to S f X f with after that we went ahead and substituted this control expression in this third equation.

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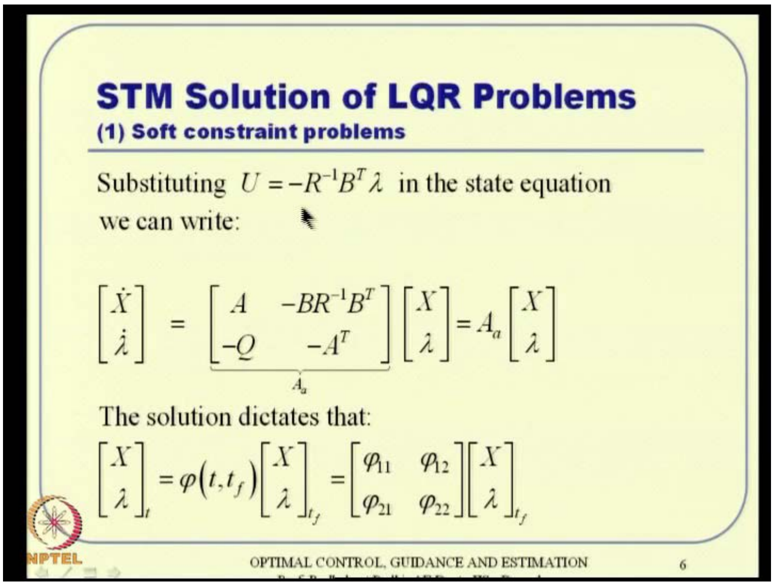

And then noted I mean we could notice that this can be written as something like X dot lamda dot is a coupled side of equation like this where this is a nothing but the

Augmental system I think. And because it is a linear homogenous linear system we can also write the solution in the form of equation and matrix. And hence you can write lamda X and lamda at t is nothing but phi of t t f and X and lamda at time t f. By the way just a comment some people intent to write it in something like a initial condition also. And other words it will be phi of t t 0 and then X lamda of t 0 again.

It depends on your situation and then which is the advantage something like that sometimes in the literature you may see something like that both are correct actually any way. So, writing this way then we could partition this matrix as some phi 1 1 and phi 1 2 phi 2 1 phi 2 2 sort of thing and then.

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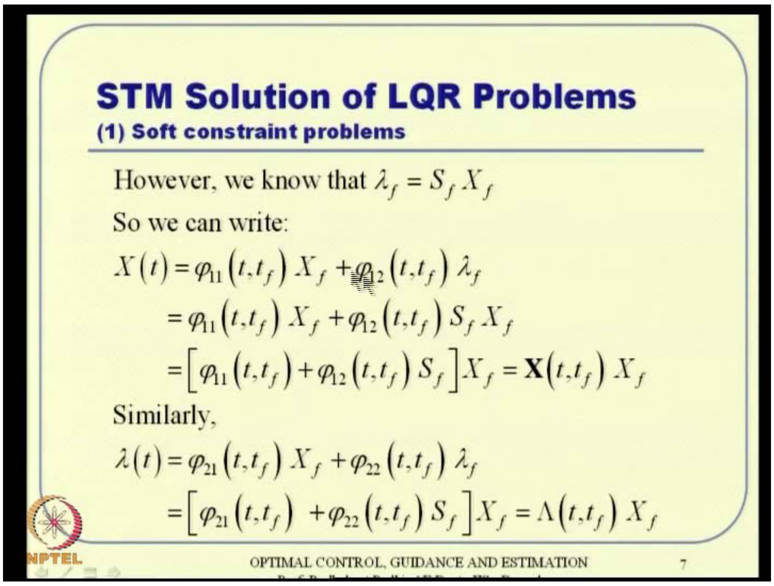

You could notice that using this boundary condition lambda f equal to S f X f. You can also write this kind of equation where X of t is dictated by X of f through the state transition matrix where X of t t f this algebra we discussed last time already. And similarly, you can write also write lambda of t in the second half of the equation and the through a fairly similar algebra leading towards that actually.

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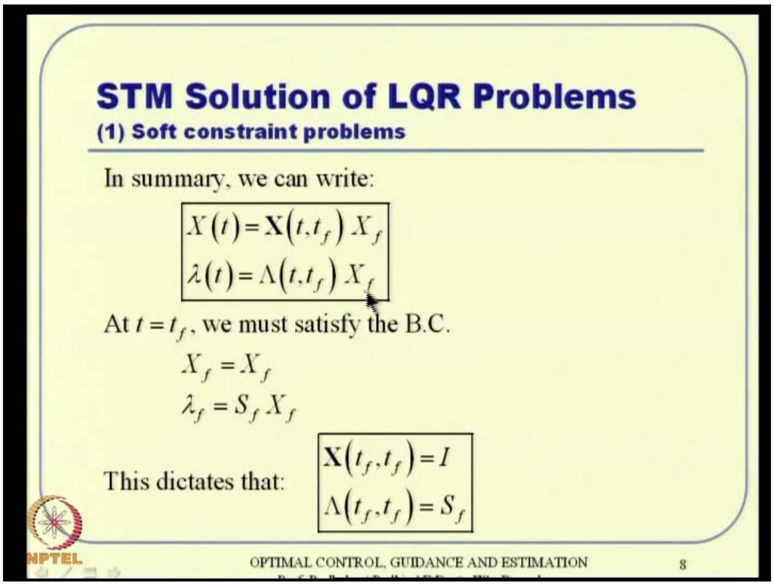

So, when you do that when we write it something like this then I expect X of t to be a X t t f (( )) times X f and lamda of t f to be matrix times X of f this is not lamda this is X of f. So, t is equal to t f it must satisfy the boundary condition X equal to X f and lamda f equal to S f X f. So, we can get this final boundary condition as well actually. So, using this boundary condition this turns out to be the boundary condition for this.

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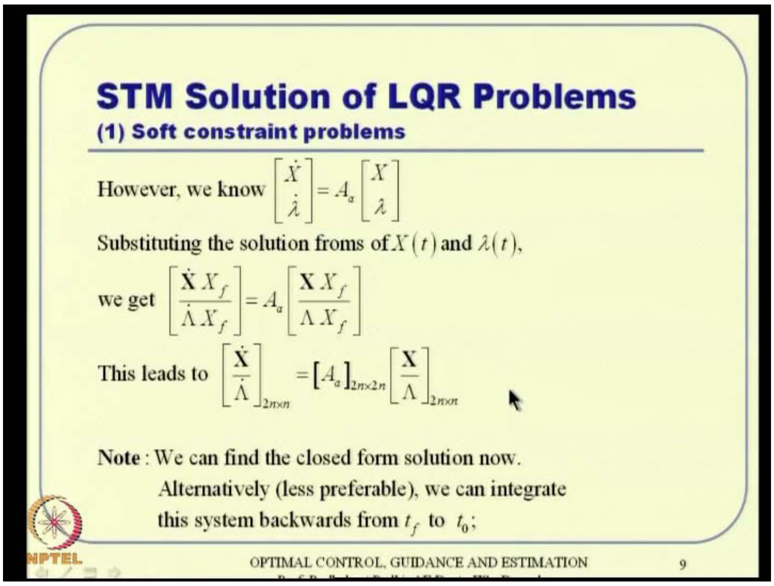

And also not that the this state transition matrix will satisfy the same differential equation. So, we have a differential equation and we have a boundary condition. So, essentially we can integrated by chords and store the solution and think like that actually. The problem; however, is X f is not known basically because ultimately after getting the solution for the state transverse matrix still the solution for X of t lamda of t requires the information about X of f, X of f is not know. So, how do you handle that now it is easy for because the state transition matrix representation allows us to write it this as well. X of t 0 equal to X of t 0 t f into X f and that is valid because this expression is valid for any time including initial time.

So, you put initial time condition here then you will get X of t 0 is equal to X of t X t f times t. So, that is what you write it here and typically state transition matrices are never singular. So, you can always invert it and get it in this way. So, X of f information is available now. So, you can go back to that and then write X of t and lamda of t something like this because of X of t and lamda of t is already there as an expression. Now, you got X f so, using that expression you can substitute here you get expression for X of t and lamda of t like this it is.

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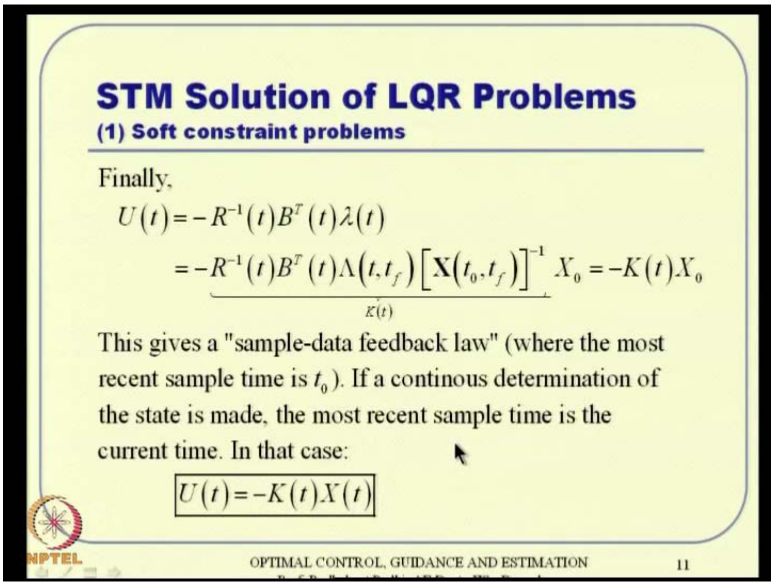

So, finally we have the control expression equal to minus R inverse lamda. So, you can substitute this lamda of t you got lamda of t here like that. And then you can always define this matrix what you see here as some sort of a gain matrix way of t and hence you can write minus K t times X 0. And again it gives some sort of a sample data feedback law and if you consider wherever you are I mean wherever the condition is that mean initial condition then you can represent this control law or something like U of t is equal to minus K t into X t that is how you get the gain matrix for this actually.

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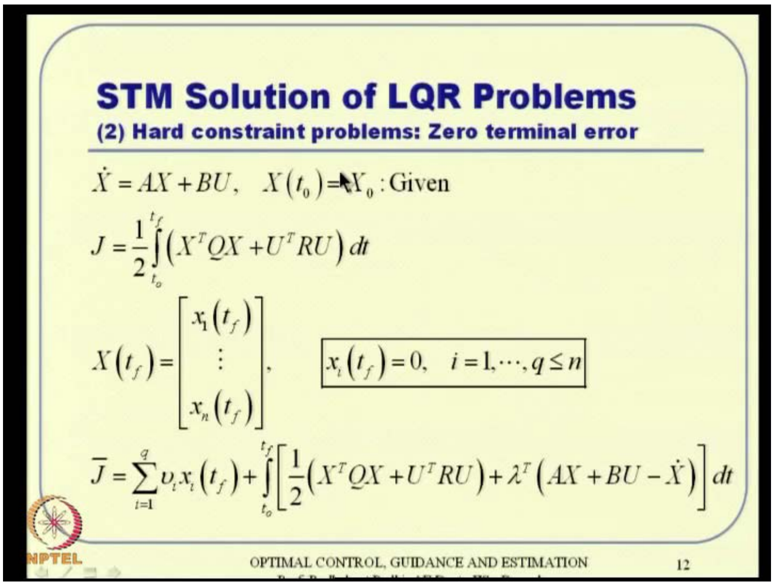

This is all about soft constraint problem how about hard constraint problem we also discuss about that. And it tells out that this class of problems demands that there are part of the state vectors is equal to 0 at t equal to A. And hence we have this sort of a formulation which talks about I mean the regular standard L Q R problem over the boundary conditions demands that part of the states is equal to 0. And that dimension of that subside can be as n actually in other words entire state vector can be constraint if it is necessary do that.

Then the regular I mean the analysis we discuss about that in last class tells out the because of the hard constraint we have to augmented I mean cross turns tells out to be something like this where I happens to be additional logarithm multiply sort of constraint itself.

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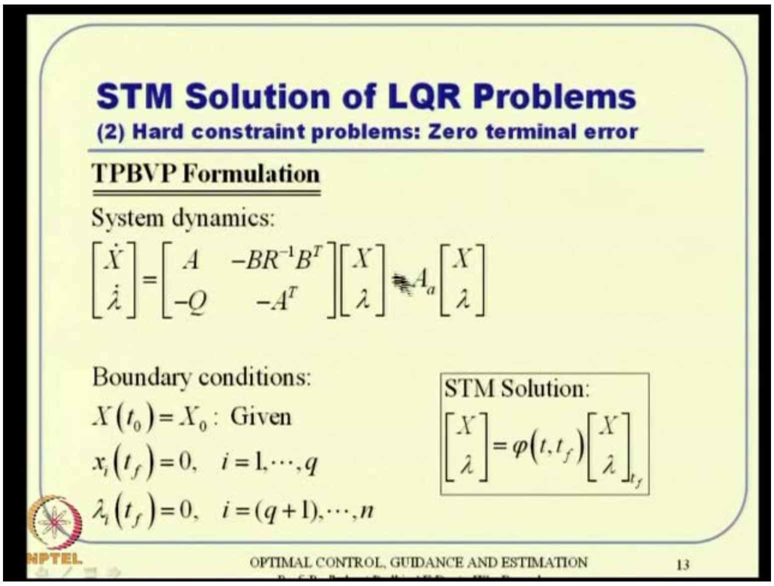

Then we carried out the similar algebra but the boundary conditions turn out to be different. And in other words we have this first 1 to q except t f equal to 1 but after that q plus 1 n except t f equal to 0. I mean lamda of t f is equal to 0 now account for all that in the solution procedure then you now, but at it is we collected the terms of these quantities which are non 0 and then define this mu vector or something like this actually. Then using these mu vector we could write this X of t and lamda of t something like this and then the analysis were very similar to what we had before in other words the state transition matrix will satisfy the same differential equations.

However, the boundary conditions turn out to be something like this actually. The retail analysis and derivation and all we have already done in the last class actually. So, because of the boundary condition difference if they very different solution actually. So, now it terms out that we cannot always claim that the this X of t 0 t f remains a singular non singular, but if it remains non singular then this expression is valid. And in that case we can always right this actually; that means, the formula in the solution is fairly similar to what we had before. But the real actual values numbers will be different because the boundary conditions are different.

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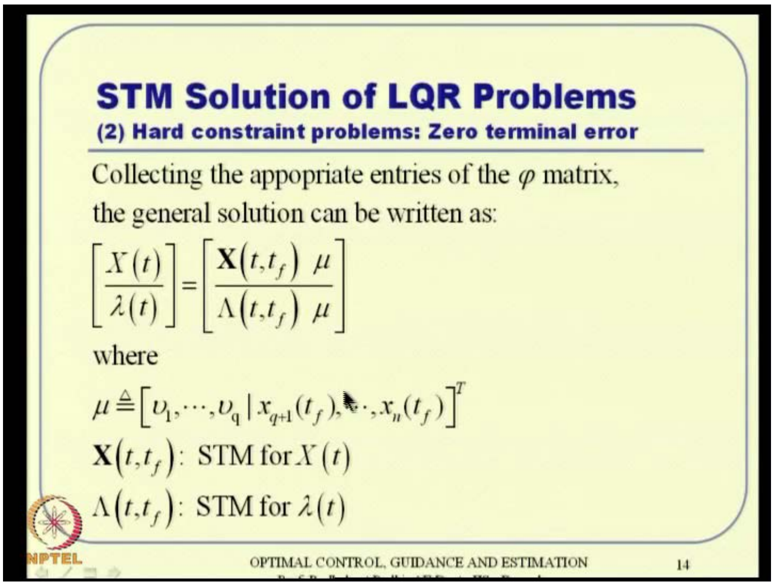

Again similar to previous analysis the continuous data in case of a continuous data; that means, we have this t 0 verses t and all that the small type probably alright. So, that is the small one this for again if you take wherever the states I mean initial condition then you can write the control or something like this. Now, here the problem happens that s t goes to t f then this approach is that but X of t f t f you see the boundary condition it tells a one type of a row actually ones of rows which are 0. So, essentially X of t f t f is nothing but a singular matrix. And hence there is a problem at the end actually but that is expected because the ambition was little very high in other words we insisted on 0 terminal error.

So, I mean if you have a soft constraint this problem typically does not arise, but you have a hard problem this problem does arise. Then the I mean the usual approach is after some towards the very end when t approach is very close to t f then you do not update your gain matrix. So, use the previous gain matrix and all that actually and so, that can be some sort of a what I called a linear actually. So, in practical problems typically you can do it and recognize that way. So, anyway this was the summary of discussion last time and this class like to use this concept and then apply this through an example. A good example which talks about optimal missile guidance through state transition matrix solution of L Q R.

So, let us see how the formulation is and how and where it leads to and think like that. Now, what is the fundamental problem of tactical missile guidance situation in a two d plane in the plane of engagement assuming that it is some sort of a two d engagement towards the end we can write it this way. So, there is a reference line typically parallel to I mean to this origins sort of thing horizontal line and then there is a missile changing the target it is a this is missile and then there is a target. So, it is changing the target, target is running away with the velocity V T with an angle theta t with respect to what is as called line of sight this yellow is stands for line of sight.

And similarly, the velocity of the missile is something like V M which is a which is different angle theta M and the way it corrects the direction of V M is through applying this lateral resolution a M. You can think of something like a centrifugal force and think like that way. So, the movement we have some sort of a acceleration that is perpendicular to V M then it will try to rotate the V M vector. So, remember we do not have a thrust vector control and like that I mean do not have a thrust manipulation control. This V M with typically is not controlled because there are several reasons for that the thing it is not a very good control manipulation thing you cannot manipulate it as you wish in a fast derive itself.

So, V M is typical left as is it sort of thing, but assuming that V M is larger V T; that means, the V M is moving fast compare to V T then there is a only the direction correction of V M is will typically lead to engagement. And this concept relation something like rotation of line of sight vector I mean this line of sight vector what you see rotation of line of sight vector. That means, if the line of sight vector does not rotate then ultimately it leads to something called a collision case. And it leads to collision triangles sort of thing this is what happens actually. If it does not I mean if you start rotating the same vector properly we manipulate properly.

Then this is what will happen in other words if this is your initial L O S after sometime this is your L O S that is your L O S. Like that this all these after sometime they remains parallel and a target appears larger and larger; that means, you are approaching the target anyway. So, ultimately it leads to collision at some point of time actually that is the whole idea. The whole idea tells us that somehow it applies some sort of a lateral resolution this a M is called lateral resolution to V M. So, that it the turning will take place V M turning take place and ultimately leads to a collision actually, that is the whole idea of typical missile guidance there are several variants that of course.

And then one of the variants will talk about that A M should be applied perpendicular to L O S instead of perpendicular to V M, that is geometrically more correct and then more precise also. But\ applying a perpendicular vector to V M is practically easier than compare to I mean let us compare to applying thing which is perpendicular to L O S actually. So, there are again mechanization issues and all that and there are also results which show the does not really matter. What will matter is the magnitude of a M will vary in such a way that ultimately it both will lead to the similar sort of results. Now we will compare in our something like this.

And the result tells us something called proportional navigation guidance which is there for a long time really tells us that a M is nothing but N V M lamda dot. So, that means, if I apply a lateral resolution perpendicular to V M something like these where it is proportional to V M as well as proportional to line of sight rate actually. I mean this angle rate the rate at which it changes I will make it proportional to V M and proportional to lamda dotmas well. And ultimately it leads to this kind of a expression where N is known as navigation concern actually. And traditionally it has been shown before that if N is equal to 3 then it tells some sort of a optimal performance. That is not the major observations before in a classical since by many researchers.

Then it turns out that formally it can be shown that N equal to three nothing but an optimal guidance actually. That is what the main motivation of this particular lecture I am going to take you through this example and prove that N equal to 3 is nothing but a linear quadratic optimal missile guidance actually. Anyway so, this is the problem and we want to solve this problem without knowing this we want to come up with some sort of a lateral resolution trajectory. Lateral resolution history rather which will leads to this collision actually. Then we will correlate this expression with that and so, that this can be represented and something like this actually.

Anyway the system dynamics that is accounted for is quite simple very simple rather it tell tells us the missile and the target here this situation is lightly reversed. I have actually taken this example from (( )) so, that book uses some sort of a notation like this I thought I will and that is why. So, anyway missile is here it moves with a velocity V and there is target here. So, this is L O S actually line of sight what we are applying is lateral resolution perpendicular to V which direction a and this is the velocity vector capital V and this is a velocity vector along the line of resolution basically. So, and if you have this one this is V and t f minus t is something called t go. So, V into t f minus t will take you there actually assuming V is constant.

So, the velocity missile velocity assume to be constant and also on the way we will assume that this angle that sigma what you see is line of sight sort of things that remains small actually. So, the whole idea here is to somehow close this y t, y t as to be nullified and if t f computation is proper; that means this fellow is no other in mean this missile it cannot escape it as to go to the target actually. X equal to t f is going there the question is by the time the what this is called something called (( )) actually, by the time (( )) becomes zero your heights also becomes zero then the point lies on the target itself. So, how do you do that we will consider the system dynamics or something very simple.

Secondly, we have been this two sort of kinematic equations rather v dot equal to a in this direction and y dot equal to V; obviously, the cost function what you are interested in is to minimize y f as much as possible. So, this (( )) outside the integral and we are interested in some sort of a lateral resolution minimization also primarily because lateral resolution leads to turning. In the movement some something is moving on turning the projection through L Q R, is larger and it essentially leads to something called induced stall. Induced stall is primary factor for reducing the velocity and all that you want to enter too much of lateral resolution.

For that you can preserve the velocity; that means, you can preserve the energy actually, essentially it leads to the high impact velocity as well as larger range actually. Think like that those are the benefits by having a minimum lateral resolution. And additional benefit as additional benefit is as a for a minimal if a lateral resolution demand is minimum. Then obviously, through the autopilot loop and then the control loop and all you are will also turn out to be minimum actually remember are typically bounded by certain values and all that actually there rates and there values and all that. So, having lateral resolution as minimum as possible helps us in several other things actually.

So, we account for this a square minimization on the way but the primary motivation is to minimize y f actually. So, y f should be x los actually that is the reason why this cost function is selected. So, the problem is the premium this it is not a hard constraint problem it is a it is actually a soft constraint problem. And then this soft constraint problem tells us that y f square possible at the end while on the way lateral resolution should be a square actually. This optimum L Q R formulation as this cost function usually the system dynamics.

Now, to precede further first we have to formulate exactly put it into the l q r frame work and for doing that we define a state vector v. And I mean v and y that is a state vector, U that lateral resolution which is nothing but a that is our control vector or control scalar rather here. And once you define that then you can represent the system dynamics as X dot equal to a X plus v u where a and v turns out to be like that. And similarly, because there is nothing in the state side here the Q as happens to be all zeros 0 0 0 where as R happens to be 1 and this and S f happens to be like this 0 0 0 c remember the waiting is c times y f and y f is a second component of a state vector.

So, it tells out that c should not appear here, but it should appear here and that because the definition of state vector is v y and not y v like that. And also this time to go definition is what I told t f minus t that is the time to go definition actually; that means, how much time is left from here to go and the target actually. So, then I mean following our results that is we discussed I mean the few slides earlier we first we go back to this augmentation matrix x dot lamda dot which is given something like that. Now, we have all these matrixes right a we know b know q we know r we know s f we know. So, using all these this augmented state matrix I can write it something like this and then A I mean if I substitute all that it turns out to be like that.

And solution happens to be x of t lamda of t is nothing but phi of t t f into x t f and lamda t f actually. So, essentially you remember this is a linear time in variant systems. So, phi of t t f is nothing but phi of t minus t f that is all results in linear systems theory you can do that as long as the system matrix is time in variant actually. So, what is the thing I mean we want to compute some sort of a because the time in variant case we can also compute this phi of t minus t f actually. First we compute phi of t and then we will substitute t whatever t verses t minus t f actually. So, what is phi f t in the in this scenario is nothing but the power a times t this standard result. So, we already we know this is the system matrix.

So, what happens to be like lead a to power t is nothing but (No audio from: 23:23 to 23:32) so, e to the power A t turns out to be like something line I plus A t plus A square t square by 2 factorial plus A cube t cube by 3 factorial like that actually. So, we have to actually evaluate the in this case we have A so, every where is it A a sort of thing. So, first we have to evaluate A a we already know it is a matrix. So, A a square is how much and A a cube is how much like that actually you have to evaluate. At this point of time I also like to tell that this is evaluation of A to the power of A a t using these polynomial expansion sort of thing are infinite series expansion is not a very efficient way of doing things.

There are other ways of evaluating e to the power of a t as well, but we will follow one approach here to demonstrate the idea there when somebody wants to follow some other than it can also be done. For example e to the power A t there is another standard reason is that nothing but laplace inverse of s I minus A inverse actually. So, those of you want to follow this way I welcome you to do that also the resolve the problem taking that e to the power A t is nothing but laplace inverse of s I minus a Inverse. So, if it is A it is to be A also alright (No audio from: 25:01 to 25:08) anyway we will proceed with one approach.

So, A a we know we will just evaluate A a square and A square happens to be something like this which is actually good to see. Because once you see all zeros here including the diagonal elements one triangle is completely 0 then it is something called adding important matrix. That means, you keep taking more and more powers that some point of some power will become 0 and hence all other powers there on words they all become 0. That means, the series which is actually A infinite series trunked at some particular power; that means, the expression does not contain any approximation errors actually so, this is the situation here. So, A a like this a square turns out to be like that and A a cube turns out to be like this and a fourth is all zeros.

So, from there onwards a fifth sixth seventh everything is 0. And hence we can write the polynomial e to the power A a t up to third power only this is I plus A t plus this term and all that are and you know A a square also we know a q as well. So, you can put it all that and turns out to be something like this and remember this phi t what I have Interested is phi of t t f; that means, phi of t minus t f is here. So, wherever t appears we substitute that as t minus t f sort of thing. So, this is what our state transverse matrix in this particular situation actually. So, using this we want to do not want to keep on taking t minus t f carrying forward and all that and we it appears severely in missile guidance literature also basically.

So, we define that something like t go time to go and this is by definition is nothing but t f minus t. So, wherever this t f minus t appears is actually it is t minus t f it is nothing but minus of t minus t f sort of thing. So, wherever that term appear we substitute by t actually. So, for example, this is nothing but minus t go this is actually t go like that actually. So, substitute this t go minis t like that and get it something like this. Now, we have got this partition; that means, it started with a two (( )) so, this four dimensional matrix four by four. So, we partition that and get this phi 1 1 phi 2 1 and phi 2 2 actually. So, X of t can be written as like this; that means, this can be always written as something like that is what we define before something like state transition matrix first and per X.

Now, we have phi 1 1 of t t f t minus t f rather whatever and if 1 2 minus t t f as well actually. So, this is there available that is available S f we know because S of f turns out from the system formulae this one actually the problem formulation. So, everything we known so, we can evaluate this the state transition matrix X of t t actually. So, if you put it back all the expression that we know it turns out to be something like this phi 1 1 is this expression this part and then phi 1 2 is that part here we put it together phi 1 1 phi 1 2 and S f, S f is that part that is called. Now, put it together and then evaluate the expressions it turns out to be something like this and similarly, for lamda the state transition matrix for lamda can also evaluated through this expression.

And phi 2 1 is available nothing but all zeros and phi 2 2 is also available this is the matrix. So, we put it and then turns out to be like this and hence it is all like this. So, X of t t f is somewhere like this and lamda of t t f turns out to be like this. And hence lamda t this is the expression I mean that we could notice that. So, lamda of t nothing but lamda of t t f into X of t t f inverse times X t that is the standard result that we have. Now, we have lamda of t t f which is nothing but these and X of t t f is nothing but that we can put it back. But remember there is an inverse operations still going on here you too take symbolic inverse here to get the solution actually anyway.

So, this is what is lamda as a function of X and hence finally, the control U is nothing but minus R inverse B t transverse lamda where R we know B we know. And lamda we completed just now we substitute this expression for lamda expression for R inverse B transpose taking together sort of thing turns out to be like that remember where R is nothing but 1. So, R inverse is also 1 here and V transpose is what you get here. So, this is what it is and then we put lamda is nothing but that so, substituted here and simplify this 2 by 2 matrix inverse and can be done symbolically rather easily. Here this one over determinant and then take over a I mean adjacent elements very perfectly very easily by I mean one over determinant.

Then what will happen is? The matrix comes out that if this diagonal element will exchange the place and half diagonal element will sign actually. So, you can use all that or you can go ahead and compute the inverse symbolically yourself starting from first principle and all that. Then it turns out that U f t that is lateral resolution takes this form finally, that is means if I know v and y I can write it this way and also I need to know t go as well. So, finally, the solution of what we need to apply to get it there to catch the capture the target happens to be these expression.

And especially if c goes to infinity; that means, what is the implication there when c goes to infinity it all means that y f goes to zero is all constraint actually. We will least bothered minimizing the lateral resolution on the way that component is not our constraint all. In this situation remember one over c term here that one over c will go to 0 and then we can further simplify the expressions t go square will get in terms of things like that way. And it will arrive at the this expression actually. Now, just hold it hold down for a just second just will note it down for a second and proceed further for the different approach actually. now what happens here let us assume that sigma is turns to zero sigma is small so, I can write 10, 10 sigma is nothing but sigma sort of thing.

So, under assumption this angle for this clock wise thing. So, we put minus sigma sort of thing. So, 10 sigma rather time of minus sigma whatever you can call is nothing but this angle also same thing this happens to be something like this by this basically. So, minus sigma is nothing but this divided by that, that is nothing but v into t minus sorry t f minus t and we assume v is constant. So, we can take it out and then this expression turns out to be like that. Now, what is sigma dot then this is sigma which is with a negative sign and all that. So, now we can express the sigma as well so, minus 1 by V, I will take it and then take the derivatives of these which happens to be these turns the derivative of that y dot minus y times the derivatives this minus 1 actually.

So, if you substitute all that and then get it somewhere like this. So, what I notice here is like what we notice if we just take V into sigma dot and multiply with 3 then it turns out to be this expression. V into sigma dot is that and 3 times V into sigma is nothing but 3 times that and what you obtain here is same thing basically. So, what you turns out this expression what I have here and this entire expression I can substitute as 3 times V sigma dot that is what I told you before that N equal to 3 actually leads to an optimal guidance law essentially we started with this expression right this 1 it happens to be N times V times lamda dot are in this particular sigma dot actually.

And I told that N equal to 3 leads to some sort of a optimal reason and that is what we just told where N equal to 3 is nothing but optimal guidance actually. But remember this P N guidance is an optimal guidance provided there are several conditions needs to be made in general on guidance there is navigation constraint N. So, under the situation like this; that means, we considered linear engagement dynamics and we also considered non maneuvering and stationary targets. That means, very slow moving target; that means, the missiles velocity advantage and the L O S angle is not high we assume that this expression right ten sigma sort of thing.

So, L O S should not be high and L O S angles should not be high then induced drag minimization; that means, through lateral resolution sometimes lateral resolution will also called as latex short form of lateral resolution sort of thing. So, induced drag minimization; that means, through lateral resolution minimization that issue is ignored we do not bothered about that. And lastly this N equal to 3 as to be used under those situations P N guidance is nothing but an optimal guidance actually. So, this is just one way of designing the P N guidance relationship through optimal control. Obviously, there are other way approaches as well and probably we will see 1 or 2 approaches as we along with it actually.

As we proceed further in this course sometimes come down we will see how the other technically routine and then we can get better on results also basically we do not have to really assume. Let us say that target is non maneuvering or stationery we can actually assume that target can move target can maneuver. Then what are things that you can talk about? We will talk about something called augmented P N and think like that. So, we will see some of those things as we go along through this course probably later. Now, in this particular class as I told we are also seen some frequency domain interpretation and then we will study this robustness margins sort of thing.

Robustness margins derivation I may not able to do this class we will just talk about the results that are available in why this L Q R is popular? Is one of the reasons is also because of this it gives some sort of a very good robustness margins actually. Anyway so, before going there we will study this frequency domain interpretation first. So, let us start then so, what happens the optimal trajectory; that means, the close form trajectory happens to be something like this. Now, I assume that everybody known is what I am talking now that X dot is a X plus U, but U is nothing but minus R inverse B transverse B times X.

So, I substituted that and then this is again matrix R inverse B transverse B. So, equivalently I can write a minus B K times X actually this is the close loop system dynamics and hence the optimal trajectory is dictated by the this static equation sort of thing. Assumtions on the way A B is stabilizable A and square of Q is observable. So, that is the standard assumtions for L Q R solutions and all. And now you have define an open loop characteristic polynomial where is nothing but s I minus a determinant actually this a very standard linear systems concept. If I take s I minus a and evaluate the determinant that will turn out to be a (( )) of polynomialness basically that we talk about open loop characteristic polynomial, s is the laplace variable actually.

Now, what is closed loop characteristics polynomial then it tells out that, that is what lamda c of s. What is defined that way is s I minus A minus B K; obviously, this is your lamda open loop lamda I mean sorry delta open loop delta of s s A minus A. And the closed loop characteristics polynomial is delta c s actually that is defined something like this s I minus A minus B K because that happens to be system matrix actually. So, open of the bracket you can write it this way as s I minus A plus B K sort of thing. And this same thing can we further manipulate it and it turns something like this see I considered s I minus A to now to get sort of thing then B K I would multiply s I minus A inverse s I minus A.

That is nothing but identity s I minus A I will put it here one component and B K times identity is nothing but s I minus A inverse times s I minus A this is algebra manipulation of thing. Once we do that turns out the s I minus A that I can that appears to the right hand side that I can also assume an I here identity here. And I take out that and also here the standard result determinant of A V is nothing but determinant of A into determinant of e. So, using that result I can take this one this determinant is nothing but determinant of this one the first one into the determinant of that one. Now, what is determinant of s I minus a that is nothing but open loop characteristic polynomial so, I can let you this way.

So, closed loop characteristic polynomial is given as something some matrix times the open loop characteristics polynomial. Now, and this particular matrix what you see here is something called return difference matrix is a very standard terminology sort of actually. And on the way people also use something called loop gain matrix and all that that is defined as something like this actually only that part with a negative sign that is the loop gain matrix. If you put I plus and all that that becomes return difference matrix and all that actually. Now, Kalman equation in frequency domain is what you are interested in. So, we will start Riccati equation start with Riccati algebraic Riccati equation.

So, we write it this way and then there is further algebra manipulation here the to begin with what we do is add and subtract this term s times P. P is nothing but the Riccati equation matrix so, s P something like this and minus s P. So, we are not (( )) that we are adding it and subtracting as P times actually. So, when you do that this said not happens P into s I minus A here and s I minus A transverse into P from minus s I minus A transverse multiply by P and then this K transverse R K is equal to Q actually you can manipulate that also because K is nothing but this R inverse B transverse B actually. So, if you notice that then this entire thing R inverse B transverse P is nothing but K and then you can multiply I mean you can insert some terms and all that try to this way actually.

So, keep it aside and then we further we define something like a phi of s is nothing but the s I minus A inverse and with this definition phi s minus is nothing but just put substitute minus s I wherever s exists you put minus s there it turns out to be like this. So, phi f minus this transverse is nothing but that the inverse and transverse can you have been can compute. So, you can put the transpose inside and take the inverse out and then this transpose turns out to be nothing but A A plus B transverse is nothing but a transverse plus B transverse if we use that s I is a symmetry matrix.

So, transpose does not matter so, A T minus s I minus a transpose whole inverse actually. So, that is the observation what you have here why we need that we this already it term like this. So, somehow we want to kind A interpret in that way and proceed further basically. So, now what you do is take right this expression in terms of this phi s and phi f minus s and then pre multiply by this one and post multiply by that one both sides actually. If you carry out this terms in algebra it turns out to that this equation can be written something like this.

Now, this k is defined as like that. So, R K if I take multiply both sides premultiply both sides by R turns out R K equal to B transverse P. And if I take transverse of both sides then it happens to be K transverse times R (( )) matrix city is equal to nothing but B transverse and again P, P is again matrix as well times B. So, this is nothing but K transverse R is nothing but P times B. So, using all these and adding R on both sides whatever we have you had all both sides it can be written something like that actually. So, this is I mean if you write it this way fine or you can also write it this way because phi of T now you can substitute what is phi of s phi of minus something like that.

Now, essentially this equation leads to something actually that is by substituting the definition of phi of s and phi of minus s actually. This particular equation that you see here is nothing but the same Riccati equation written in the frequency domain sort of thing and this is not called as Riccati equation. But it is called something like Kalman equation Kalman is the one who comes up with this is and this is called as Kalman equation. In frequency domain actually or control design by the way Kalman equations can be very famous for (( )) this comes from this algebraic equation for control design. So, this is actually Kalman equation for control design actually.

Now, the beauty is we can use this equation to come up with the gain matrix or in a particular example problem. And we do not need any of those I mean (( )) solution of the Riccati equation and think like that actually go through an example make our ideas as little more clear. Now, if you take a double integrator problem x one dot is x two and x two dot is u sort of thing; that means, x one double dot is nothing but u set of that is kind of a double integrator problem. So, performance index which I (( )) under L Q R Index something like these. And this anyway that may be there is small mistake here probably we will start from there now this is the problem this is because function and this is a system dynamic equation actually.

So, you have to first identify various matrices. So, a happen to be like this b happens to be 0 1 Q happens to be identity both the terms are here x 1 square plus x 2 square and R happens to be 1 u square is available. So, all these A B Q R is available now what happens remember here we want to apply this term actually. So, we have b available Q available r available A available like that actually. Now, the Kalman equation is nothing but the whatever you see here you can be written as something like this. Where s I minus A inverse happens to be like this because s I minus A is available A matrixes is available.

So, you calculate s I minus a symbolically and then take s I minus A inverse symbolically also and will turn out to be like this. Now, we define gain s k 1 1 k 1 2 few more gain as to be row matrix in this particular case in single input system anyway. So, gain is to be k 1 1 k 1 2 we do not know the values of k 1 1 and k 1 2 yet we want to compute it using Kalman equation actually here. So, what you do we substitute that in this equation especially directly and we can we can now retain whatever k 1 1 k 1 to symbolically and this ultimately after simplification of the matrix multiplication and like that will lead to us to these kind of an expression actually.

And here we can actually equate the coefficients of various power of this 1 by s square 1 by s cube s fourth like that and ultimately we will see that 1 is equal to 1 anyway. This is otherwise it is 2 k 1 1 minus k 1 2 square is nothing but minus 1 here and then k 1 1 square is nothing but 1 it is if k 1 1 if k 1 2 k 1 1 square equal to 1 then k 1 1 is 1 equating these to that there coefficient 1 here 1 in k 1 1 square nothing but 1. So, k 1 1 is 1 and k 1 2 of 1 2 once you have this k 1 and this expression happens to be minus 1 basically. So, this is minus 2 and then this is minus 1 and then take it other side it becomes 3 for that is k 1 1 2 sorry k 1 2 square equal to 3.

So, k 1 happens to be square root of 3 now, obviously we can always argue what happens to the plus and minus things and all that actually then turns out that if it relate k 1 1 1 and k 1 2 is square root of 3 then it is leads to a stabilizing control over it actually. All other things not lead to stabilizing control and which happens in the Riccati equation solution directly also. If you will have a multiple solutions for Riccati equations then it select the 1 which is possible to definite solution anything and then only it lead to stabilizing control. So, similar ideas exist here and out of eliminate this minus roots and we take the positive roots and come up with the gain matrix. And this gain matrix 1 square root of 3 is that matrix can be also derive by solving this algebraic Riccati equation formulation.

And I encourage all of you to do that using your pen and paper and all that do not really have to go through this method of formulation and all that, but quickly you can also use of do not have time want to have quick answer. And think like that it can also use for a function of L Q R method of (( )) and just type in this values A B Q R and the immediately the gain matrix will pop up that will nothing but 1 in square root of 3 basically. So, this is about this how do we use this Kalman and how do you these I mean how do you make use of this Kalman equation rather for control design problems actually.

But this is not the main motivation for having a Kalman I mean Kalman equation actually people come up with this and then to the further only tells us that how do what is the frequency response actually. That means, what we know is some sort of a (( )) and think like can we really do can really do that actually. And uncertain then what happens is the like what we had do in linear system analysis we substitutes s equal to g omega anyway and then using that analysis will proceed for the then derive this root diagram everything actually.

So, that analysis I will not cover here, but those of your interested actually this part of the thing I have taken it from B S Naidu book and those of your interested you can see the book for the further analysis actually. But interesting results turns out to be something very good and it tells out that this entire result L Q R formulation as a very good robustness property. And the gain margin happens to be minimum half and maximum there is no bound actually it can be infinity also. And phase margin can be at least 60 degree it can be more than that, but at least 60 degree is guaranteed actually all that of course, happens with exact state feedback. If the exact state feedback is not there; that means, you use some sort of a estimated state and think like that.

And then this margins are concern it does not satisfy the margins then there are concepts of L Q G and L T R and think like that will be probably see it around the line actually. But assume that exact feed backs are available and most many times it does available by the way as long as you talk about controlling your own system dynamics. Without the need for a (( )) Input for controlling other words if you talk about missile guidance problem we need to know where the missile is what angle it is going whether it is stationary or moving away and thinking like that. Those information cannot be declared by the good sensor and the in the target actually target users tells us to do it.

So, and so, but it can reveal it is own strategy and think like that for that we have a some sort of a (( )) and think like that there it is an some sort of estimation in the loop actually. Otherwise now for many costly systems; that means many good systems. So, we have a good sensor equipments and all available. So, if the system is heavily kept with good quality sensors then it can actually get all the state feedback without estimation in the loops sort of things actually. So, then you can actually implement and you can so that gain margin is minimum half and maximum infinity and phase margin is at least 60 degree. Anyway by this little bit interesting discussion here that I mean earlier day is this this control gains and all were actually synthesize using analog devices.

So, there were some chance of a having this gains what you intent to do intent to give it the system the actual gain would have been something different actually and that is no more a constraint because it is I mean now a days it is all digital control actually. So, we actually compute it in a computer and hence we feedback to the system; that means, there is no I mean there is no relevance of gain being inaccurate and think like that. But still gain margin is a very important concept primarily because any amount of system parameter in accuracy will ultimately reflect in some sort of an equivalent gain in the close look system.

Because a system parameter which is not what you assumed in the control design something else then it can be equivalently describe a some sort of a (( )) in the gain (( )) actually. So, that is how it is still important and even though you have digital computers actually. And similarly, phase margin as a time domain interpretation of what is called as delay margin, and delay margin is I mean if you remember this phase margin essentially comes with this time delay input of the signal actually. So, then delay is an inherent phenomenon you can have sensor input I mean sensor output delays you can transport delays computation delays something like that.

And with the advancement of all these technologies and computers the gain margin I mean the phase margin or time delay margin is the high timed delay margin is probably not required. Because things are things can be done in a very efficient way s; that means, that good margin requirement from phase point of view and very stringent now a days. But we certainly need something some gain margin and some phase margin as well because we no matter how fast is the computer no matter how fast is the I mean whatever it is there will be a some amount of transportation delay and some amount of computational delay and think like that.

So, the system must have some sort of a positive phase margin for successful operations actually there is another there are many concepts ideas like that. For example, these gain margin and phase margin then they can have to be positive for a stable system and they happen to be both 0 at the same time. It cannot have like 1 0 1 or 0 and the other 1 becomes later and all that they can start with something and, but then when they happen to be 0 there to be 0 at the same time actually after which the system go unstable actually that way.

So, anyway those concepts of gain and phase margin are classical control concept interested students can find many typical books around that to get lot more ideas there the whole point here is if you have L Q R design it is optimal control design. And essentially it not only does a lot of jobs that compare to let say typical (( ))control or port placement design because remember port placement design is provided you have single input if you have a multiple input then there are not. So, easy things to do and not so, good things to do actually then we will talk about control (( )) think like that, but then it is not scientifically done the one way of controlling is through L Q R control it is. So, likewise there are lot of other things lot of good properties associated with that and hence it is been probably in a use as well actually.

So, with that I think I will stop this lecture and we will see the further things in the next lecture thank you.