Advance Control System Design
Prof. Radhakant Padhi
Department of Aerospace Engineering
Indian Institute of Science Bangalore

Lecture No. # 33
Construction of Lyapunov Functions

This particular lecture is all about Lyapunov theory again and then the outline of this particular lecture is like this. First we will go through the this review of Lyapunov theorems then it is we will see that whatever we discussed was the end of last class this is a LaSalle's theorem and all.

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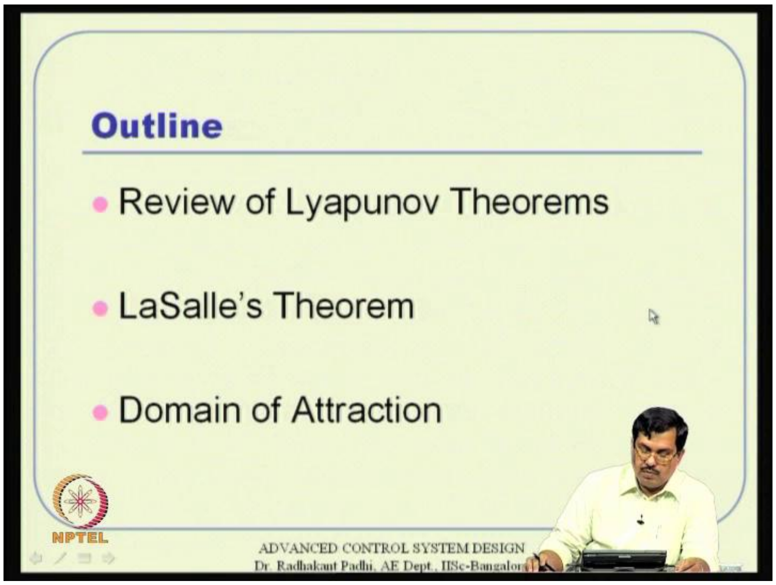

And then we will try to use this LaSalle's theorem to something to estimate something called domain of attraction. And any asymptotically stable equilibrium point obviously as a neighborhood about which it cannot attract trajectories. And then particular domain is called domain of attraction is in general not easy to estimate, but once we estimate there are lot of potential usage. Actually, in other words, you can think of like stopping your control x on once the trajectory enters this domain of attraction because that is one of the usage that you can think about actually. So, anyway the so let us proceed with that quick review of Lyapunov theorems and some concepts that we discussed last previous two classes and all.

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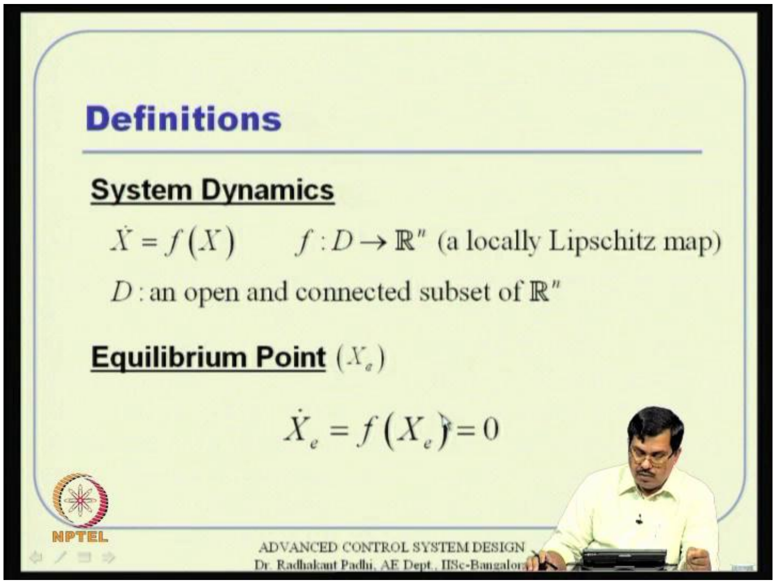

So, first we what we discussed is the system dynamics is an homogenous form and this is also like autonomous form. So, in other words explicitly time does not appear in function as well as control also does not appear in function. However, if the control is in a state feedback form then this homogenous form also applies for close loop system dynamics actually. Then was we are interested in this open and connected sub set of R n and the function is define from D to R n actually. And we are interested in analyzing the equilibrium, I mean stability behavior around equilibrium point primarily. So, equilibrium point is defined as some other state derivative are equal to 0. So, whatever the solution shows about each equilibrium point you wanted to analyze the stability behavior.

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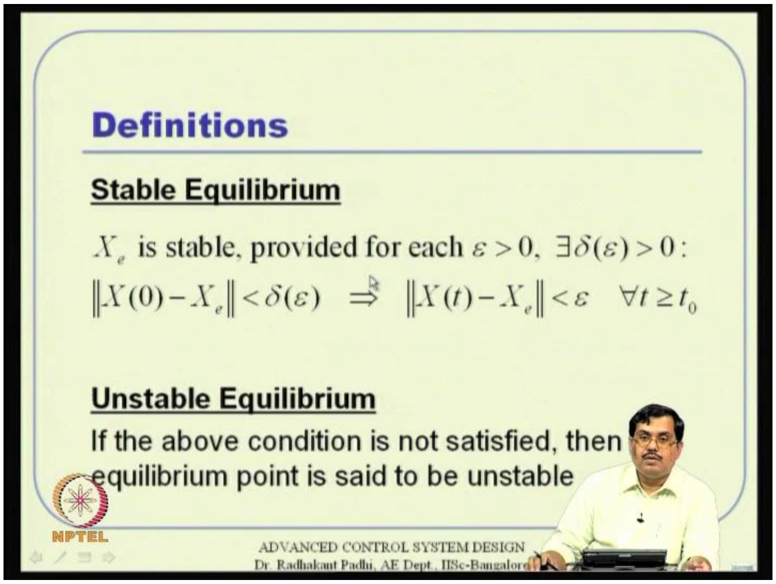

The once our definition that we studied and this first thing is what is called is stable equilibrium. In other words if you start with some finite wall then for all time your trajectory remains in an another finite wall and the radius of this walls can be function of each other also. Actually in general so that is the kind of weakest notion of stability and if we does not satisfy that also then it is a unstable actually.

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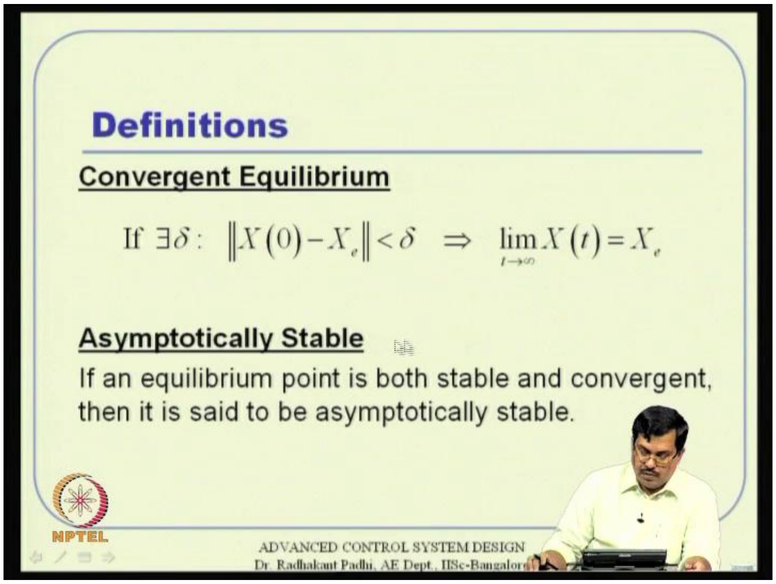

Or if it satisfies the weakest notion then we are further interested and for that we need this idea of convergent equilibrium. That means eventually the trajectory and goes to the equilibrium point, no matter where it travels on the way then it is convergent equilibrium. And obviously if I mean if an equilibrium point satisfies both stability and convergent behavior then we tell it is asymptotical stable equilibrium which is always desirable to have actually.

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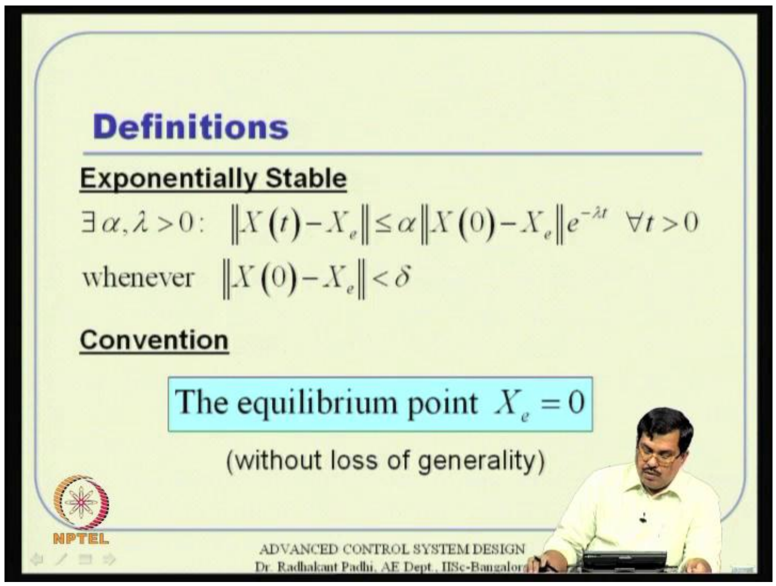

Then even further stronger notion of stability which is the exponential stability so, in a trajectory not only goes to the equilibrium point or but the rate of d k is also exponential, this is the strongest notion I have all that. And for a every practical purpose we tell we can also do change of coordinates and all that so, we take this equilibrium point to be the reason actually.

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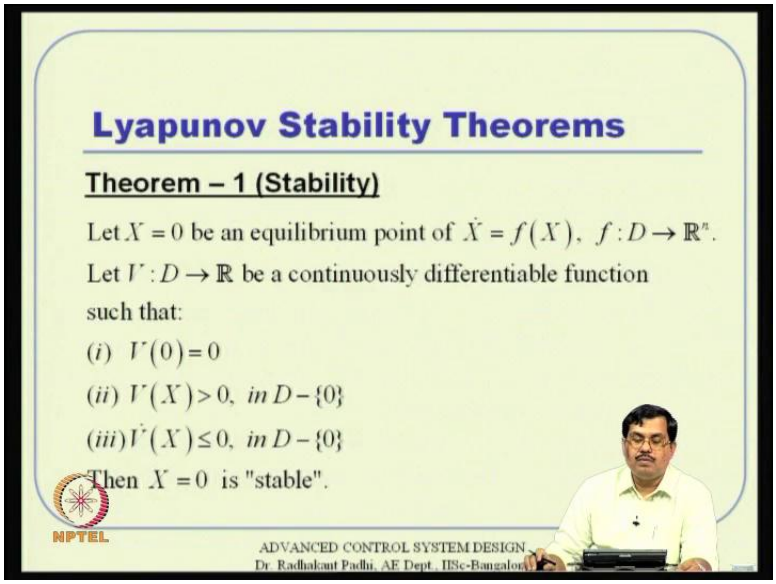

Then with that we had some bunch of theorems Lyapunov theorems and all that Lyapunov stability theorems and all. The first theorem told that if this is X equal to 0 is an equilibrium point of the system provided this V is a continuously differentiable function so such that these things are happen. But if it is a continuously differentiable function and in addition to that point number one and two are also satisfied then it is also a something called a positive definite function. So, we start with a positive definite function and also I want to make sure that the rate of change of that positive definite function is less than equal to 0, it is negative semi definite function. And if that happens then X is equal to 0 is stable in the weakest notion actually. Now, obviously weakest notion is something that we typically do not like so much, we want to conclude further.

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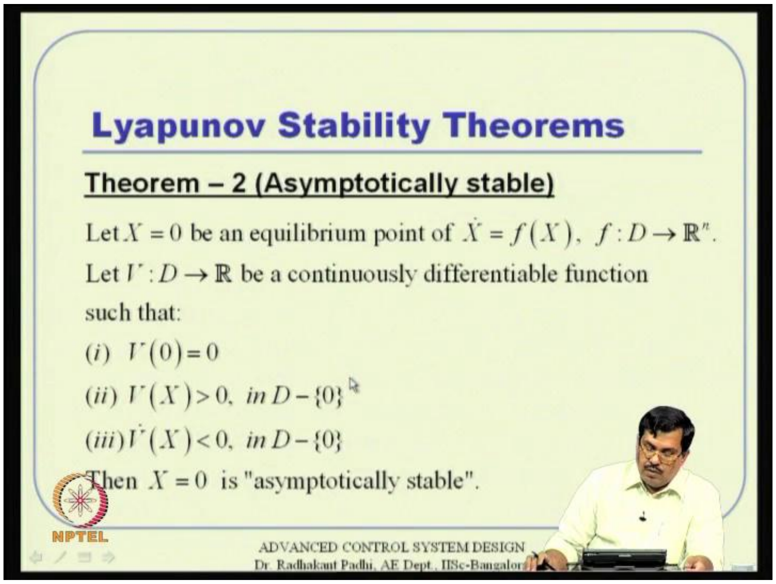

And for that we have this asymptotically stable equilibrium conditions sort of thing. And that is for that the only change is if you mark here, this in equality becomes this I mean strictly any strictly less than 0. All other things will remain exactly same so you , but the V dot needs to be negative definite function. And that is as we discussed is a is an order of magnitude complexity higher in complexity because to conclude negative definiteness V dot needs to be your function of all components of state variable. If it is one components left out then also it we cannot conclude that, we can conclude only semi definiteness. So that is as to difficulty what we throws up actually.

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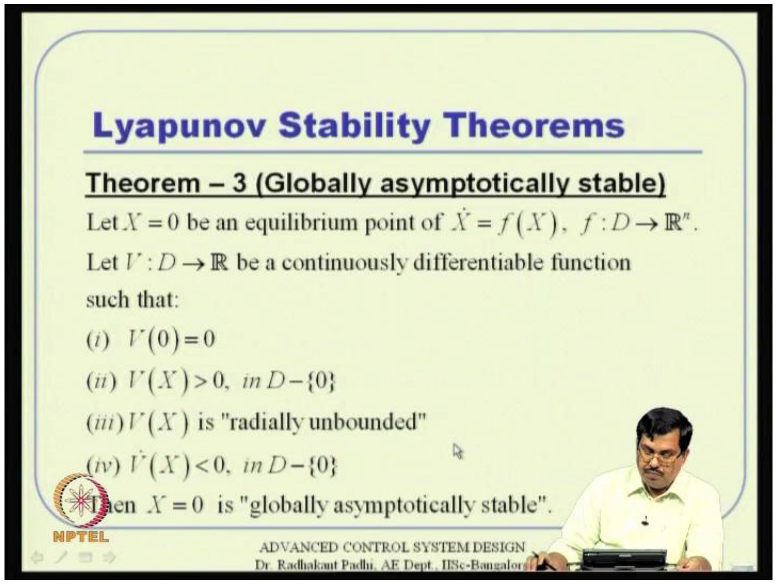

Anyway, so the theorem three in this regard is like how do you call that with these things only guarantees you is some sort of locally asymptotically stable we are used. That means around the neighborhood of the equilibrium point then only I mean you can conclude something , but how about global behavior? If you want global behavior then in addition to all those conditions we have one more condition which is the V of X value of definite function is to be radially unbounded. As well actually that is as X goes on further and further away from origin then what about all directions V of X is to keep on increasing actually. Now, that is V of X initially readily unbounded then only we can talk about like global asymptotically stable behavior actually.

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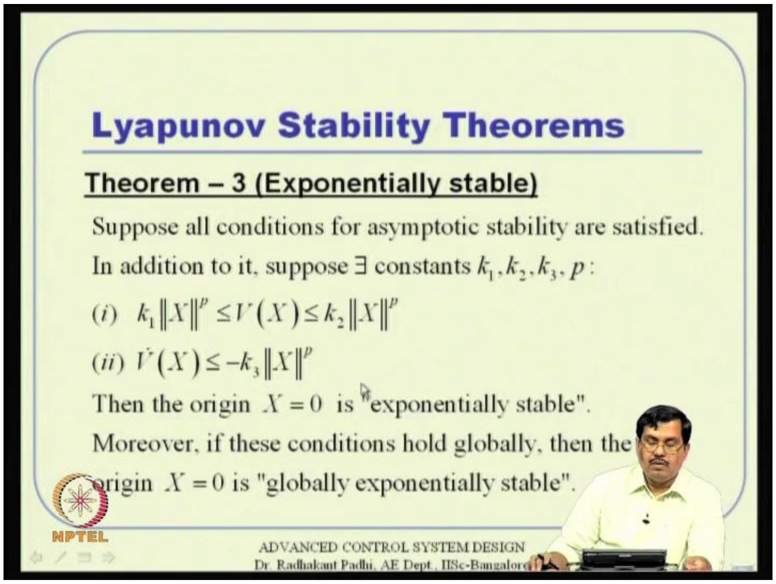

Then exponential stability conditions so in addition all the condition that we discussed about asymptotic stability. We needs to have this further equations I mean in equality needs to be true that V of X needs to be bounded by exponential functions both below and above. And then V dot also needs to be bounded above by this negative V 3 times norm of X actually. So V of X needs to be bounded from both sides with exponential functions and V dot needs to be bounded above by this negative constants sort of thing. Remember this self is a negative number what you have in the right hand side. So, if this two conditions are satisfied in addition to whatever conditions you talk here then it is locally that exponential stable. Or if it also satisfies this conditions where V of X is readily unwanted then it will become global exponential stable. So, what as we discussed also previously this conditions are not in general for non-linear system this condition are difficult to utilize also.

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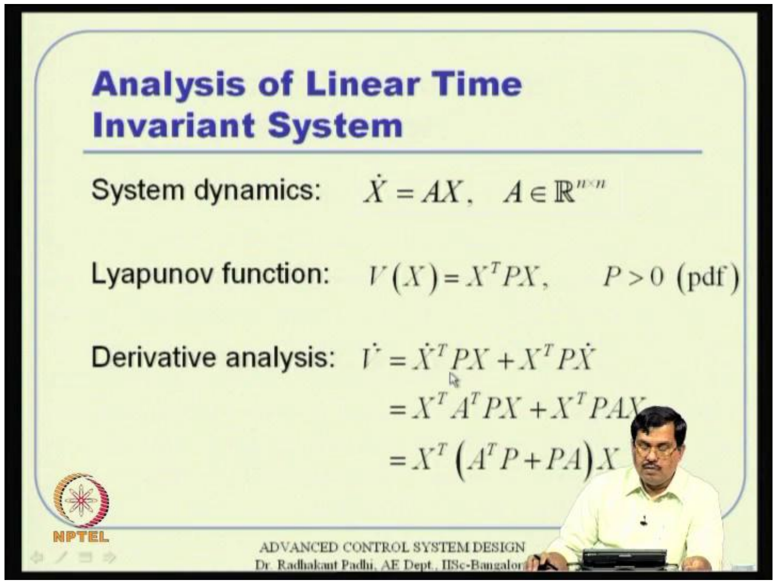

For linear system, it is possible by using this reline equality and all that actually as we discuss before. Now, coming to this linear time in variant system as a special class of system we also discussed about that. And analyze the stability behavior of that particular class of systems we took Lyapunov function candidate like that where P is positive definite matrix. And then we saw that V dot is nothing but if you start with that and then apply this derivative rules and all that then X dot is substitute with A X and X dot transpose is X transpose A transpose like that, then it throw something like that.

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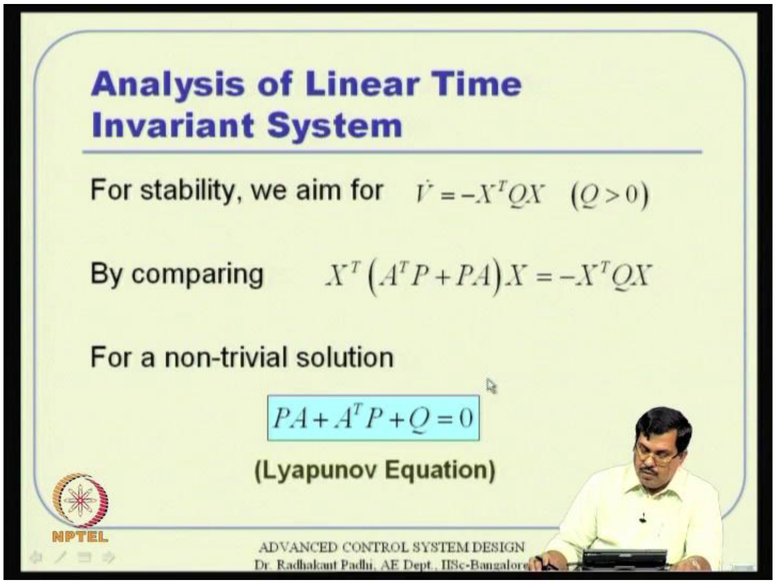

So if you have to this V dot is strictly negative definite then we can aim that V dot needs to be something of this form where Q is a positive definite matrix. If you happens then this we equation and whatever we have as V dot here it needs to be equal and if that happens for non trivial X solution and all we have to have this Lyapunov equation has to be satisfied. And this Lyapunov equations we being a linear equation, it is a function of A and Q where P and Q both needs to be positive definite matrix.

So, one way of looking at that I will probably select P and then solve then evaluate Q, but that does not help us too much because the solution can be like the conclusion can need not be one sort actually. But you really want to have one sorts so I mean check and all that then we start with Q, positive definite and e Q and then solve for P actually. And if it P happens to be also positive definite then we have done and if does not happen to be positive definite then also we are done actually. And this is how this primarily because we are talking about linear systems actually.

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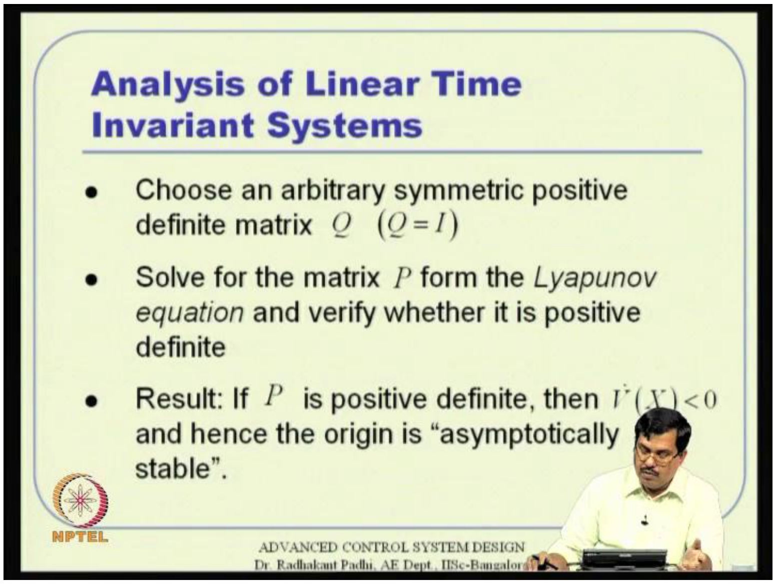

Anyway, so that is the procedure so you can choose an arbitrary symmetry positive definite matrix Q and if I can select any matrix then probably I will go and it and select I identity. And then we solve for the matrix from Lyapunov equation here and then you tell the whether you check whether this P whatever the solution proves of we just check one time whether P is positive definite or not. And if P is positive definite obviously the I mean both V and V dot may V is positive definite then V X V of X is positive definite, Q is also positive definite since V dot e is negative definite. So that is how all this things are satisfied and you have this clear cut result whether the equilibrium point is asymptotically stable or not. That is the way to utilize this Lyapunov theory therefore, analyzing linear system equations the I mean linear system dynamics.

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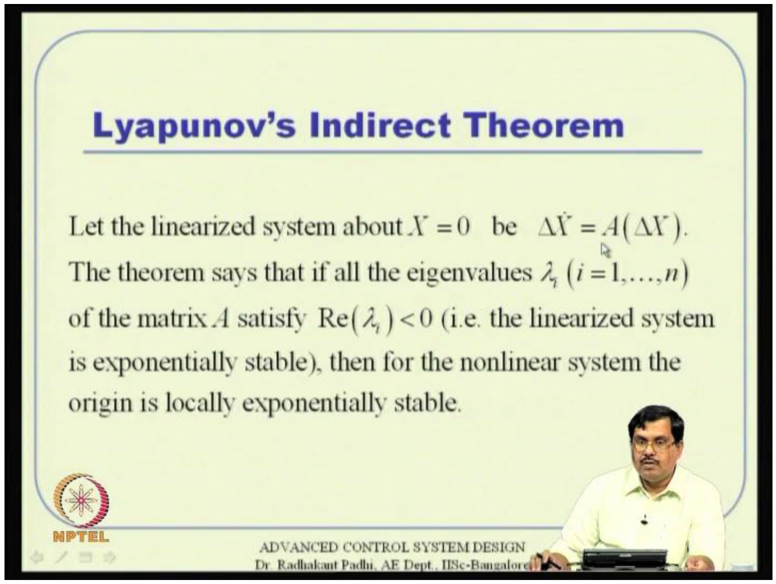

That is the great theorem which is like Lyapunov sender theorem also we discussed about that. So, what it tells is around the equilibrium point we linearize the system dynamics and study the behavior of the linear system dynamics. That can be rather easily studied from Eigen values and all that. And if it if all the Eigen values of this matrix A or in the left hand side of the complex plane then obviously, we know that the linearize system is exponentially stable. But using this conclusion it also turns out that the non-linear system is also local exponentially stable.

That means that the region may be small, but still guarantee is there that if the linearize system is stable when the non-linear system is also stable. And that is a beauty of the Lyapunov theorem and if you ends very direct straight forward approach to test whether the (( )) the system is actually local I mean kind of a exponentially stable or not. If you really want to that kind of very quick conclusion, all that we need to do is linearize the system dynamics and then study the behavior of the Eigen values of the matrix actually.

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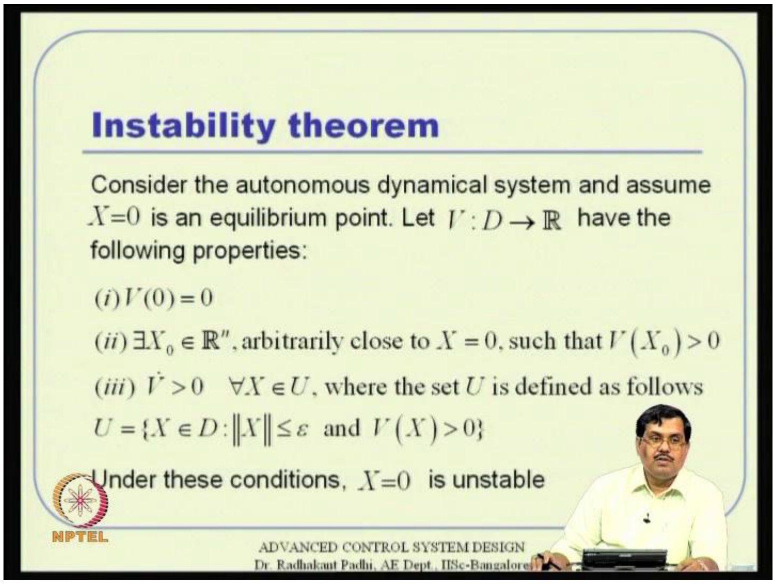

Then on the way we discussed about instability theorem also why because this the Lyapunov theorem happens to be this sufficient conditions and all. So, we whenever see your what you are showing whether if one see one selection fails then we never seen whether some other selection will succeed or not so in for because of those difficulty and all. There are also like equally thought of what is in stability theorems in the sense, that if the system is really unstable no matter how to whatever attempt you do for selecting good Lyapunov function you will never succeed actually. So, because of that there are theorems that we discussed about that and then essentially tells you that if I select a V of X which is positive definition and V dot happens to be also positive definite then I have done.

But we do not need those wrong condition you can simply need like one point X 0 plus to this origin such that V of X is positive and V dot also need to be positive in this particular set actually. So, this is what we discussed before actually so, this is which is one of the in stability theorems. There are many other in stability theorem as well actually.

So, if somebody can show this system dynamics is really unstable using one of the theorem then also we are done actually. Also remember instability theorem are also sufficiently conditions only basically. And we also discussed last class about some construction Lyapunov function ideas because standard Lyapunov functions you know this quadratic functions that X transpose X or X transpose X X transpose P times X where P is positive definite that is a very standard way of starting this basically. If it does not happen then you can probably other alternately idea is to select the total energy that kinetic energy plus potential energy. But other than this two we are kind of line or what you take actually. So because of that we studied about some ideas of construction Lyapunov function and one of the idea happens to be the variable gradient method.

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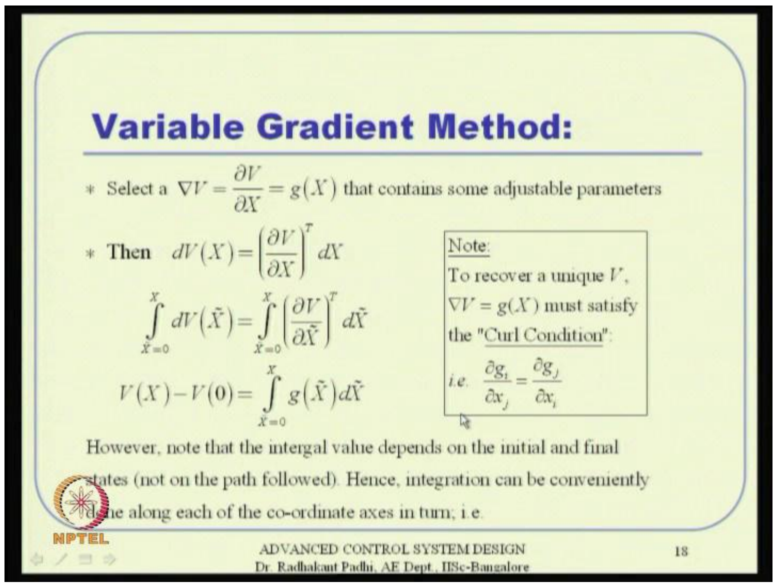

Where you do not select the V, but we want to select gradient vector del V by del X actually. And once you select a gradient vector del V by del X will be able to solve for the from the expression actually. And integrating one direction one component direction at a time sort of thing actually, but the point here is this g of X must satisfy some sort of a condition. That means if I take del g by del X that matrix needs to be symmetric actually.

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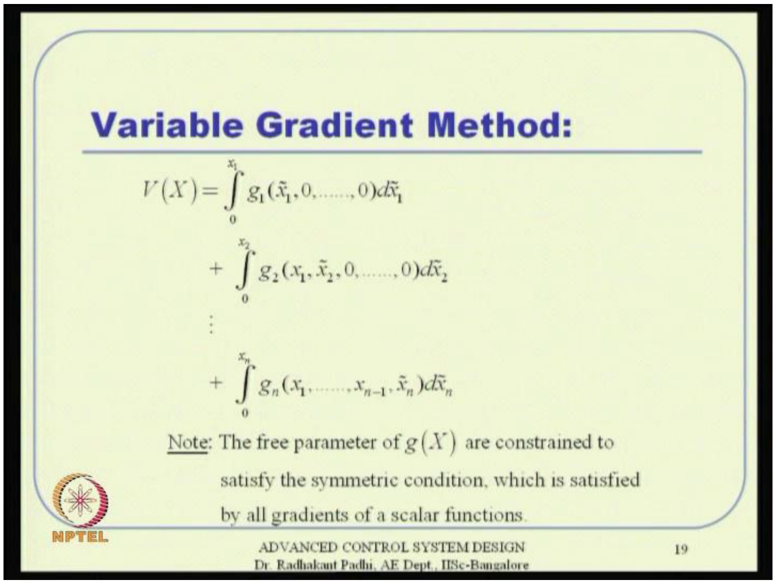

So, the integration can be done component by component that is what we discuss before. Once you once you know that g 1 g 2 or g n and all selected already then V of X you can compute it that way actually. So, you can start with some sort of a free parameters and all that. And then those three parameters can be existed to see that if g of X needs to be positive I mean sorry this del g by del X needs to be symmetric actually. So, using that you can satisfy some of those thing we have seen in many examples on the way also, if you review this previous two lectures and all many examples where there to clarify some of the ideas actually.

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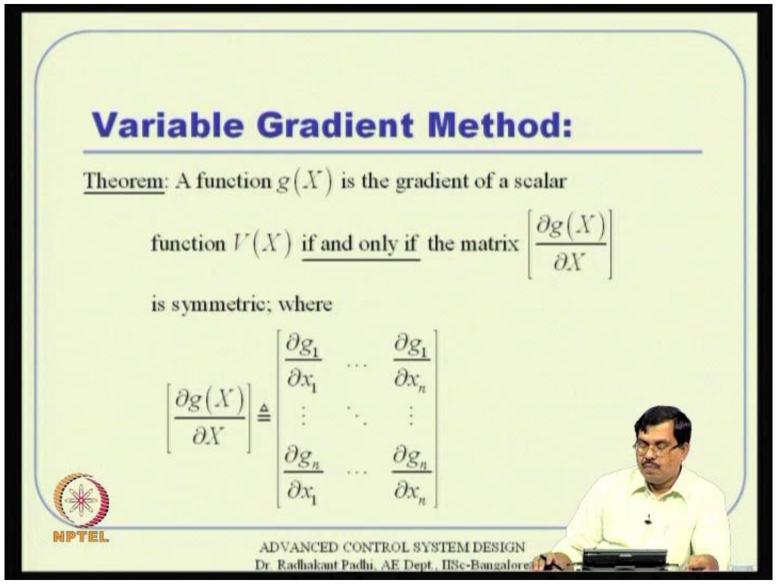

So this the condition that I was talking, the restriction that we are placed here I mean we have to work with that is del g by del X matrix needs to be symmetric actually. So, with that selection we will be able to solve for d of X. And then you proceed with your selection which are evaluation of V dot and thing like that actually. So, then we also study something like method and all that.

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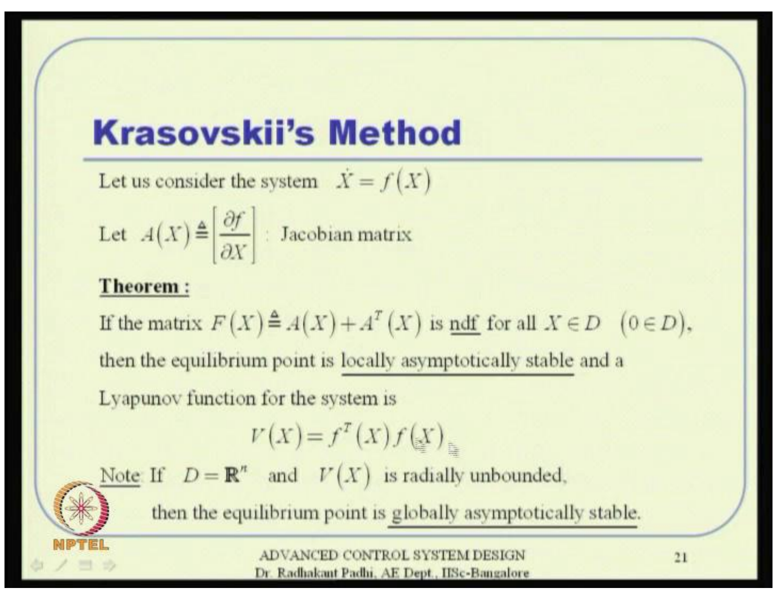

Where there are the I mean I one version of the theorem is very straight forward all that. It tells you is given system dynamic you formulate this A of X, this is not linearization, very simply computing the Jacobian matrix in a symbolic form. And once you compute that then A plus A transpose you select as another matrix something f of X and then tell this is this equilibrium point is local asymmetrically stable if this big F of X is negative definite actually. And the corresponding Lyapunov function by construction of the f transpose of f rather, so that is very easy to see. And in addition to that if this happens to be gradually unbounded of course, this f transpose f we are we do not have direct control on that because f is a system dynamic actually. But if it happens to be radially unbounded then it is also globally asymptotically stable. But as see this I mean realizing this one is I mean intuit feeling this one this intuitive feeling and all very difficult.

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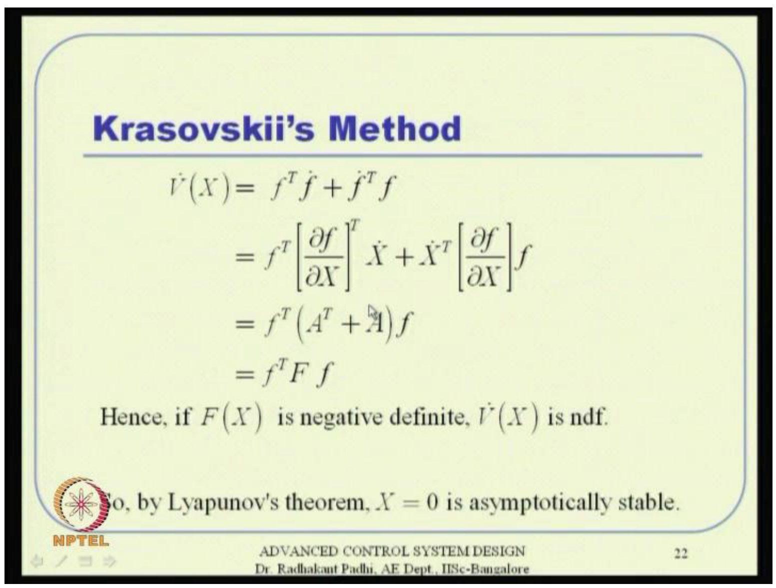

Because if we see V of X is that they f transpose X can be X dot is nothing but that and then f dot is nothing but del f by del X transpose X and X dot and all that, but it should work in both the I mean all the expressions there. And I will tell if the V dot is nothing but f transpose big f times only. So, if this happens then obviously very clear that f of X has to be negative definite for V dot needs to be I mean for V dot to be negative definite. So, what he tells you is you given a system dynamics is a strike out with this simply come evaluating A and then f which is A plus A transpose and then makes your that this big of f X is big f of X matrix rather is negative definite matrix, as seen some example also actually.

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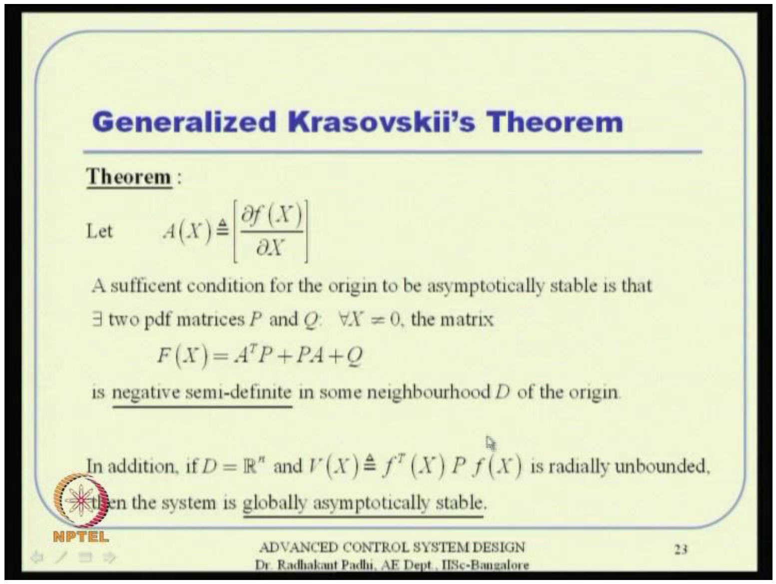

Then there are generalized Krasovskii's theorem also we discussed. This restriction turns out to be too much here because we are all dealing with the system dynamics that is given to us, we do not have any direct control on the system dynamics here. System dynamics is something and the V of X turns out to be X transpose X so there is no freedom actually. And if it happens that f of X happens to be negative definite we are lucky otherwise not actually. That means restriction conditional placed on this is relatively higher side so, we want to kind of relax those and all that.

So that way it will leads to this generalized theorem which tells you that is still evaluates this A of X is similar way or rather same way. Then you contrast f of X not just I mean that otherwise this A plus A transpose sort of thing, but you construct it something like A transpose P plus P A plus Q with two positive P D F matrices also A to select along the way actually. You select P and Q two positive semi definite matrix and then construct this f of X and then you make sure that f of X is not really negative definite, but it is negative semi definite. So, not only it gives you a freedom of selecting Q 1 Q and hence it gives you lot of freedom of constructing this f of X.

But also tells you that f of X needs to be only positive semi definite sorry, negative semi definite. So that way negative definite to negative semi definite is lot of relaxation of condition. In addition to that it gives us freedom of selecting P and Q. So that way it turns out to be much more general and convenient to it actually. And again if this f transpose P times f of X is radially unbounded obviously there is also R lower actually.

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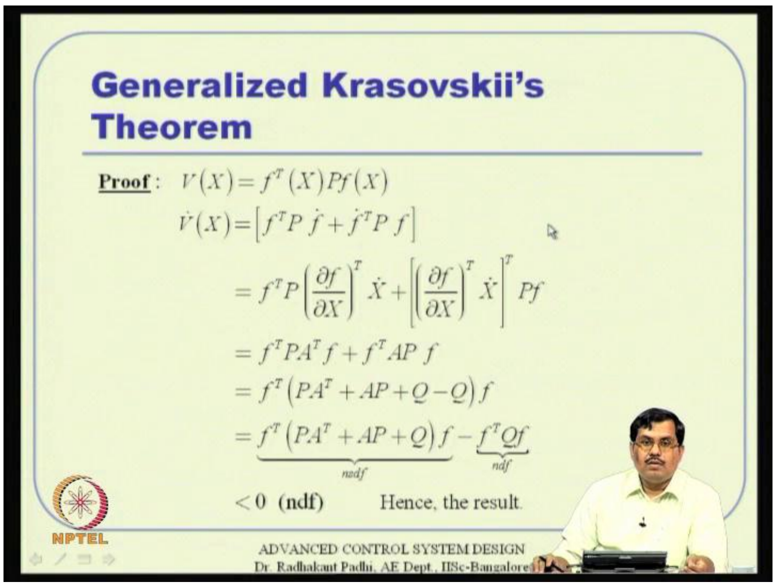

And that is also not that difficult to show I mean you can start the V of X then call then evaluate this V dot which is f transpose P f after plus f dot transpose d f and all that. Then f dot is substituted again similar way and then here is the trick you plus you do something like plus minus Q actually. Once you have that then this the this turns out to be like this minus that and because Q is a negative definite matrix then f transpose Q f is already negative definite. So, all that we need to make sure is this fellow I mean this part of the equation needs to be negative semi definite. So that is how it gives us a much more flexibility sort of thing. But this is a like to two method one is variable gradient method another is Krasovskii's theorem and all that. Then we also discussed something called invariant say it limit sets and all that.

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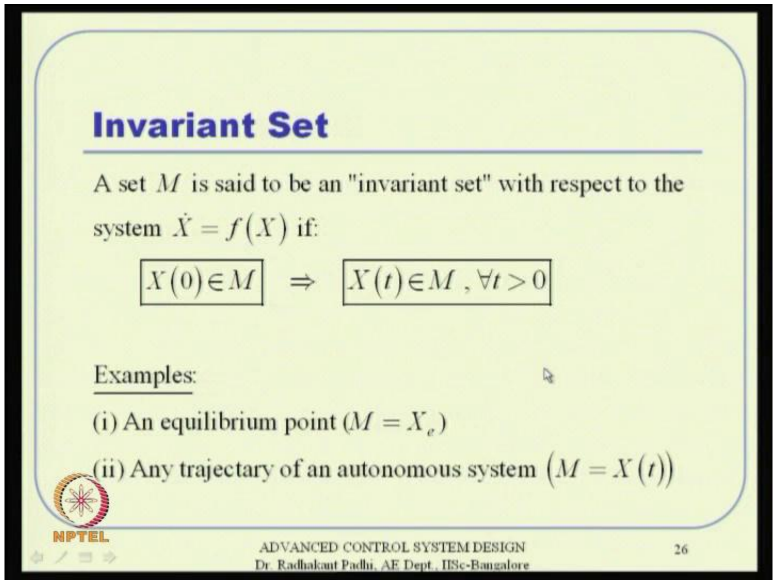

And what is that? The invariants set turns out to be like, if my initial condition is within the set then for all time my trajectory will remain within the set actually. So, obviously then the set remains set is something like invariant set. And an equilibrium point obviously then variance set trajectory autonomous system within a invariant set many other examples we have actually. Limit cycle is an invariant set like that.

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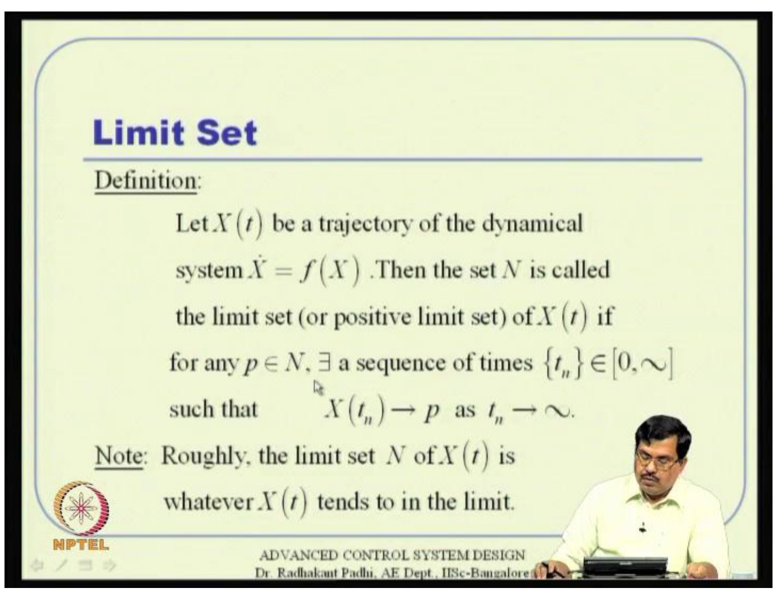

Now, limit set I mean this is the set whatever X of t tends to in the limit actually. Starts with something, but as t goes to infinity then it will go to say I mean all trajectory will go to that particular set actually, so that is that particular thing you called limit cycle actually.

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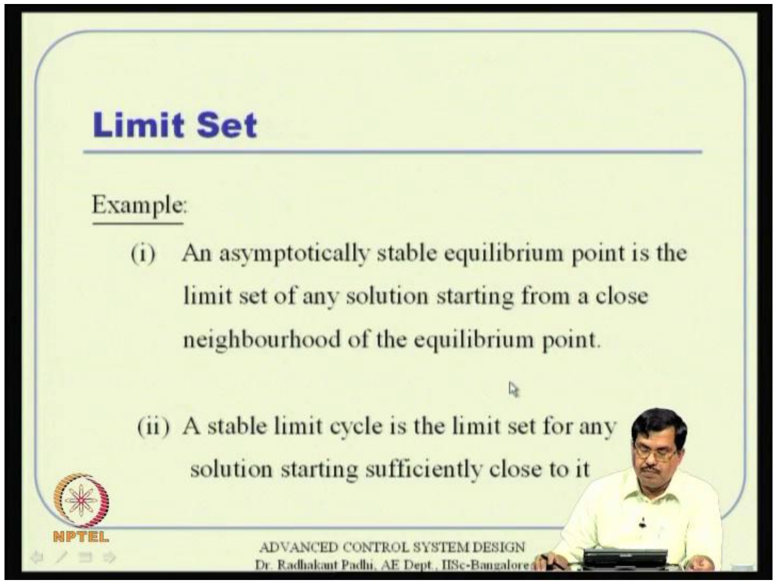

So, I mean examples sense asymptotically stable equilibrium point is a limit set for any solutions starting from a closed neighborhood of the equilibrium point. If we starts with the closed neighborhood obviously, the trajectory is going to go to the equilibrium point. So, obviously that becomes limits set actually the where t goes to infinity under the limit this X of t we will go to that particular set. And stable limits cycle is a limit set of any solution starting sufficiently closed to it actually so that is definition sort of thing. Then we also studied some sort of LaSalle's theorem that is what we ultimately let towards. And before studying the LaSalle's theorem in full we actually studied subset of this theorem.

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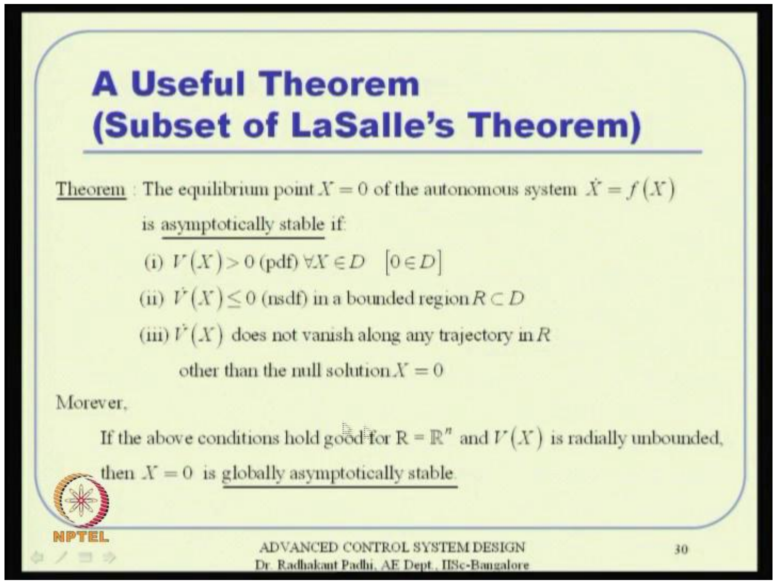

Which told us that even if V dot happens to be negative semi definite in a bounded region R of course, the region is to be bounded actually which needs to be subset of D also. And then V dot equal to 0 happens only at the equilibrium point nowhere else on that particular region R. Then what I mean what did means is V dot of X does not vanish along any other trajectory in R other than all solution X equal to 0. So, then it can still conclude that X equal to that origin X equal to 0 is globally asymptotically stable actually who had a V of X is radially unbounded. We are not radially unbounded then you can always tell it is stable or local asymptotically stable basically.

So, what is the I mean utility sense we saw that two examples and all. And one example I mean this theorem is actually very neat because most of the time what happens again is a will be able to serve V dot V dot of X negative semi definite rather easily. But showing negative definiteness is difficult, but using this particular idea of LaSalle's theorem makes our like much much easier actually.

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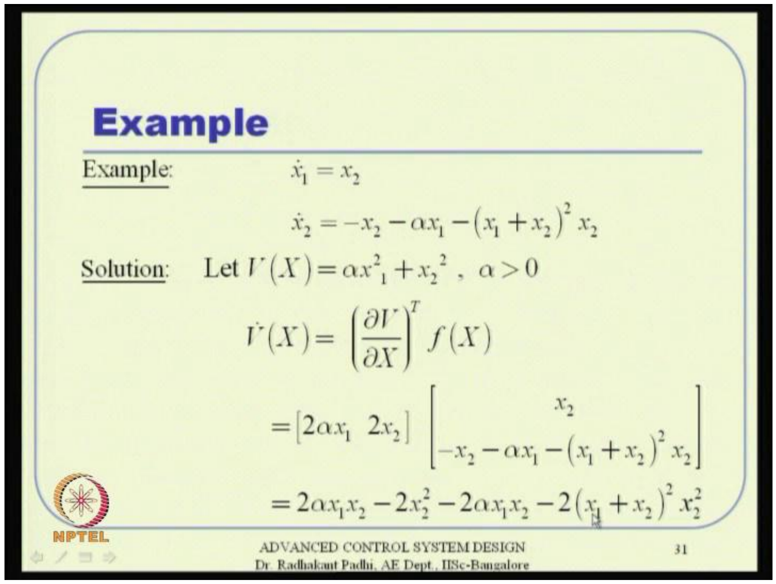

So that is what we saw one or two example one example was like that. We started with this second order system and all that. And then we choose this V of X to be like this where alpha is positive so again this V of X is suddenly positive definite, then V dot happens to be like that del V by del X transpose times f of X. So, del V by el X if you evaluate this is 2 alpha times x 1 from here and 2 x 2 from here and then f of X is like this x 2 angle and all that so we put it there, then evaluate this expression actually.

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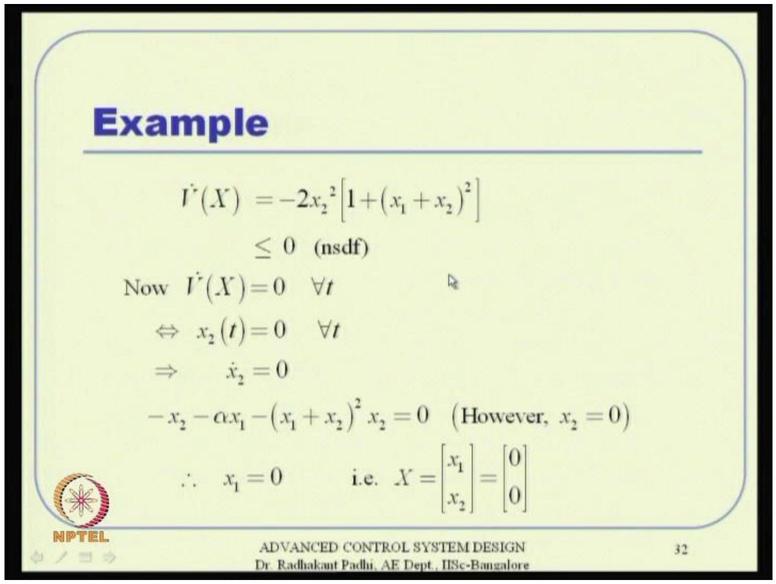

Then you tell V dot of X I can variance in this two in this form and it certainly happens to be negative semi definite actually. Now, if it is a negative semi definite where we talk because we can only conclude stability in the weakest nozzle, but we need to be continue further so, we tell let us analyze this situation V dot of X equal to 0 for all time. Let us see where it happens actually and if it happens only at the k region or equilibrium point we have done actually in a way. So, in that actually let us to the conclusion that V dot of X equal to 0 happens only at x 2 equal to 0. And if x 2 happens to be 0 for all time then x 2 dot equal to 0. And x 2 dot happens to be like this so that substituting that point and then also noting that x 2 is already 0 and that is anyway 0.

So, you are left out with big X 2 remains this one is cancels out, this one is cancel out anyway so you are left out with notes say other than x 1 equal to 0. So, it happens that x 1 and x 2 both equal to 0 whenever this V dot is equal to 0. So, V dot equal to 0 happens only on the equilibrium point and hence using this theorem we tell V dot that this even if V dot is negative semi definite the origin is still asymptotically stable, that is what we are able to solve. So that is what it is written here, but in addition to that you can also see that the choice of selection of V is actually really unbounded say quadratic function after all for any as positive alpha, it is radially unbounded. The moment this quantity gives an increasing V of X also keeps on increasing anyway actually. So, if that is the that is true then obviously there is that means origin is globally asymptotically stable basically.

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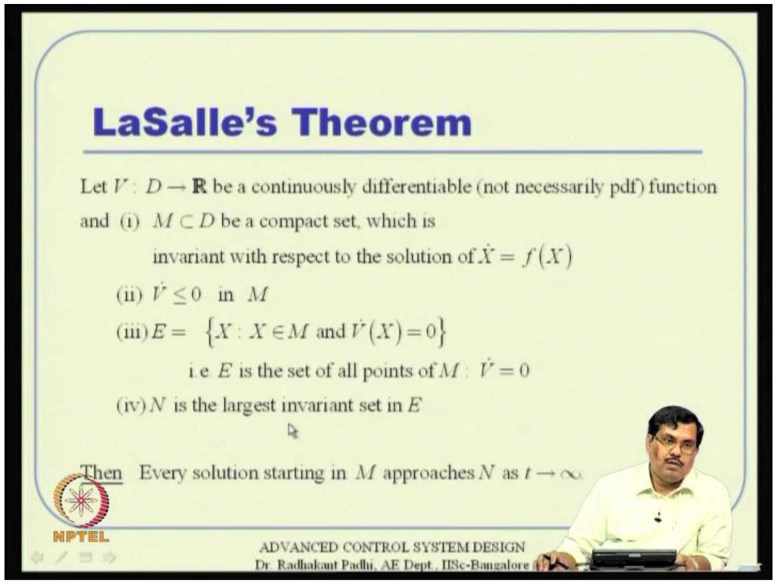

Then it was end of last lecture we talked about this LaSalle's theorem sort of thing. In I mean that is the theorem that LaSalle gave slightly bigger theorem than what we just saw as a subset sort of thing. And this theorem tells that this V can define between this domain D to R needs to be a continuously differentially function again it need not necessary with positive definite function actually. It is simply necessary that it makes to be continuously different that is all. And then this following condition holds good actually.

That is those what are those conditions? First condition is there is a set M which is subset of D is a compact set compact is something like closed and bounded set. And not only it is a compact set, but this particular subset M is invariant set with respect to the solution. Once the solution enters to the same, it cannot leave actually and within that M it is also true that we dot is less than equal to 0. And then we define that sets set E subset of that particular M rather what it tells you is let me select only that particular set for which V dot is equal to 0. M can contain everything for which V dot is less than equal to 0, but I will contain I will select that particular set E which is actually V dot of X is equal to 0. And then I will select the largest in variant set of what this particular set E because E can have multiple solution sort of thing.

Like for example, I will also take you through an example where it gives you E is union of two sets two subset where equilibrium point as well as limit cycle actually. So then you have to select one that particular per set subset which is largest invariant set in that particular E actually. So, if you satisfy all this conditions 1 2 3 4 then the conclusion is every solution we start with this M, we will ultimate approach N, as is as t goes to infinity. Hence, it is very easy to see the previous theorem that we discussed here, it will satisfy all those conditions actually so, we will see that. So, essentially what he tells you that you start with some subset some set M for which I am already on the solution trajectory basically, with that is what it tells invariant with respect to the solution that actually. Then in that M it also turns out that V dot is less than equal to 0.

Then I define V which is equal to 0 less than I will throw out and then I will tell N is the larger invariant set M E, so then it tells you that if I start with M then I will approach N ultimately.

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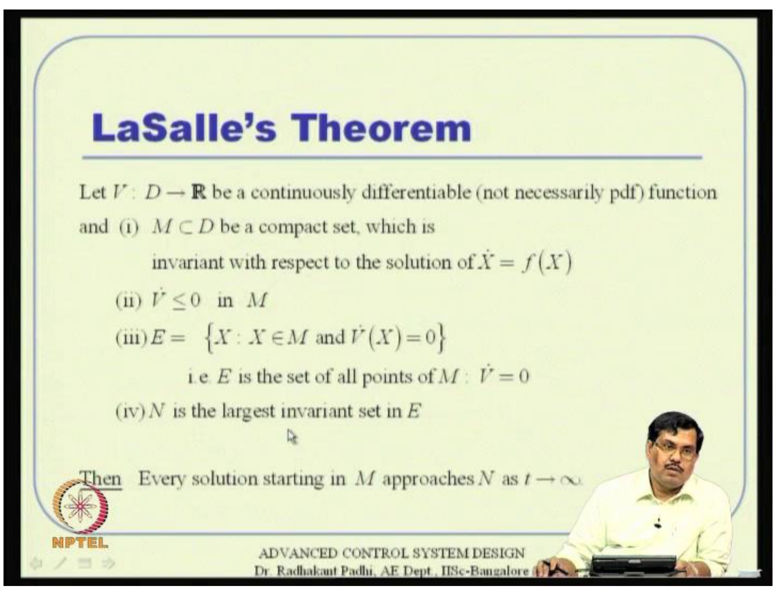

So, let us see if that some one example sort of thing and then we will ideas will be slightly more clear. But before that there are couple of remarks here and first thing is as I told already V of X requires only to be continuously differentiable function, it did not be positive definite function. And as we also discuss LaSalle's of theorem applies not only to equilibrium points it also it is more general analysis tool. And we will see one example which will be talk will be able to talk stability behavior of limit cycles also. Then we all know that earlier theorem that we discussed that was all that is obviously some sort of a subset of these theorems. In other words, it can be derived as a corollary of this theorem actually.

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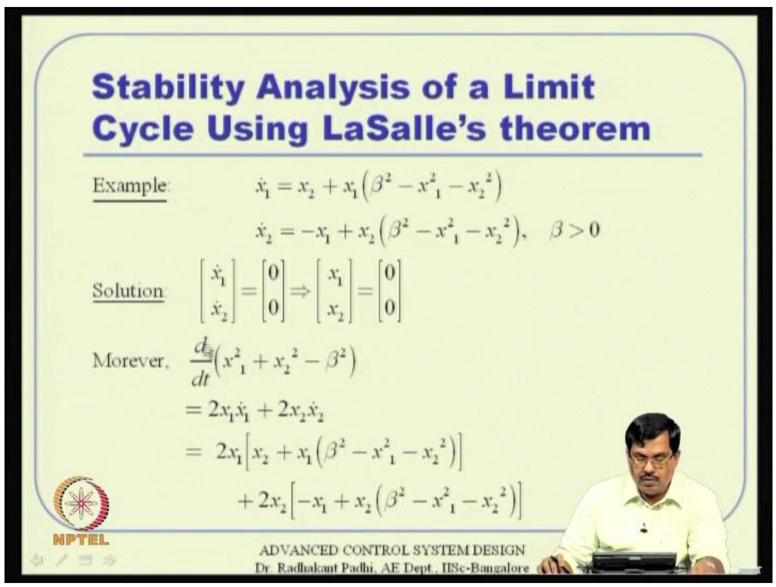

Let us see that example and then ideas will be probably more and more clear. So, we will let us talk about an example where x 1 dot is given here like this and x 2 dot is given like that where beta is greater than 0. And if it is like that suppose I put x 1 dot equal to 0 x 2 dot equal to 0, then let us say beta is strictly greater than 0, that way then what happens actually? x 1 dot x one solution ultimately is x 1 and x 2 both equal to 0, if x 1 and x 2 both equal to 0 both are 0 anyway. But there is a interesting I mean that is the only equilibrium point concern thing actually. The equilibrium the as for as equilibrium point is concern we have only origin is the equilibrium point I mean no question about that because it does not throw any multiple solutions sort of things actually.

However, it can also analyze let us analyze this expression with respect to this system dynamics, this is the same expression what we have here with negative sign probably. So, if you take d by dt of this expression x 1 square plus x 2 square minus beta square where beta is constant anyway, then it turns out it is two times 2 x 1 times x 1 dot plus 2 x times x V dot this is 0 anyway then x 1 dot you substitute this and x 2 dot is substitute in that expression here to be then carry out the algebra and it will turn out that I can nicely combine this d by dt of this expression is something like two times this 1 x 1 square plus x 2 square into that again.

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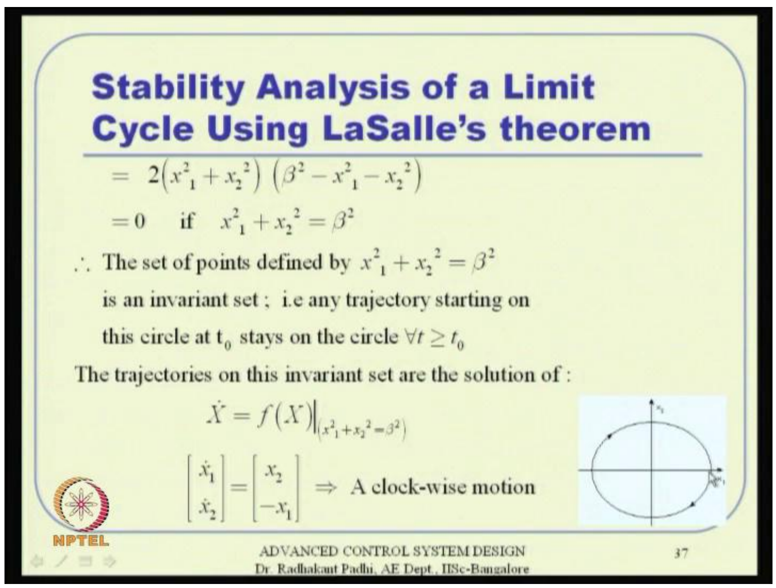

So, this expression what you are talking about is 0 as soon as this happens this will never be 0, then the equilibrium point so forget about it. But other than equilibrium point this can happen actually obviously and if it happens then this expression whatever you are analyzing d by dt of this expression happens to be 0, that means what actually. This particular expression what you are talking, if I satisfy this equation in the beginning itself then I will keep on satisfying this equation because it is the derivative later changer derivative I mean later changer this expression then 0. The condition is this again this expression x 1 square plus x 2 square minus beta square. If I take x 1 square plus x 2 square minus beta square equal to 0 then the rate of change of derivative of that particular expression is again 0. So that tells us that we have got some sort of a limit cycle actually.

If I start with any point on this particular trajectory we satisfy this equation then I will continue to stay with that particular I mean that particular locus on the basically this x 1 square plus x 2 square minus beta square. That expression will remain constant it will never change actually. But x 1 and x 2 keep on changing because x 1 dot x 2 dot is not 0, x 1 dot x 2 dot is still some value and that value is if you go back to and see that only this portion is becoming 0. So, you still have x 1 dot is x 2 and x 2 dot minus x 1 so that means this variable they change. But they change in such a way that this expression remains constant actually and that actually gives us some sort of a limit cycle behavior.

If I prop that in phase plane x 1 and x 2 there, when x 1 and x 2 are changing, but the cost I mean the way they change the way they interact with the each other is that this expression remains constant actually. And as long as I satisfy this x 1 square plus x 2 square equal to beta square, that is my expression. That means x 1 will change like x 1 dot is x 2 x 2 dot id minus x 1, what are that I tell you let us say if I am in the positive quadrant then x 1 dot will change positive x 1 will keep on increasing. But if I have x 2 negative which happens in third and second I mean in forth quadrant x to negative then x 1 dot will be decreasing.

So, in this third and forth quadrant x 1 will start decreasing, decreasing means x 1 value will keep on becoming lesser and lesser basically. And that and then in the first two quadrant it will keep on more and more actually, x 1 will keep on increasing. So, obviously we will also have one more conclusion here that not only we have a limit cycle, but the limit cycle rotates in a clockwise direction basically. So, these are like further analysis and things like that actually.

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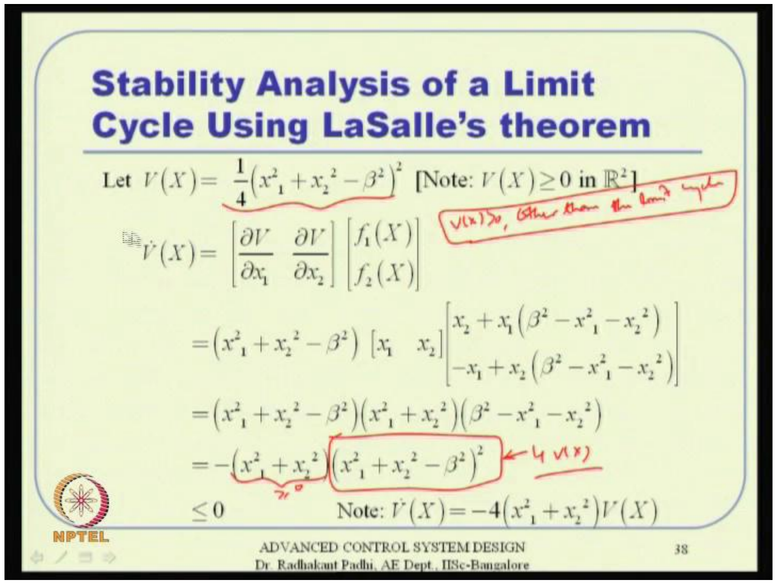

Now we will go back to this Lyapunov function and try to kind of see what is going on there actually. So, all this Lyapunov function that we started with I mean so for you are not started with any Lyapunov function like this, these are simply analyze this expression. Now, let us start with an Lyapunov function I will take the same expression with whole square term actually. So, obviously this happens to be positive semi definite in R square so that is because the LaSalle's theorem does not demand that V of X is positive definite, just needs to be continuously differentiated. So, do not I mean even though we can call that has Lyapunov function something like that it is really not a Lyapunov function, cannot it is just a function can did for which we want to apply LaSalle's theorem actually.

Now, let us saw we analyze this V dot of X and when I mean having said that what it can turns out that the it is convenient to work with Lyapunov function also. You can this particular example we can think of that as a Lyapunov function candidate around the local neighborhood of the limit cycle. If I because this particular expression, if it is non 0 it gives us some sort of a distance measure from the limits cycle. If it is 0 we are of we are moving on the limit cycle. If it is non 0 we are moving away either the either inverse or outwards either way so these actually gives us some sort of a distance measure quantity basically.

So, if I take this particular function is not only continuously differentiable or it is also, some sort of a greater than equal to 0, as long as possible we are not on the limit cycle that is the only condition actually. Probably, you can if you want to be more precise then V of is greater than equal to 0, other than you can complete this statement other than the limit cycle. Now, what I mean is the V of X is strictly greater than 0 as long as it is not on the limit cycle. Beyond the limit cycle it is still equal to 0 which is still true actually.

So that the statement is not wrong in general V of X is better than equal to 0 that is it actually. Or as long as you are not on the limit cycle any direction any other direction you are going then V of X is positive actually. So then talk about V dot, V dot happens to be del V by del X x 1 del V by del x 2 row vector. These are column vector or do the regular algebra what you whatever you are having and remember del V by del x 1 is del V by del this expression whatever expression into this expression del I mean del by del X plus this expression inside.

So that means 1 2 and 4 will cancel to be half and then when it takes two times x 1 that two will also go. So, you will be left out with this expression actually into x 1 of course, the first quantity and this expression is this one into x 2 and then this two expressions are there so you try to put them all together. Remember this is scalar quantity, this is a row vector, this is a column vector. So, it is a 1 by 1 and then this is 1 by 2 and this is 2 by 1 so, ultimately you are working with the scalar quantity. So, if you carry out this algebra and then try to simplify and all it turns out to be like that. So, you need f n should like that then because this is actually nothing but some sort of a 4 V X actually.

This expression what you see here, this particular expression what you see here if you compare to that one this kind of thing is nothing but 4 V of x. One forth in 4 we can go actually. so, what you are telling? With this particular quantities anyway greater than equal to 0 always and this quantity also greater and equal to 0 always so obviously you are with an negative sign of course. So, we will end up with the case where V dot is actually very negative semi definite basically. And this is also like a side comma and that V dot id nothing but minus 4 times this expression times V of X actually. That is some further conversations and remaining which I will not discuss here and here. Anyway so V dot happens to be negative definite I mean negative semi definite basically.

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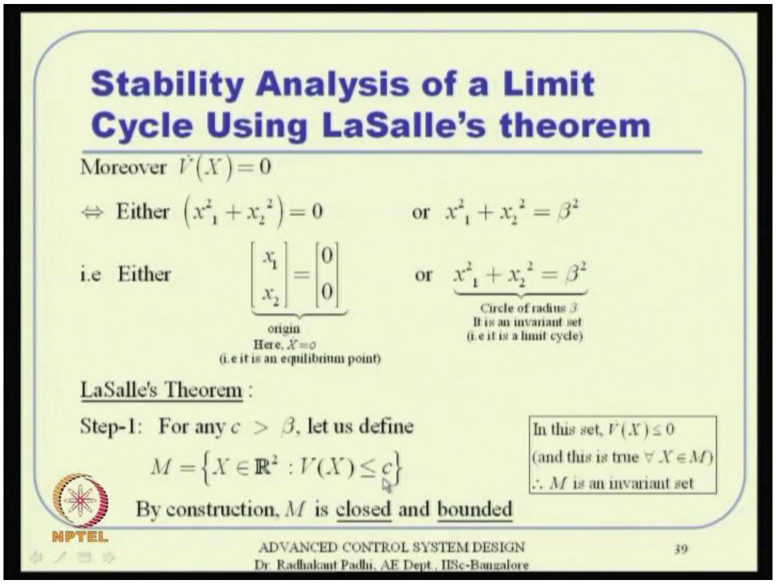

So, now we will study when this V dot happens to be equal to 0. Now, if V dot is equal to 0 remember V dot is this expression anyway so, this V dot happens to be 0 either I mean x 1 square plus x 2 square equal to 0 or x 1 square x 2 square equal to be beta square. I mean this expression clearly tells us that. So, if it happens to be equal to 0 then we will end up with this origin I mean x 1 square plus x 2 square equal to 0 will happen only at only when x 1 x 2 both happens to be 0. That means this condition will give us originate the thing means equilibrium point. And this particular condition will give us a circle which is nothing but limit cycle actually, limit cycle of radius beta basically. Now, it is we are kind of ready to see this meaning of this LaSalle's theorem conditions and all that actually so, first thing first we define say invariant set some sort of M.

So remember go back to that and tell sorry, we define a complex atom which is invariant with respect to the solution basically. So, this is obviously if we define M with some sort of A V of X is less than equal to c then this particular set I mean we dint talk about whether it is invariant or not yet. But let us define it this way and it again tells us that why construction this M is certainly closed and bounded, why it is bounded? Because it is a equal design anyway. So, everything it is a like bounded below b of X happens to be a positive functions anyways and then it is less than equal to certain quantity. So that is rounded, why it is closed? Because it is regular expression first of all, we say continuously differentiable expression with this I mean this equality also taken in to account. So, boundary values are also included basically.

So, M happens to be like closed and bounded and hence it is a complex. That is the first thing that we need to worry about. The M has to be define in such a way that it is closed and bounded so it has to be compact actually. Then we have to see whether is invariant with respect to solution or not actually now, we go back and see. I mean this particular thing what you are define within this set if you start with the solution I mean with start with the solution of the system then obviously it will remain there only. Because V dot happens to be negative semi definite V dot V is whatever V dot is ha happens to be always less than equal to 0. So, V is bounded above then the solution has always satisfy this actually.

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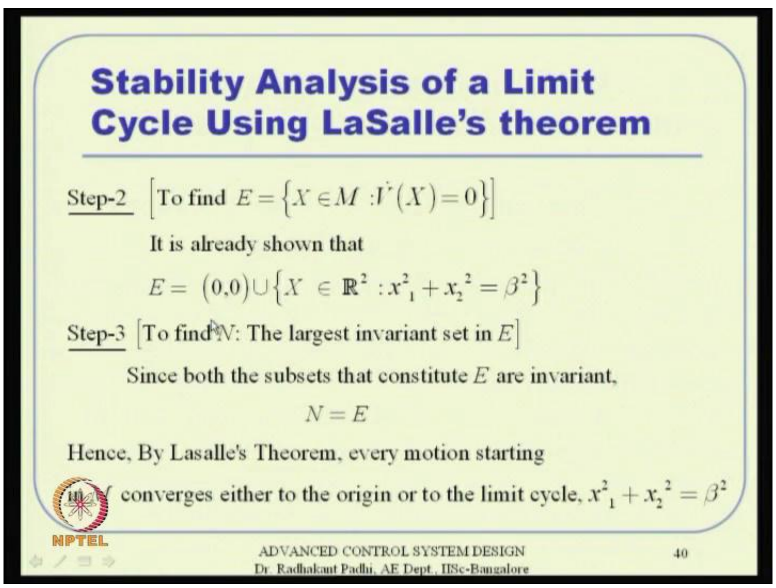

So that means it is invariant with the then we have to find the set E for which V dot is equal to 0 and V dot is equal to 0 happens either on the equilibrium point or on the limit cycle. So, if I take union of this two then obviously tells where this E is nothing but the union of those two actually. Then we have to find the largest invariance set of that and obviously the largest invariance set is certainly E itself. Because in this set this if you trajectory starts with the equilibrium point, it states on the equilibrium point. If we starts within limits cycle it states some limit cycle also basically. And the union of that is actually largest set actually, no other possibility does arise, one possibility is starting with the equilibrium, another possibility starting on the limit cycle.

And if I take union of that is it and obviously that is a largest one possible actually. So, N what you were talking is nothing but equal to E so, LaSalle's theorem if you apply, now what it tells you is every solution thus starts with this set M will ultimately converse to this set E set N basically. That means either it will converse to the limit cycle or it will converse to the equilibrium point that is all it will tells you.

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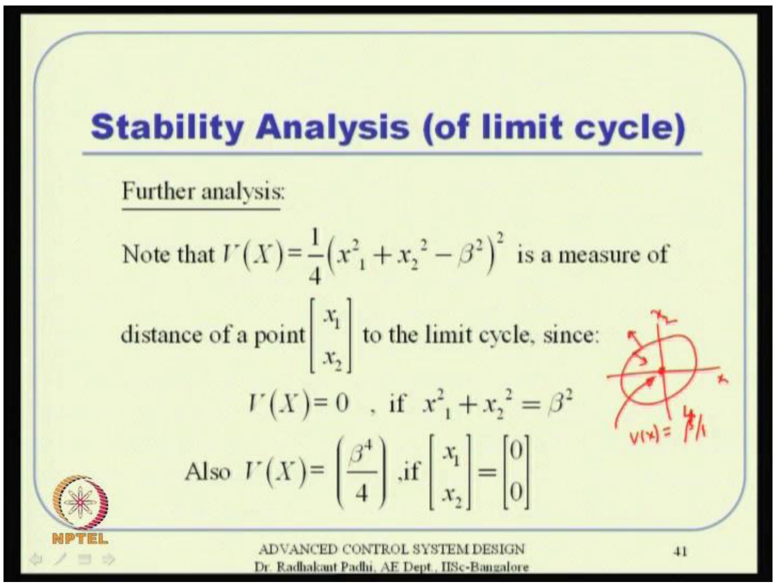

Now, let us do some little bit further analysis we want to wind up this idea of whether it is either or and think like that we do not want that actually. Now, further analysis we will see I mean the we have no that this particular function that we selected is a measure of distance of point x 1 x 2 to the limit cycle basically. Hence V of X is 0 provided this condition holds good actually and if x 1 and x 2 is equal to 0 then V of X takes this value. So, if I just pictorially draw a little bit then what happens here x 1 x 2 is there somewhere. And on the limits cycle Lyapunov function is 0 and as I go along more I mean radially inward or outward, it keeps on increasing. And if I am on the equilibrium point this V of X that I have selected takes this value of beta 4 by 4 actually. On this beta 4 by 4 is on the origin basically.

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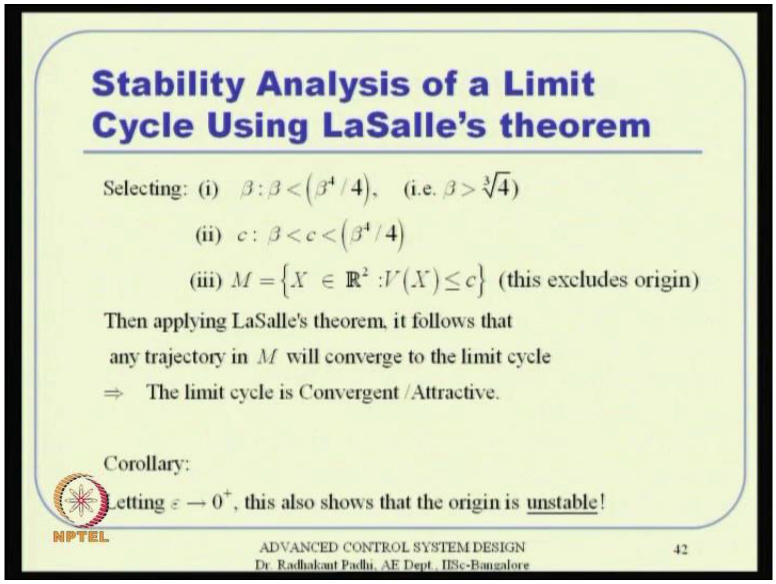

So, let me select a beta such that which satisfies this condition, this beta is less than beta forth by 4 sort of thing. That means I limit my beta to be this is like if I had the meaning then beta has to be greater than that value and all that. Then I define a c which is a in between these values and all that. Then when I tell if I apply this LaSalle's theorem and all that then ultimately, it tells me that it will all trajectory that it start with M will eventually converse to that was that is the meaning of these.

So, if I define a cycle like that why I am doing all that because I a excluding the origin actually. If I exclude the origin anywhere else I start with then my V of X happens to I mean that largest invariant set contains only the limit cycle. Because I am not considering region to begin with where origin is also included I am excluding that. So, if I exclude that then everything will go to the equilibrium point actually. Now, it also tells us that if I just the inside of which is just slightly close to 0 that means I am almost touching equilibrium point what I am away from that then also I will go to the equilibrium, I will go to the limit cycle only.

If I start anyway very close to the if I start wherever I start very close to that ultimately, I go and converse to the limit cycle actually. And not only this side I can start with that side also and then ultimately I will converse there, this both side there is a finite domain for which I can think that the limit cycle is attractive actually. In other words you have already prove using LaSalle's theorem that the limit cycle is I mean this stable actually. And also as corollary if I start very close to equilibrium point then also I am going to at the limit cycle actually. So, what you tells? What you tells is that the origin happens to be unstable actually, origin I mean that also can be proved using this instability theorem and other things what you discussed before.

But as a corollary of this particular analysis it also tells that not only the limit cycle is stable, but the equilibrium point is unstable as well. Both. So that is the thing of you that is the beauty of using LaSalle's theorem. So, if you see example, but the most of the time we use the corollary version of that the small version of the a smaller version where you tell equilibrium point is asymptotically stable or not. But the lossless theorem itself if you want to apply in a generic way, it can be like you can derive further conclusion and it is more general actually.

So, before end before ending of this discussion of Lyapunov theorems and all that and also remember all the thing that we were discussing in this particular course here is Lyapunov theorems as applied to autonomous systems sorry, in this now if you have this X dot equal to f of t X u I mean t X sort of thing actually. So then it is a different wall game all together actually where to talk about slightly different notations and all that where things can be different actually.

So, there we talk about uniform stability and thing like that so, we will not h we will not able to discuss that we have those have advanced concepts. And we probably do not need that also most of the applications we are only with X dot with f of X not f, not necessarily X dot is f of t x basically. So, last topic is domain of a attraction. This as we discussed in the beginning of this lecture every x dot stability stable equilibrium point. By definition is attractive that means if I start sufficiently close to that equilibrium point because the equilibrium point is stable. By definition it will the trajectory toward that. So, obviously there is domain of a attraction around that. Now, the question is we can define it properly and if possible can we estimate that property.

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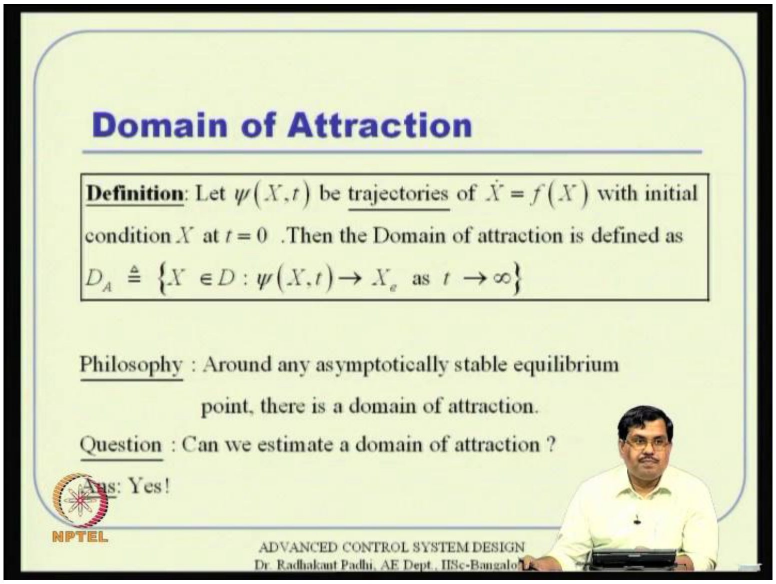

So, domain of attraction definition is like this so let psi of X t be trajectories of X dot is f of X with initial condition X t is equal to 0. Remember you are not talking about a particular X 0, we are talking about that X 0 itself can be varied. So that will get number of trajectory with different different X 0s and all that actually. So, if you have this particular thing then that particular set of trajectory I am defining a psi, then the domain of attraction as you define as X belongs to D such that the psi of X t goes to X c. Because also such trajectories that I start with the different initial conditions and all that everything will go to 0, I mean go to this equilibrium condition, equilibrium point, as t goes to infinity.

Then this D of A that I am defining as subset of this D turns out to be the domain of attraction actually. Remember X dot f of X is valid in a domain D and this D A is a subset of that domain for which every trajectory that starts with this D A will ultimately go to the equilibrium point, I mean that is very intuitively what we discussed in actually here. So, philosophy is as I told around any asymptotical stable equilibrium point there is a domain of attraction. And the question obviously is can be estimate in domain of attraction? Very fortunately uncertain answer to obvious, but also unfortunately the answer turns out to be very conservative. That means the domain of attraction can be larger, but you may not be able to estimate the larger, will be able to use it. As will have to be happy with a smaller bound estimation and all that and largely. Because this Lyapunov theorems are sufficiency conditions again anyway.

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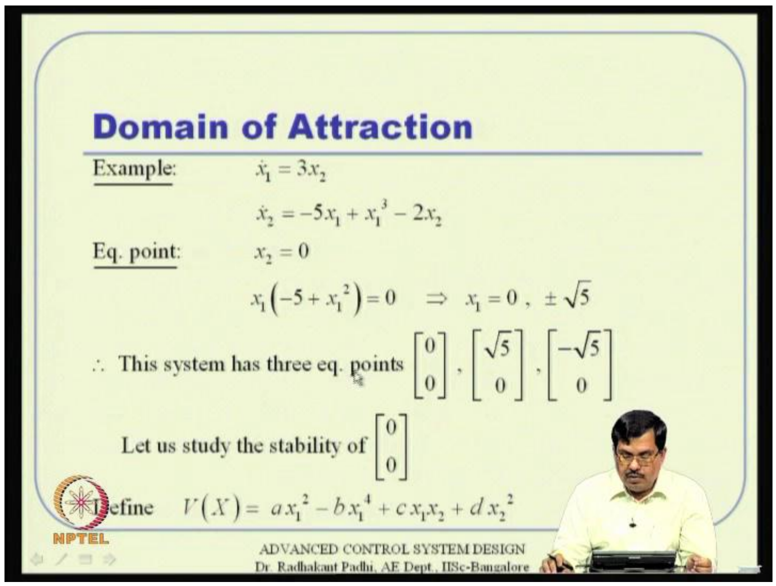

So let us go through a example and try to get our ideas slightly clear here. So, let us talk about again a secondary system x 1 dot is 3 x 2 and x 2 dot is like that minus 5 x and plus x 1 and minus 2 x 2. So equilibrium point again I put them to be 0 x 1 dot x 2 dot is 0 so, obviously x 2 is 0. Once x 2 is 0 x 1 I am left out with this can 5 minus 5 x 1 plus P equal to 0, that means x 1 is either 0 or plus or minus root 5. So, the choice is that I have here is either equilibrium I mean either origin or plus root 5 by plus root 5 and 0 or minus root 5 by 0 and 0. So, we will not worry so much on these two let us see, but we will worry about this particular thing. Let us study the stability behavior around origin. then start with some Lyapunov function candidate and all that. And this Lyapunov function candidate with respect to that let we analyze V dot actually.

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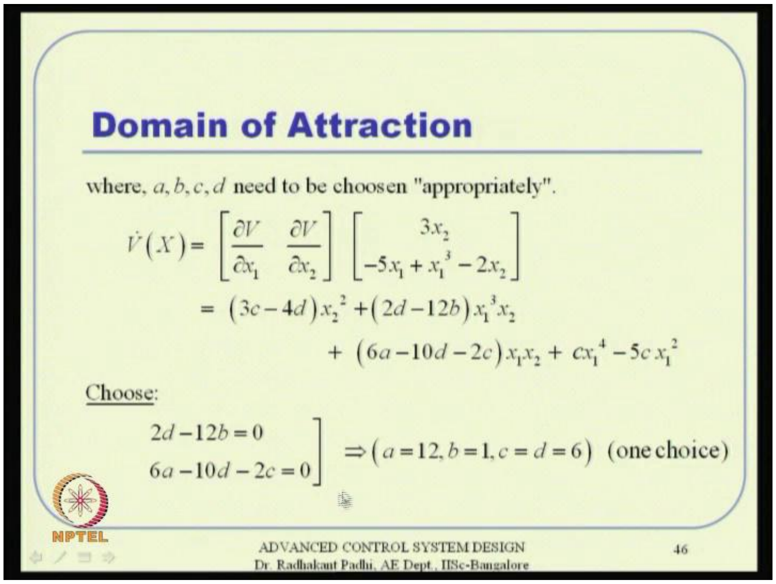

So, V dot happens with again del V by del X del b by del x 2 and all that if you put f 1 f 2 and then analyze this two. And for conveniently because there are several free parameters here that we need to select we need to select it in such a way that let us say these co efficient which are odd co efficient for us to analyze, there are 0 basically. While select this one to be 0 that one to be 0 so, I really wanted to bother about this particular x 1 2 x 2 x 1 x 2 all that actually odd terms to (( )) really so, will be left out with only even powers of x 1 and x 2. So, once you do that you these two conditions again are three variable I mean actually a b c d four variable and two equations so obviously it is under constraints, you have infinite solutions. So, one solution will select it that way so that we can work further actually. So then it turns out that V of X turns out to be like that with that particular selection remember a is 12 b is 100 and all that in this expression.

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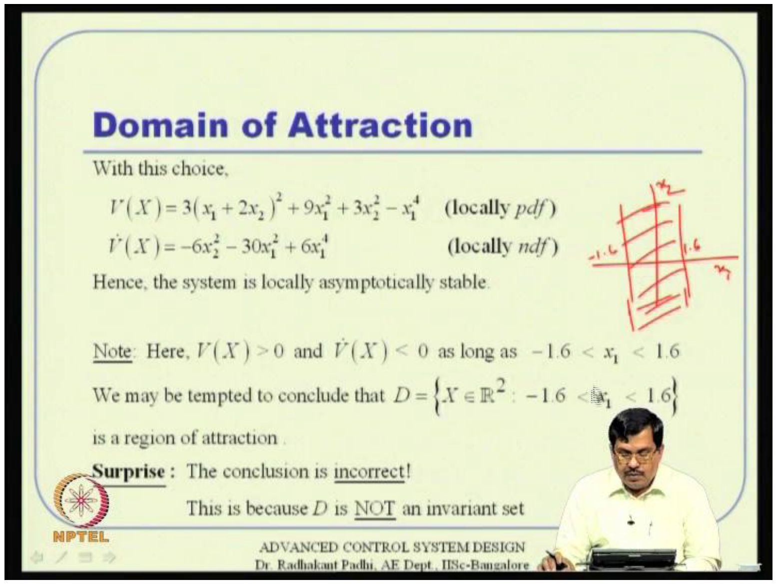

So, if you substitute there then b of X turns out to be like that then V dot turns out to be like that. And if you analyze little bit carefully then remember x 1 forth is less powerful when x 1 goes to 0 close to 0. So, this one will not be very this negative x 1 forth will be very actually so obviously, tells us that it is locally function this is particular thing. And then V dot whatever is this here remember this thing this minus 30 expands there, but plus 6 x 1 4 is again this is less powerful will tell it is locally negative definite. So, this is locally p d f and this is locally n d f and hence the origin is locally stable. Now, the question begins that lets analyze little further and tell when this condition is universally true?

Let say so, V of X is greater than 0 and V dot is less than 0 as long as this x 1 satisfy this condition. Remember it is a because the trouble making things are x 1 only, x 1 is everything is positive, positive, positive here only minus x 1 4. Everything in negative where plus 1 x 1 4 so, trouble making values are only x 1 terms actually. Now, we want to analyze the condition based on x 1 bound let us say and turns out that as long as x 1 is bounded like that, bounded below and above with minus 1.6 and plus 1.6 then V dot is grantee to be less than equal to I mean strictly there is an 0 that means V dot is negative definite. So, we may be tempted to conclude that if I define a reason D, which is like x 1 is bounded like that, that means I define a reason something like this.

This is like x 1 and x 2 so x 1 is bounded between that so it is 1.6 here and minus 1.6 here as in basically minus 1.6 here. So, it tells you I will define this region into this region appears to me that if this particular region infinite strike really, it V dot is negative actually. So, we are very kind of a tempted to conclude that in this region my V dot is because my V dot is negative all the time, this is my domain of attraction. But the conclusion turns out to be incorrect and the correct thing will see actually, why is it incorrect? Because if this domain D what we have discussed I mean what you have defined here this infinite strike that is not invariant set actually. That means a if you start with any solution here I mean any point here, you may not remain within the point I mean this region actually this trajectory remain. And because there are kind of we will see that there are kind of local minimum something like that actually.

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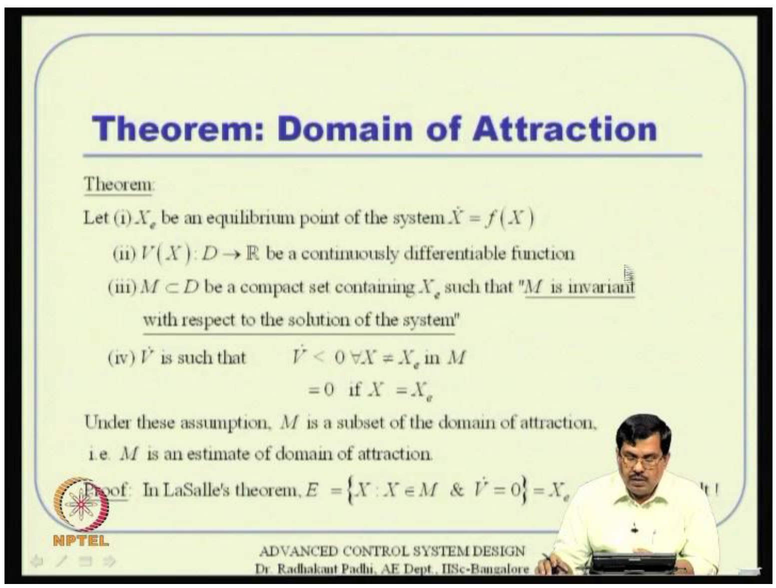

We will see that so, theorem domain of attraction tells you that the X e needs to be an equilibrium point first of all of this form. And V of X D to R needs to be continuously differentiable function then M needs to be complex needs to be X e such that M is invariant with respect to the solution; all these are compatible with theorem basically. And V dot is strictly less than 0 for every point other than equilibrium, but all equilibrium V dot is equal to 0 so, these are like important thing actually. V dot is equal to 0 only on equilibrium point, V dot is otherwise it is less than 0. And this M that we were talking about subset of this b domain compares it, containing X e equilibrium point such that M is invariance set with respect to the solution of the system. That has to be guaranteed sort of thing actually.

And under these assumption and this is a M is the subset of M whatever we are talking about is not the domain of attraction, but M happens to be a subset of the domain of attraction. That means you can that the thing that I was talking about that our estimate can be very conservative. What will be able to estimate is some sort of M? Which is actually a subset of domain of attraction. So in other words domain of attraction can be larger, but will be able to conclude only a subset of that basically. So, LaSalle's I mean proof part of it is rather easy, you can end in lossless theorem you can simply take E equal to that and then the result will come actually. We will continue with that example so, example what you are discussing V and V dot happens to be like that. And V dot happens to be less than 0 as long as this condition is satisfied.

Now, let us examine a little bit further, take V of X like that and V dot is already like that and we already know that happen so, what you do here let us consider x 1 is equal to 1.6 and x 1 is equal to minus 1.6 also later. So, what you are telling here is you are examining this situation on the boundary line what will happen will let us examine and all that. So, on the boundary line it happens to be like x 1 is equal to 1.6 that means it is a function of only x 2 now, x 1 is already evaluated at one at 1.6. So, the V happens to be a function of only X on the boundary.

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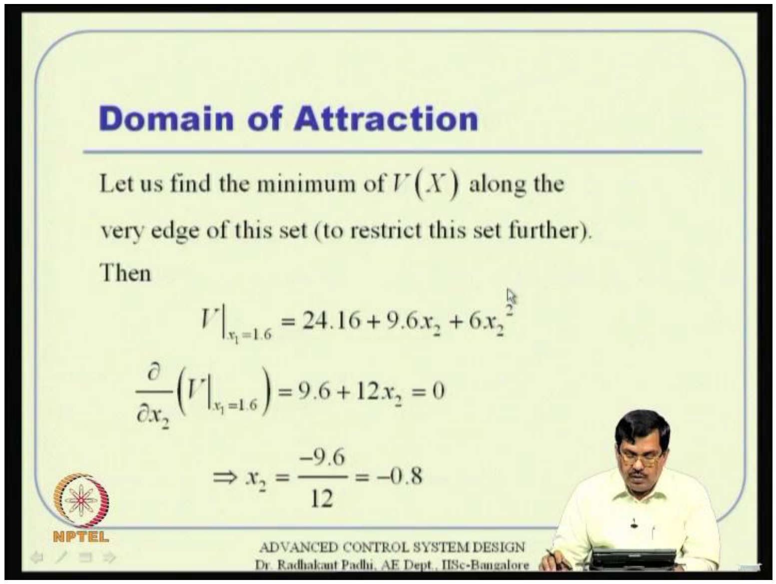

So, what happens on the boundary whether there is a local minimum x 1 and all let us try to analyze actually. So, if you take first order partial derivative this for analyzing local minimum maximum that needs to be equal to 0. And hence x 2 if you evaluate that happens to be minus 1 point minus 0.8 actually.

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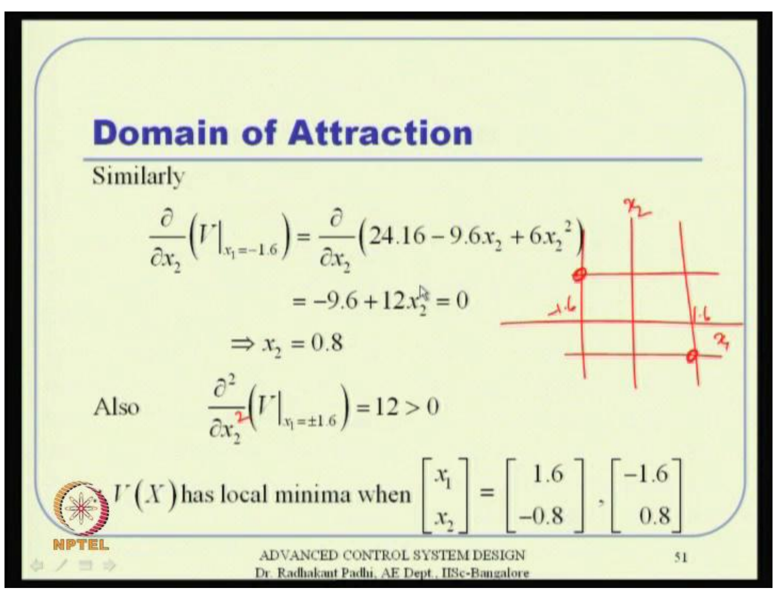

So, and similarly, if you take the other one x 1 is minus 1.6 it is again if you do this same algebra, it turns out that it is plus 0.8 actually. And it turned also turns out that del x 2 sorry, this is this can happen both for both minimum and maximum so, you want to maximum what is really happening. And for that we need to have partial second order derivative sort of thing, but anywhere if you take the second order derivative then whether it is plus 1.6 or minus 1.6 is happens to be plus 12, this is greater than 0. This plus 12 only, here it is plus 12 x and there it is also plus 12 x a constant will go. So, it turns out that at both this points we have double derivative is positive and hence we have got a local minimum actually.

So that there is a thing that we discussed before that this strikes sort of thing and all that this is this type x 1 x 2, this is x 1, this is x 2, this is 1.6, minus 1.6 this strike that you are talking, now we are having some x 2 values also. And if it x 1 is plus 1.6 x 2 happens to be minus 1.6 minus 0.8 actually. So that means you are talking about the point here and we are also talking about a point here. So, one these two points happens to be local minimum actually so, then that is solution can get rough there it will never go to this region actually. If it happens if it is stopped actually it never go to origin actually. Anyway so what it tells us is because this V of X is local minimum on the way and all so let me try to ignore this, let me try to explore those points and all that.

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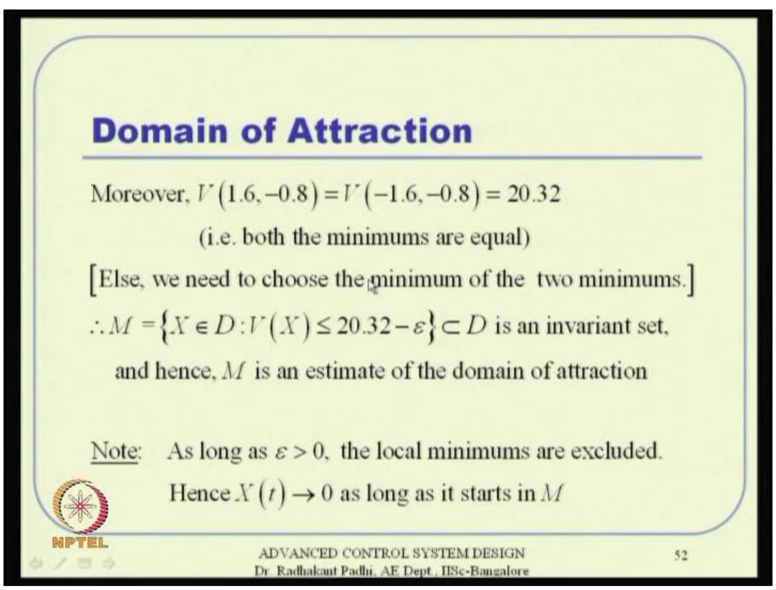

And then I will define some sort of set which will avoid that actually, it will not go towards the local the minimum. So, then I will evaluate this Lyapunov function both at the both of Lyapunov function and both the points happens to be like that so I will simply try to avoid from there actually. So, in other words, if I define a set for which this M, M is a subset of D for which the V of X happens to be just a little bit less than this point this value. Whatever value is there, there is just a little bit less then that minus x 1 and X is small positive quantity. Then it turns out this local minimum values are local minimum point are exploded from this set and one that is a excluded x of t has no other go about to go to 0 actually. That means if I define a set properly like that then it turns out that is an estimation of my domain of attraction here. So, I mean that is all about this conservative results and things like that actually. So, before we stop this lecture there is a another interesting results sometimes it is useful so let us study very quickly that tells you if my V dot happens to satisfy some week is any function of time.

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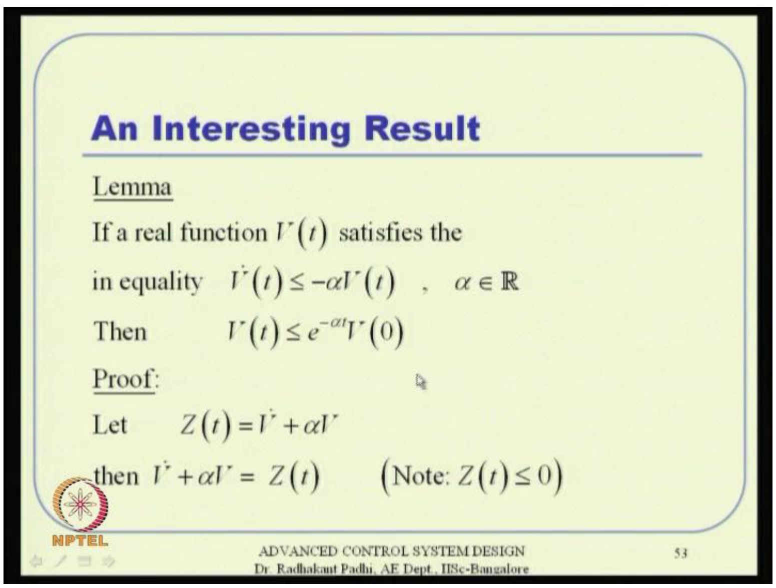

And if it satisfies this in equality then for all time in this inequality also satisfied. Why is that because if I define this Z of P or something like V dot plus alpha V and then consider that as a linear differential equation remember Z of t is less than 0 by construction this is inequality.

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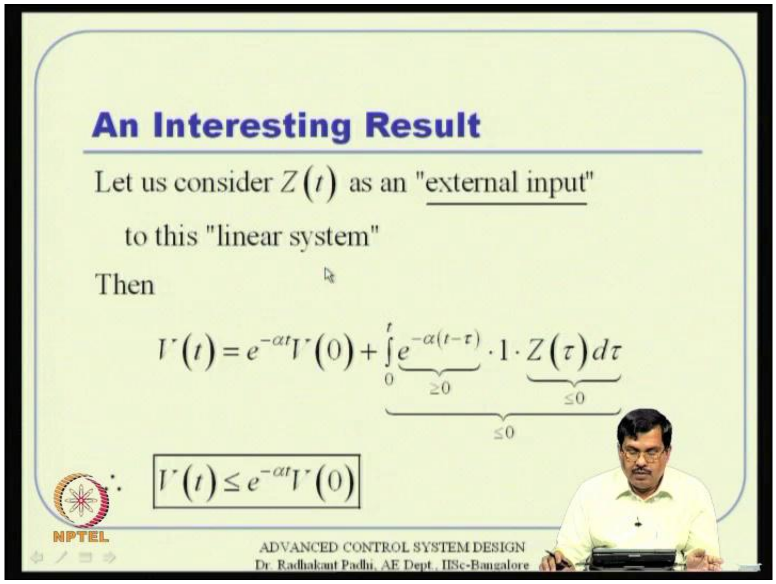

Then if it is a linear system equation I can solve it like a linear system equation. Then it turns out that this integral thing is in this is less than equal to 0 and this is always greater than equal to 0, this is a positive integrals sort of thing so, the entire thing will be less than equal to 0. And hence V of t will be always less than equal to 0. So, if I somehow if you can show that this in equality satisfied then V of t will ultimately go to 0 and that is the point actually. Anyways so that is the interesting thing I thought I share with you, but all other things are important. This references are already there, we discussed about that many other things, I have taken from first and second primarily. We can see some of these references for Lyapunov theory. Further classes we will see the utility utilities of this Lyapunov theory in different, different contest actually. Let me stop here. Thank you.