Advanced Control System Design

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Lecture No. 32
Lyapunov Theory - II

Lyapunov theory concepts and some basic definitions as well as some direct theorems and all that and so one or two examples. Now, continue there we will continue that discussion further and see some of the further concepts in Lyapunov theory. Now, one of things that we saw there in this Lyapunov theory in the previous lecture is I am mean the measure issues being to one is how do you see the term and how do you find the Lyapunov function, which will do the job for you. And second is here is like if I have a negative semi defined what we are do with that, because many times it is nice to see negative definiteness. So, that I can conclude more than what I mean otherwise can conclude.

So, we will try to address some of these issues and then try to see further concepts and think like that. So, first issue is like construction of Lyapunov function so, in some possible in some problems it is possible to construct Lyapunov function in a I mean following some sort of procedure and all that actually. So, that some of these results, we will see here in this particular lecture first.

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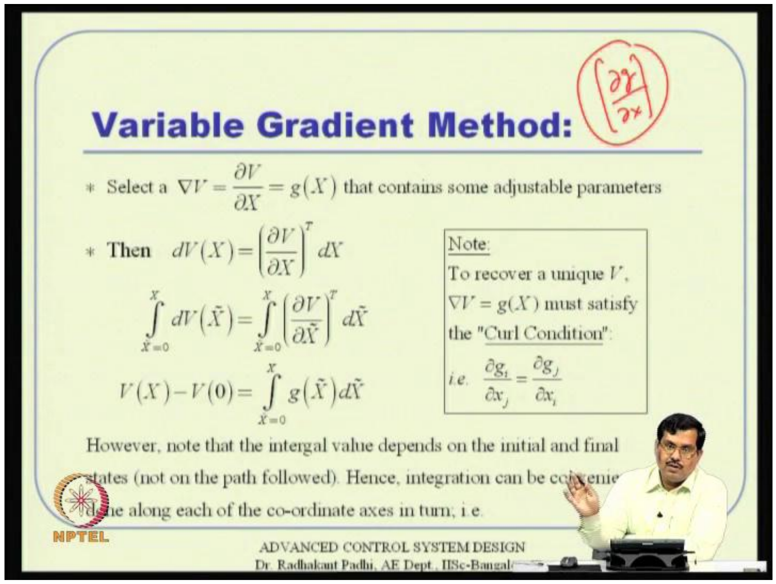

So, one method is something called variable gradient method and that, because see why this important because in Lyapunov function analysis, these are what we saw in the last class. All the theorems are happens, to be like sufficiency condition that means, if I am lucky I will keep on trying and if I am lucky one time then probably I will get the results that I want. But in general there is no guarantee actually So, how many times I will try so instead of doing that is there any standard procedure to see some of these constructions of Lyapunov functions other then the very standard thing that we discuss.

If this X transpose X and all that the quadratic function may not work out always actually, but that is a standard function that we can try out and this kinetic energy plus potential energy that is another function that you can always try out. But other than that suppose, these two also fail I mean these two regular candidates fail then what are the over things that actually that we are already brought actually. So, this is where it lead to this constructions, concepts and all that we the first method which comes to picture is variable gradient method. So, this variable gradient method tells us that let us not worry about selecting a V, but let us select delta V instead that means, we will select some sort of a gradient of V let us talk about that actually, because remember V dot is del V del X transpose into f of X.

So, instead of selecting a V and then working out this del V del X, because del y del X is something that will go to picture. If I see this previous class somewhere, we discussed about that actually, this V dot is del y del X transpose X dot so, del V del X into f of x that is ultimately V dot. So, if I have to conclude something about V dot then why starting with V and then working up this del V del X and all that, because we know f of x already. If you know f of x then I will be smart enough to select these V of X del V del X and see whether my V whatever, V I will compute from there whether this V will satisfy these or not. That is the whole idea actually, instead of starting with V and then carrying out this algebra, I will just see that what my what is my f of f and select this del V del X.

And then, but remember I have selected only del V del X these two one and two condition has to be satisfied that means, I will solve this del V by del X expression and then get V of X and see whether this V of X satisfies that or not actually. So, this kind of a little bit reverse idea sort of thing so, that is what we that is the variable gradient method talks about so, let us select a del V by del X g of X that contains some adjustable parameters. Then it obviously, adjust that depending on our f of x whatever f of x we have actually, then this expression tells me that this del V is nothing, but del V by del X transpose del X, but remember this expression is some sort of a kind of this d X contains many components of d X actually, d X is a vector quantity d V is a scalar quantity.

So, if I really have to solve for V X I have to be doing I mean I have to do this integration slightly carefully actually, but nevertheless if you start with this expression d V is that so integrate both sides from 0 to X. And then it will turn out that this left hand side is V of X minus V of 0 and then most of the time V of 0 is 0 anyway so that is what we will select actually so, sort of thing. So, this expression will give us that what is my V of X actually so, V of X is integral of this gradient, but this integration that you are talking about has to be done component wise actually we will see some procedure and all that further when you go along actually.

And also remember that this del V by del X what you are selecting this g of X it must satisfy so called crone condition actually, that means this if I see this del g bi by del c j it is satisfy this del g j by del x i. So, if I take like g j is a vector having on that so it take any component of that vector and take the partial derivate with respect to any component of state vector. And if I reverse the sequence that has to be said that that necessity fine however what that is means? This del g by del x what you see here this j is already a vector any more that is why if I talk del g by del x that is actually, matrix and this matrix has to be symmetric actually, that is all it that is all you have to be actually.

And also remember there is another thing that this integration this value depends on the initial and final states and not necessarily on the path followed actually, and if that is so and this integration can be done component o actually. That means, I travel along x 1 x s first then I will travel along with stress then I will travel along x 3 like that actually. So, I do not have to directly radial go from 0 to x I mean 0 to x that is way I can go in a component wise x 1 first x 2 next and think like that actually so that is how I will be able to integrate this. So, this is what is a near so V of X when you integrate this what that mean V of X we first integrate along x 1 axis.

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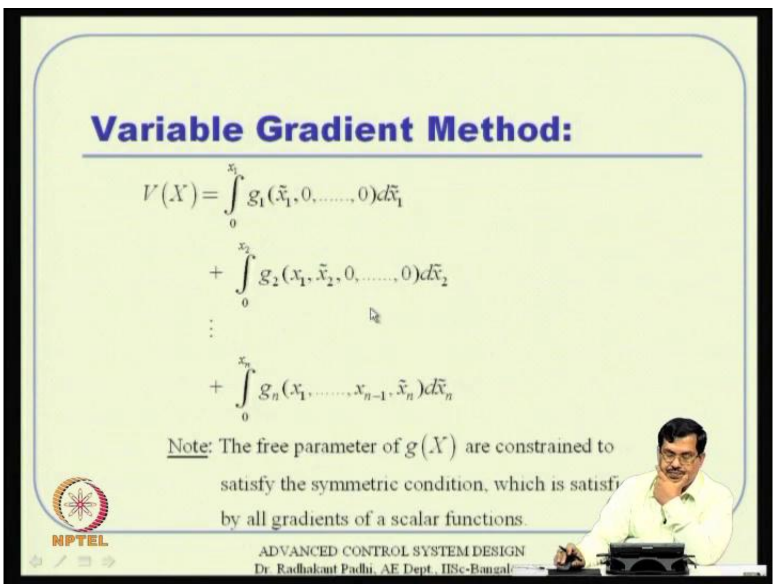

So, all other axis are 0 anyway then whatever, function comes plus where you are other x 1 remains actually, then now where x 2 x 2 will be vary from 0 to x 2 then you vary x 3 x 1 x 2 and x 3 tilde that that is your variable I mean this integration variable and all that. And keep on doing that until the last co ordinate axes actually, that is the we expression what you see here actually. So, with the assumption that we have 0 0 actually V of 0 has to be 0 for that so that is isolative proceed further actually. So, remember I have to start with a gradient of Lyapunov function g of X which will contain adjustable parameters which I have adjust later depending on my situation and this del double x g of X more satisfies some sort of a curl condition.

That means, if I take this del g by del X that concern matrix and that matrix shows to be some matrix actually then only I get a solution was or another wise I will not actually. The procedure to get V of X out of this gradient voltage is like that, if I have already got a gradient vector and which will do my job and all then I will conclude this V of X at way and then I will go back and see whether those condition as it is fall down on that, these first different or notation actually.

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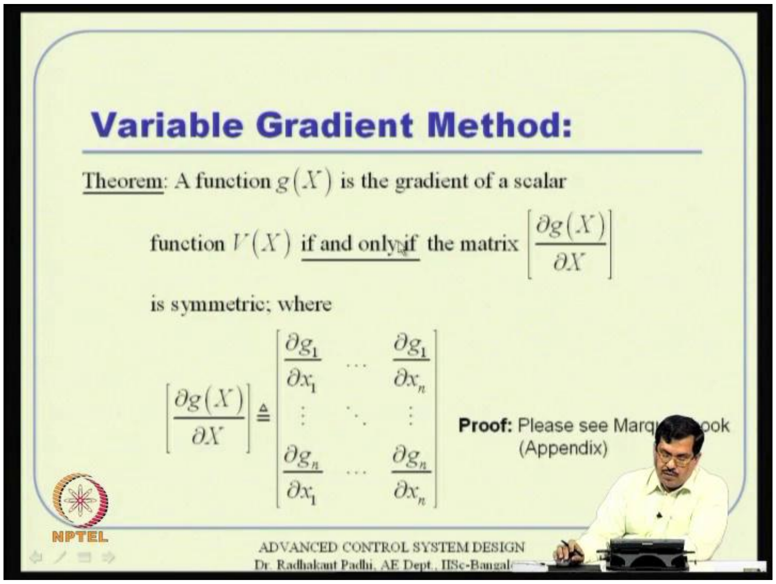

So, let us see what this theorem actually so, the function g of x is the gradient of a scalar function V of X, if an only the matrix will del V by del X is symmetry once I had told you yes know this del V by del X matrix that is the I told you this. In other words this curl condition that we are talking about is nothing, but this condition that del V by del X is symmetry that what I tells you. Now, say if an earlier condition so let see whether that that is true or not actually, in way every quickly this is this very straight forward thing also is a prove is there in office book, but I will also take you through here actually. So, del g by del X when you mean that is what you mean del g by del X del g 1 by del x 1 del g 1 by del X and all that actually so, this matrix needs to be symmetric actually.

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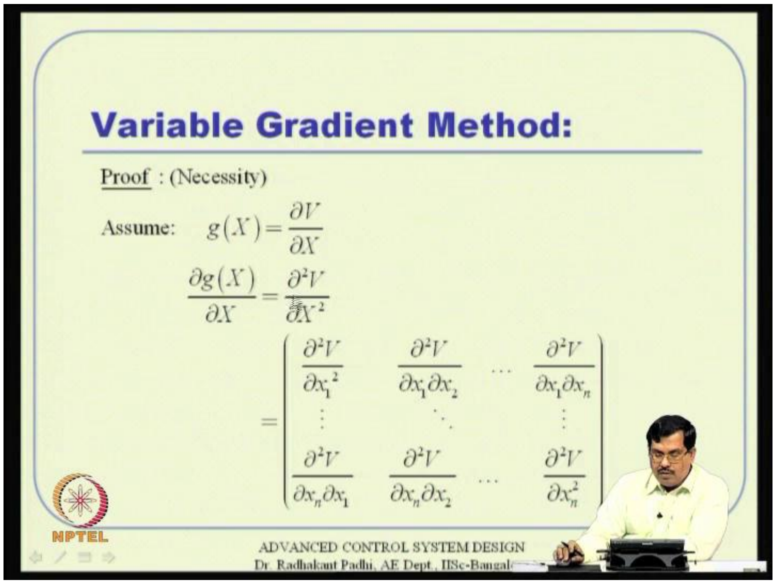

And how do you show that if only if conditions so, first let us see that this is a matrix and all that is very easy to show g of X is del V by del X y definition. So, del g by del X is y definition that and this is by that we simple expand what is going on here del g by del X and all it is nothing, but that where so del square by del V del X square actually, that will happen to be like this obviously these conditions are true.

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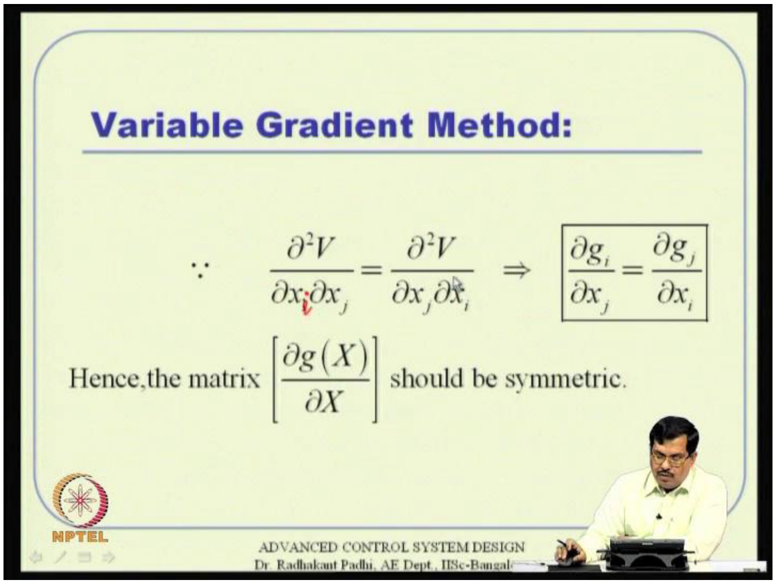

That means, del square V by del x 1 del x j this is actually i del x t say del del square V by del x i del x j is equal to del square V by del x z del x that function that concludes this is that I can take the partial derivate and any sequence or solve it I will say. So, then obviously, the my symmetry condition is it is satisfied actually so what you is symmetric actually. So, if g of X is happens to be gradient function of V of X then this del g by del X is symmetry that is what you saw now, what about the other one if this matrix is symmetry then what will happen actually, that it that is sufficient actually.

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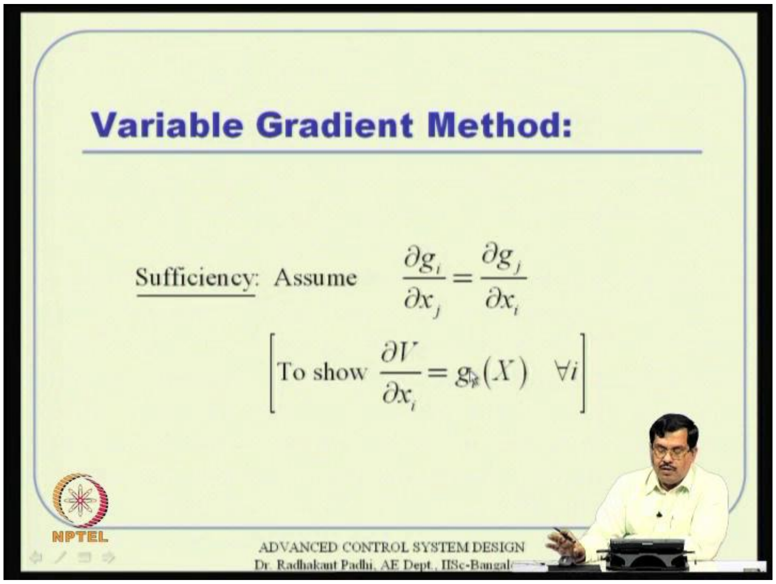

So, you have seen that this matrix is symmetric that is del g i by del x j equal to the del g by del x i. And we need to show that this del V by del x i if I take there is nothing, but those same g of g i g of X which is not in vector actually.

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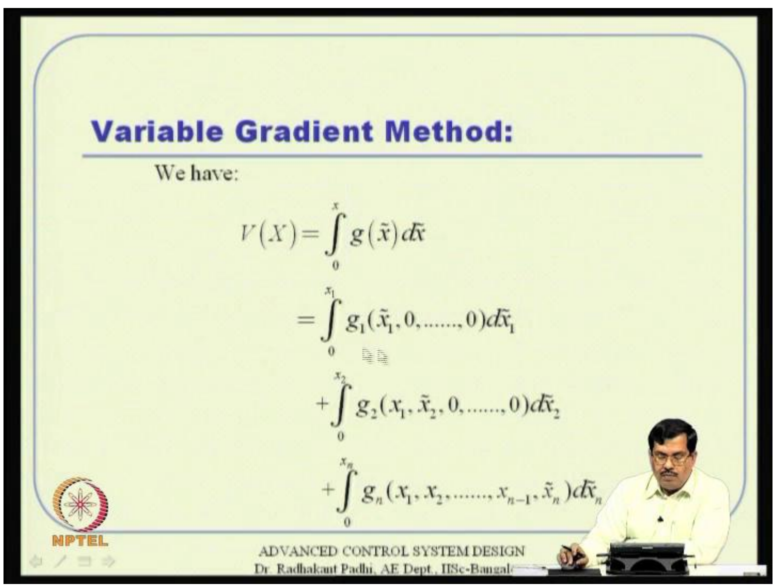

And for that we can say that this V of X initially evaluate that way we just discuss about that and then we talk about let say del g by del x 1 what is that mean actually. This is entire expression this is V of X expression I just told del V by del x 1 left hand side. So, if I take del V by del x 1 the first one is j 1 contains only x 1 by the way this expression it contains only x 1. So, I will give you g 1 g 1 is function one and then another things little the that partial derivate will be pushed inside actually.

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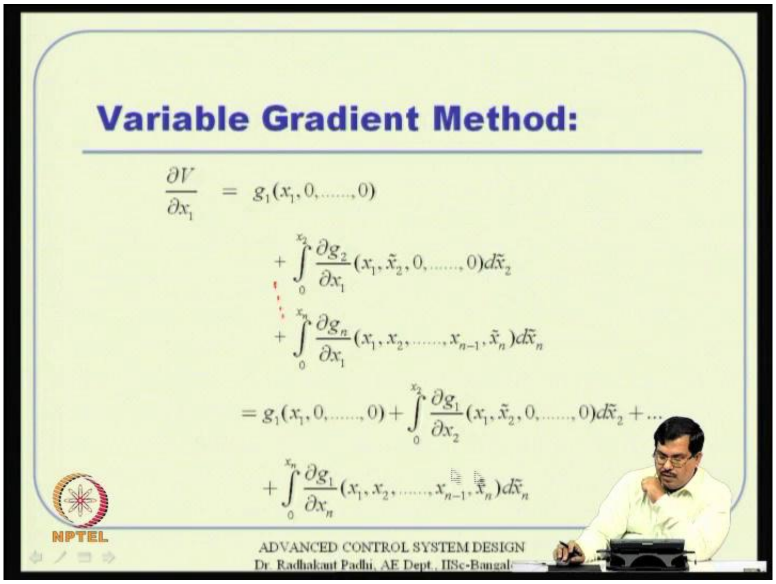

So, I will just take that partial derivate inside del g 2 by del x 1 then 3 V del x 3 like that actually so, it will continue that way and I will be able to show this and as all that partial derivate I will take you that inside actually. So, I got g 1 plus a 1 of integral show later on that then what happens? If you see this integral to be evaluated and all that actually that way then it turns out that first one is g 1 anyway second one is g 1 of x 2 tilde what is that, because if I see this I will invert this condition that this is two. Whenever, I have del g 1 by del g del x 2 I will consider that is del g 2 by del x 1 and that is what I will do here del g 2 by del x 1 is nothing but equal to del g 1 by del x 2 this expression here.

Because symmetry this is already symmetry for that condition I have anymore actually whatever condition I have here. So, if I invert that then evaluate the condition then what will happen this actually del g del g 1 by del x 2 and they integrated over del x 2 and similarly, the last one is del g 1 by del x 1 integrated over del x 1. So, that what we will simple the procedure that is what it is written here so, from this expression and to this expression I have been invert this condition actually.

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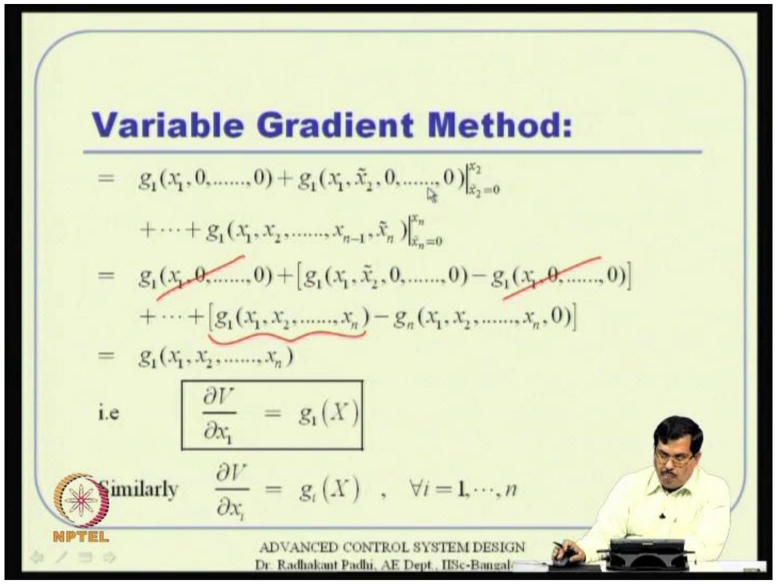

So, when I invert this condition I will be able to integrate these integral values also and that will lead me to some sort of evaluation like that everything will happen in terms of g 1 only. And then what you do this will this value what I see here is nothing but this expression minus this one and evaluate from it here anyway. So, that way so if I see this carefully what will happen this one this g 1 of x 1 and g 1 of x 1 this gets cancelled out actually, this particular thing this one or this one partial negative. Similarly, the other things will also cancelled out and you will be left out with only this follow so, that is what your telling here we are left out with only g 1 that means, del g by del x 1 is nothing g 1 only.

So, similarly, del g by del x 2 will be nothing, but g 2 del V by del x 3 will be nothing but g 3 like that actually. So, del V by del x i is nothing but g i so that is how we will able to show both actually first we showed that, if it is I mean this is the gradient vector then obviously, is I take derivate it is often to be symmetric matrix very straight forward. The other one is if you reference to be symmetry then if I take integral and all that I will able to show that this symmetry is sort in and all that is nothing but the gradient of del gradient vector of V actually and so, that is where this you can condition happen actually.

So, summary part it tells you what actually, a function g of X is the gradient of a scalar function if only if the matrix is symmetry that means, you to select is V of X first that this symmetry del V by del x is symmetry that is all you have to do. And certainly it will lead to Lyapunov function and some sort of thing later basically, so that is what you need to do here actually.

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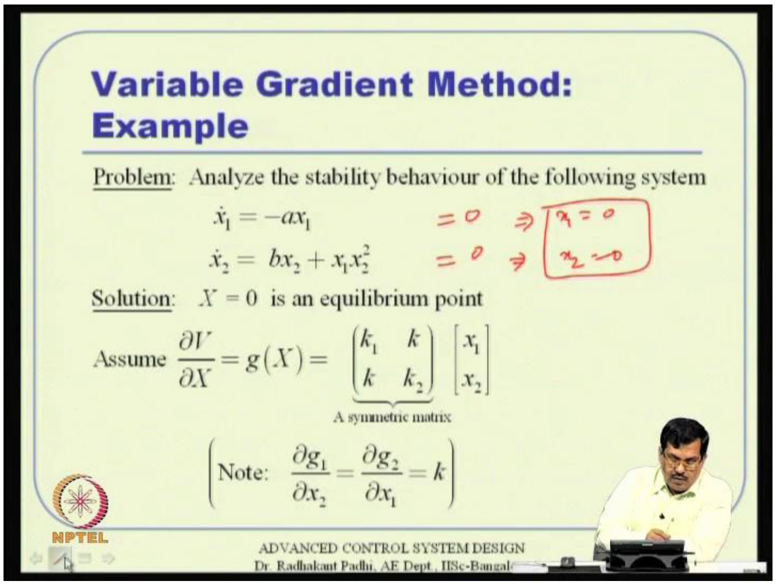

So, let us see the through in example, example is this is one example x 1 dot is like that and x 2 dot is like that. So, shall we x 1 dot x 2 dot if you put equal this is are 0 that means, equilibrium point has to be 0 that is the first condition we have see actually. So, if it is equal to 0 the next one is 0 and this is equal to 0 then this is already 0 this will be x 2 equal to 0 we want to see what is an equilibrium point actually. So, if we want to see what is an equilibrium point we put this 1 0 and this is also 0 and this will give me that x 1 equal to 0 and this will give me, but once x 1 is 0 this term is not there, but these x 2 is 0 actually. So, I got x 1 x 2 equal is 0 I do not have to do any coordinate transformation basically so, x 0 it has to be is a equilibrium point actually.

So, then this so this is now we start with applying this variable gradient method now that means the we start with del V by del X let see g of X and we assume that we have to select this g of X in such way that del g by del X has to be symmetric matrix. So, if I select something like 8 axis says del g by del X is nothing but a basically, symmetric say so if I take this matrix should be symmetry then I satisfy that condition del g by del X is symmetry basically. So, del g by del X is something like this matrix k times x basically where this k is nothing but a symmetry matrix so I start with this k 1 k 2 and k of diagonal so, obviously is symmetric matrix.

So, I have to select this k 1 k 2 and k in such way that I will able to use some of the Lyapunov theorems actually my for V of X it will satisfy the first two condition actually. So, we started with del V by del X so del V by del X if I see this this is del V by del X is k 1 times x 1 plus k times x 2 and the second component is k times x 1 plus k 2 x 2 that is what we start with actually.

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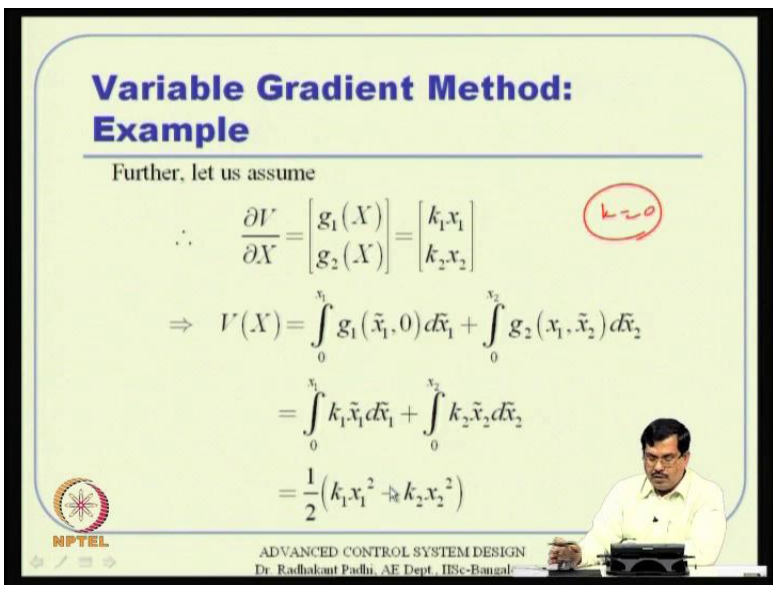

So, what it what I have taken is a this one what I simplified is I have taken k for 0 here already. So, if I take k equal to 0 I will invert selected whatever, I want actually so, I select this not only symmetric matrix, but simple normal matrix actually is also symmetric anyway. So, k is a there and with that del V by del X is like that so if I integrate this V of X I mean integrate this del V by del X in the way that we discuss before that means, first 0 to x 1 this integral then 0 to x 2 along x 2 path keeping x 1 as if variable basically. So, then this g

1 is k 1 x 1 so that I have to integrate and g 2 is nothing but k 2 x 2 I have to integrate that one actually.

So, if it is a I mean the then if you very clear that this is like that actually so, as long as I select k 1 and k 2 as positive constrains then I have and then, because V a V of X is partial derivate actually. So, k 1 k 2 positive and then V of X is positive in all X is not equal to 0 and V of 0 is also 0 that is another issue that we have the we have learn up with expression for, which if I have proven this is x equal to 0. Then I should get V of X is equal to 0 also and this is this one also satisfy this expression also satisfies that condition corrective expression essentially and what is really got is corrective expression actually, which is satisfy all the conditions anyway basically.

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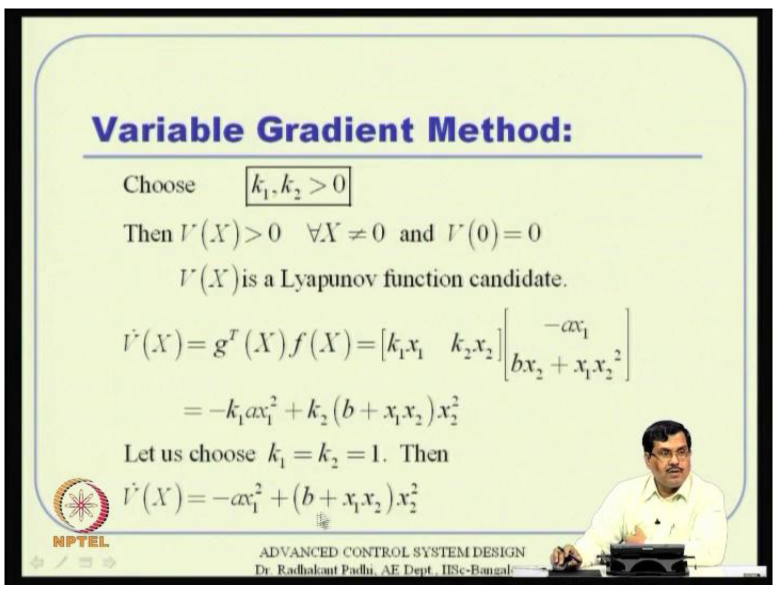

So, this k 1 k 2 positive and then V of X is positive or let us not equal to 0 so V of 0 is equal to 0 and certainty V of X Lyapunov function V of X actually. So, then if we doubt we have do not have to show that, because V of X we still have to cover work so, I mean the word we selected here it does not give an impression, but the way we need to select in any practical problem is we first have an I m this whatever, system dynamics we have and based on that we select this, which will also satisfy this symmetric condition and all that actually.

So, we will worry about V dot now V dot is that so we put that k 1 k 2 and then analyze so we will end up with some expression like that. So, nothing can be said more about that until unless we know some the k all that actually, but V dot ultimately used to negative definite function you know that. Now, we choose let take an one k 2 one here then we will end up is some expression like that still we cannot say more than that until you do some further analysis actually and what is there further analysis this expression what you have V dot is nothing but I can write it that way the same expression what I get. I take negative sign here and then both will be negative and think like that and then I will be able to write it that way actually.

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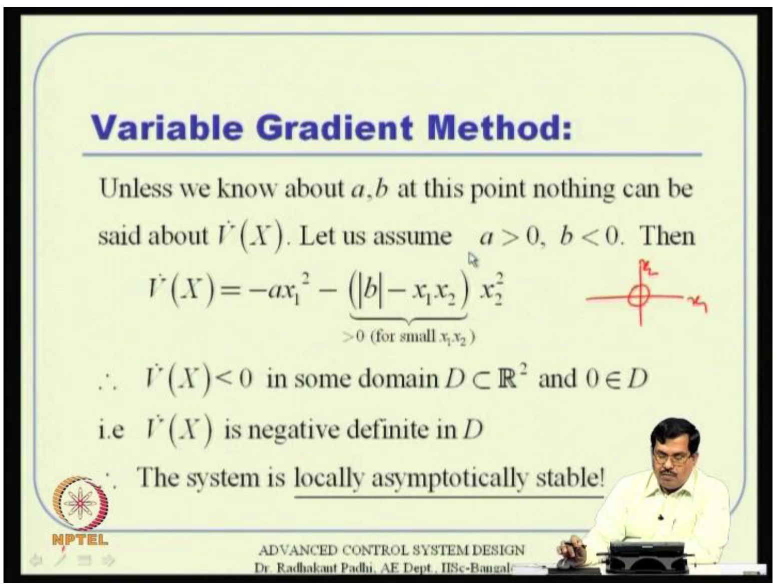

So, what that is mean actually here and this expression further assumes that this condition is true this condition is not we cannot write it that way. So, with the assumption that my original way equation what I started we satisfy this a b a b also I mean this constant a b satisfy this condition a is little and 0 and b is negative a is first b is negative. Then I will be able to show something like that and what was it tell you now, with any expression like this that means for small x 1 x 2 this expression going to positive actually.

Then this suppose, to be no more problem actually so I have got some domain around this x 1 x 2, if I say x 1 x 2 and this is equilibrium some small domain is there around 0 for which this b is larger and that one actually. If I take x 1 x 2 combination what multiple x 1 and x 2 and then multiple and whatever, that is there then this and this modulation b what I am talking about here is greater than that value. So, obviously, this expression we seen is positive one actually and once o is positive value negative of a is negative and this is also positive that mean negative that is negative.

So, obviously V dot is negative definite actually so that means V dot is less than 0 in same domain D and it is a and that domain what you are talking whatever, domain also contains a equilibrium point actually that is very more important one actually. But this domain what you are talking here domain D what a from this expression whatever, you are concluding this domain encircle the equilibrium point actually if does not encircle does not contain that then also you will fail actually. So, also remember some other things actually, but V dot of X is negative definite in D and hence the system is locally asymptotically stable.

So, this X this example is a small example of trial, but using this example who started this some condition like that over this matrix is symmetric. And all that some sort of that that they all the way we seeing this diagonal matrix and on the way you are seeing this is k 1 k 2 identity that means, we simple started with as been I am telling that we simple started with del V by del X equal to x 1 x 2 I mean this this shown in 27 component that is why we started with. Then there are with this algebra and then they look we have phases like that and then satisfy the near property V of X 1, but V dot of X when you want to conclude something over to further assumptions and all where you are seen k 1 k 2 one and then also seem that a positive and b negative.

Under those assumptions it turns the V dot is negative in some domain D which controls origin and hence in that domain at least the system is V dot is negative and hence, if the system is locally asymptotically stable. And also there is a another concept that we are going to discuss in next class or may to be this class itself or something. V dot is negative in the in the domain D does not mean that you are trajectory starting with it is domain any point in domain D will ultimately go to equilibrium that is not there. All that it tells you V dot is negative definite in domain D I mean does not necessarily V that if you start with any point in domain D will ultimately go to equilibrium point.

So, in other words this reason d that you are talking about is not necessarily the domain of attraction so, the domain of attraction concept which will give substrate of D. So, we will talk about that later in this class or probably in next class or something I mean probably in this class itself and think like that we will see that. But they actually domain of attraction that means, if you start with that domain we will ultimately go to the equilibrium and all is some concept called domain attraction that is going to be some sort of substrate of this D. And then it will invoke class 0 and all that is it is substantial heard of two to conclude that domain actually.

So, but remember that the property is that we are talking about is in some domain D, which will tell you that the system is locally asymptotically stable that is a later tell you, but it does not mean that the entire domain D the system trajectory is attractive nothing like that this was that let us remember that actually.

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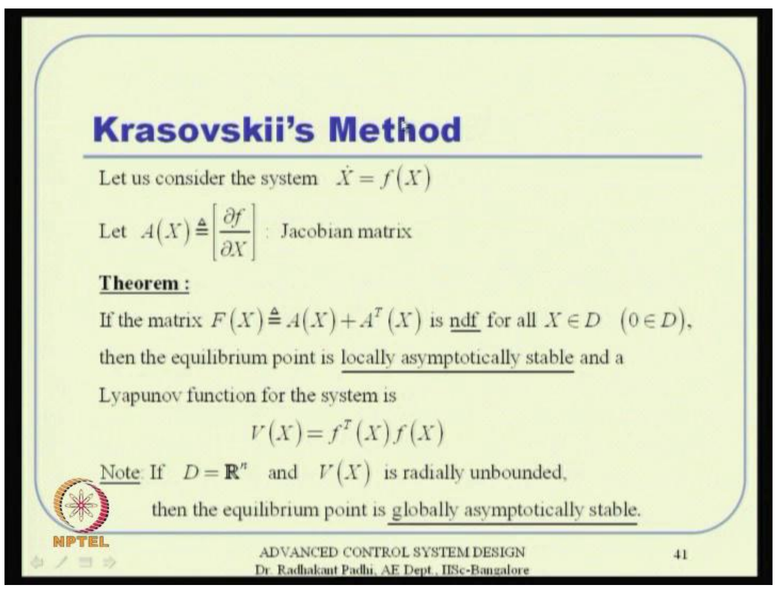

Now, that is all about the this variable gradient method what about them as this is something call karsovskii's method as very straight forward in fact actually will not variable that proof sort of it here and all that well let see if just want to understand this actually. And this is like let us assume the system same X dot equal to f of X and we conclude constrain this a of X matrix the regular way a of X is del f by del X which is nothing but Jacobean matrix. We construct the same kind of a linearization concept sort of something, but you are not really doing any linearization we are not linearizing the system dynamics. Where simple evaluating the expression del f by del X we are not evaluating it around the equilibrium point nothing actually. So, we will end up with V of X which is actually joco matrix which is nothing but Jacobean matrix.

And this theorem tell us that if we construct this A of X and then you construct this F of X which is another matrix which is nothing but A plus A transpose and if this F of X matrix turns out to be negative definite matrix for all X and D. Then the equilibrium point is locally asymptotically stable and as for the condition I mean set is actually which is concept over theorem actually, but it also gives you a construction of Lyapunov function. Because this associated Lyapunov function turns out to be like that this V of X is actually f transfers f over f is the original system dynamics matrix and system dynamics vector the two function. So, if I start with this f transpose instead of f transpose X they this gives us a candidate f transpose f actually f is vector you know that.

But the f transfer f is easily Lyapunov function candidate provided this f of X which his nothing but A plus A transpose is a negative definite matrix that is the only condition. So, given any initial dynamics I cannot jump into that I cannot tell k transpose f is only Lyapunov function candidate this is where this is can date provided this follow is actually an negative definite matrix function and that is where it is difficult actually. Because if it happens to be let say if you if it happens to be only some sort of a linear function of f is linear function and all then A of X will be some sort of a constant matrix.

Then constant matrix I will be able to tell negative definite for all X and all that the movement it happens to be a function of X then I have to very carefully in telling that whether it really happens to all X in the domain no actually that way. And that is where it as it difficult, but then this I mean this theorem is very handy also anyway. Now, so what it tell it tells me that I evaluate jacobian matrix and then I construct this A plus A transpose and this matrix function what I am talking this if f of X as to be a negative definite function for all X which belongs to del l actually.

Then the equilibrium point is certainly going to be locally asymptotically stable and the corresponding Lyapunov function is f transpose f actually, proof is there I mean I will not going to discuss proof of you can see some of the text books to see the proof actually. And the same text book that, I mentioned a in my previous lecture that they will also contain that actually. So, if del A now if D happens to be R n that means A the entire R n space and V of X happens to be radial unbounded you know there is no grantee, because f of f transfer f we are talking actually.

So, we are not designing from the we are not selecting it or sense actually that way so, whether there is no guarantee that we truly radially unbounded, but if it is happens to be radial unbounded the D happens to be the entire R n space. Then the equilibrium point will be globally asymptotically stable so, this is a karsovskii's method and it is very powerful thing in the is a direct way of evaluating all that actually.

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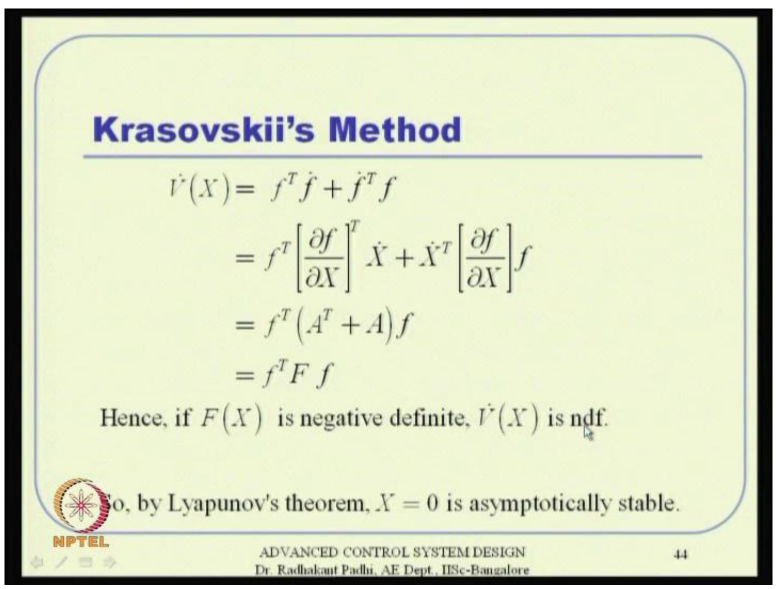

So, let us see this why it so on that actually why this a transfers A comes into picture it is very still very straight forward if my V of X is a like this f transfer f then my V dot is turns out to be f transfer f dot plus f dot transfers f. So, f dot is nothing but del f a del X into x dot this is f dot f dot is del f by del X into f dot arrived f is a function of X only so, f dot there is nothing but del f by del X into X dot. So, that is what I do here and similarly, thing I do here in reverse transpose and all that then I will end up with this when I put this X dot is nothing but f of X dot is f of X.

So, I substitute that X dot is f and X transpose X dot transfers f transfers anyway then I will end up with something like this so, my if I start with V is something like f transfers f I will end up with V dot something like f transfer f big F turns small f. And while big F is find is transpose A transpose plus A so, V prefix the big of f big F is negative definite then obviously, this V dot is also negative definite you know this expression tells me like that actually this a expression in terms of f actually.

So, if we and if big F is negative definite not three function then V dot is ultimately a negative definite scalar function. So, what is it I am so, we want to show an negative definite minus of scalar function, but we are taking the help of negative definite of it of a matrix function. So, that is actually complexity increases increase your complexity in way, but any way is was the theorem it tells you if I start with like this or one of you some expression like that and hence, if my n this f of X is negative definite then obviously, V dot is also negative define and jump down actually.

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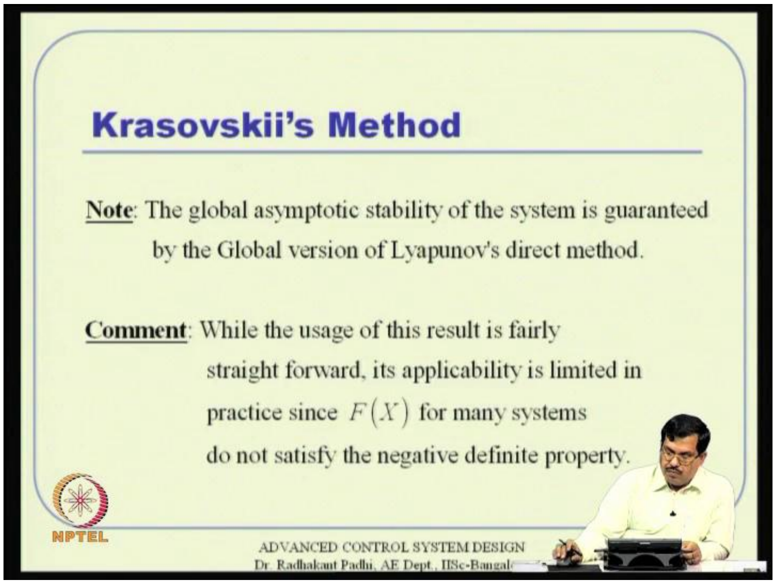

So, obviously, you think not here is the global asymptotically stability of the system is guarantee by the global version of the Lyapunov direct method actually. While they usage this result is fairly straight forward means, applicability is limited in fact is since this big F of X for many systems do not satisfy the negative definite property that is why I told in the beginning also. This is actually a nice thing to you see like, but this is complexity amplification actually, like in otherwise you want to show in negative definite like a scalar function, but you are taking help of negative definite phase of matrix function which is actually amplification of complexity actually.

But it is so happens, that you are F of X contains only small expression of X and all that then maybe we will be able to do that actually output.

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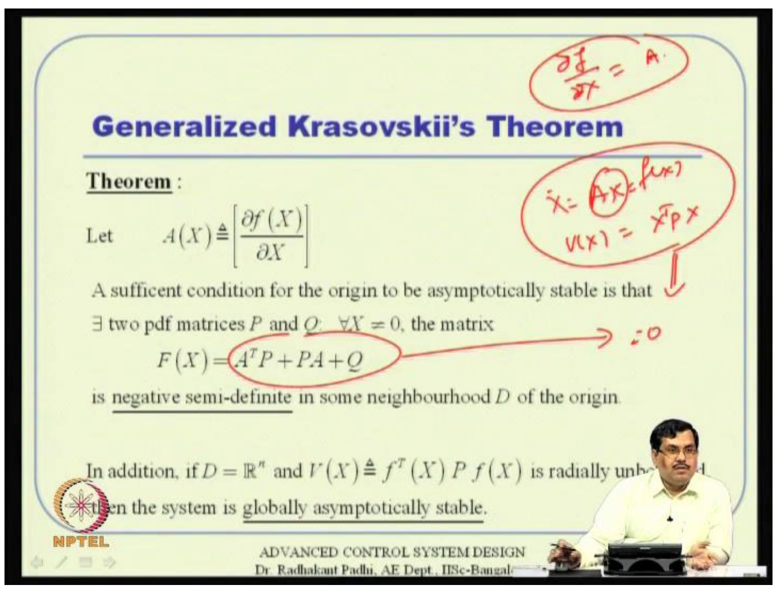

Now, there is a generalized version of that this karsovskii's theorem, which tells us the like some sort of Lyapunov equation expression sort of thing it will happen here. Let see that actually what it tells me is let me evaluate this A of X del of A del X anyway and one more sufficiency condition tells me, that the origin is asymptotically stable. If there is adjust this two positive definite matrix as P and Q such that for all X that is not equal to 0 the matrix this expression you know that in Lyapunov equation this is equal to 0, but in the del X it is not equal to 0 you know.

So, A of X A is a is an is a matrix function now, but their adjust two pdf matrices P and Q such that this expression is negative semi definite. Then in that neighborhood D of the origin then the system is asymptotically stable and if they in addition this D is equal to R n and this V of X is radially unbounded and think like that then obviously it will lead to globally asymptotically stable condition actually. So, what is that mean actually say I will I want able to relax this negative definite condition what I am demanding here to negative semi definite condition that is the more important things. So, and the as you know as of this negative definite semi definiteness is lot of asymptotic actually.

So, all that it tell us is there as to be here it tell they told what actually it simple was very straight forward you just talk about evaluating this A transfer A and if it happens well and good otherwise no actually. But here it tells it is second I can select this two positive definite matrices P and Q such that this expression, which as a additional component Q now and additional P k is multiplied and all that see the originality A transfers plus A that is all. Now, you have A transfers P plus P A plus Q where P and Q or suppose to be selected by as it is so, it gives us lot of flexibility. And it tells f of X we did not be symmetry need not be negative definite, but in x 2 only for negative semi definite was another simplification actually.

So, that is that what the krasovskii's general theorem and all that and this V of X earlier this V of X was f transfer f now, this V of X will be f transfer e times f of X when you if you tell that of this. Now, this is actually nice in way because if you think that linear time invariant system and all X dot is equal to A X and all that. And then you tell if this P is kind of let us identity to matrix and all so, V of X is nothing but X transfer X. But if it is P is positive definite then it will take the X transfer P X and that is what we did in many our time invariant systems in a last class actually.

So, then it resulted in a Lyapunov equation which is nothing but this expression equal to 0 whatever, the Lyapunov equation this particular expression that we are talking here that happened to be equal to 0 for LTA system actually. So, all this are kind of extension some sort of that what you already know in way actually and why it is so, because if I talk about this linear time invariant system this is my f of X this is my f of X this a times X. So, what is my del f by del X? Del f by del f by del X is simple a, I will end up with simple a basically so, all this a concepts are kind of nicely explained in that side actually.

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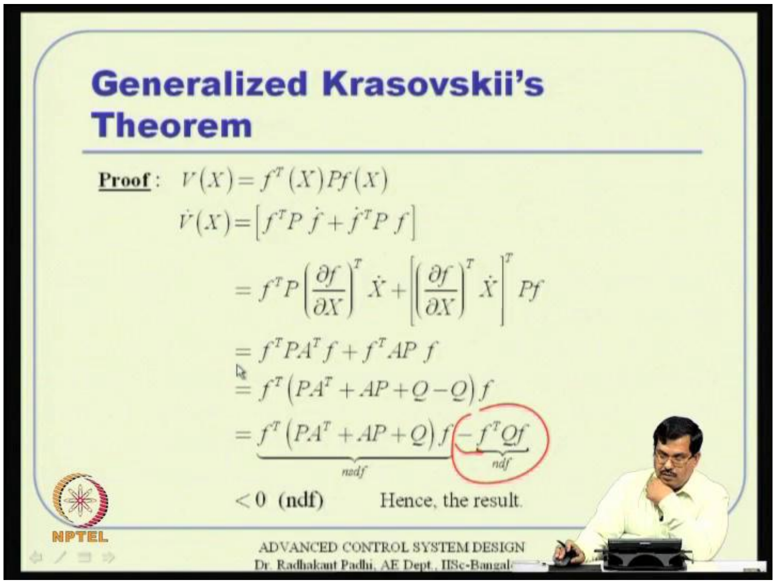

So, what is the proof part it like if I take V of X is equal to let say f transfer P f then V dot is we will follow through this algebra which was provide anyway. So, then it before, but here this time that P matrix will come into picture here and then f dot is substitute and again this is like f dot is a f actually so, will learn of it this kind of a thing. Now, you variants subtract Q matrix and we are just that doing additional algebra here so, interpret that is one component is coming from that and one component is that. Now, if Q happens to be negative definite then it will be the expression is again negative definite anyway and hence and all that you are demanding for V of f is what this expression as to be negative semi definite.

By definite what you are demanding there are just two positive definite matrices P and Q. So, if my Q is already positive definite that means, this expression not this expression I give this entire expression the starting from negatives and negative sign including this this entire expression. If Q is positive definite then minus f transfers Q f is negative definite and if so, that all that it means I have to make sure that this entire expression what I see in the left hand side not left hand side was left part of it this part has to be negative semi definite. So, that is what it tells you that this expression as to be simple negative semi definite that is why it tells actually.

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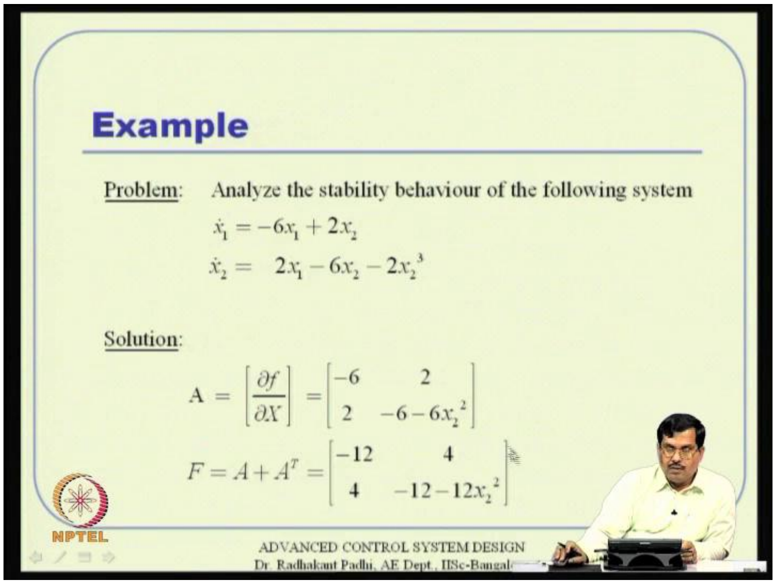

Another example you analyze this stability behavior of the following system again it is small example in today means, you remember. So, del f by del X reference to be like this however, this is no more simple number basically, if I have a linear system with limit with number actually. But if it is non-linear system I have an expression here and sense this f is A plus A transfers contains an expression basically. So, I have to make sure that analyze this F and control something over that so, I carry out with the Eigen value of f I will be able to do that, but Eigen value it is itself is a function of x 2 this matrix contains an expression of x 2 so, Eigen value will also be function of x 2 basically.

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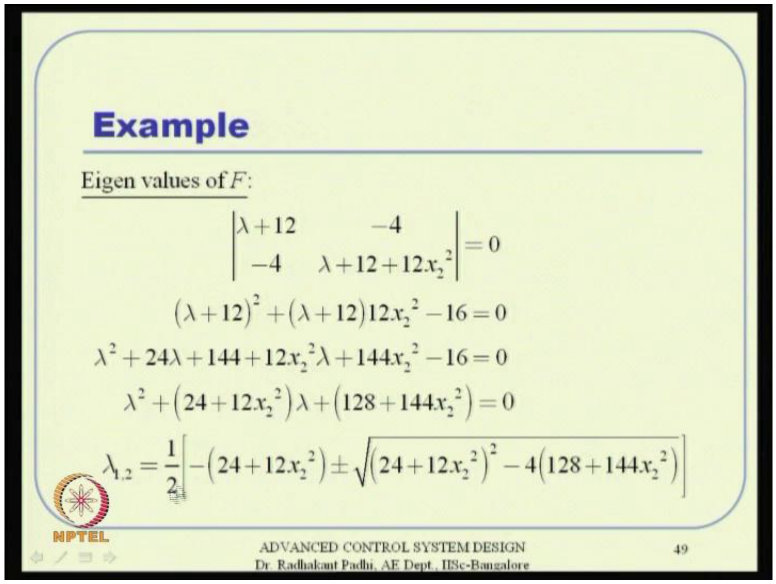

So, I will be let us put that Eigen value expression I mean this a characteristic equations and then try to analyze this so, this will lambda plus 12 whole square plus lambda plus 12 into that minus this 4 into 4 is 16 actually supported there. And then I say then spend that and then collect the coefficient and think like that so, lambda 1 to happens to be like this actually so these are functions of x 2 basically.

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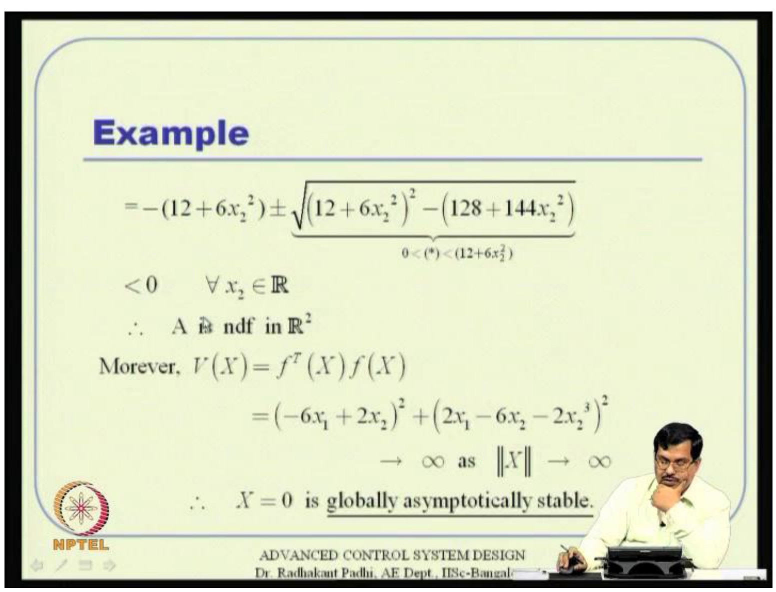

So, we will analyze this expression now seem try to simplified to cancelled out and think like that whatever, is for whatever simplification is possible here. Then you tell wait a second this expression that I have here inside is certainly going to be in between 0 and this expression 12 plus x x 2 square and because this is something and I am actually taking out something else. But no matter whatever, this expression this expression is going to be bigger than that so, if this expression is going to be bigger than that this entire expression is bounded between 0 and that value and hence, what I am telling this is plus or minus anyway.

So, this when I talk about lambda 1 2, lambda 1 2 is certainly less than equal to 0 for all x 2 n R basically, but this is the expression whatever is contained. See if it is negative is further reduction if it is positive then there was chance actually after getting a if it make it positive and all that, but if it is positive this expression is a is never going to super power this expression, because this expression is less than this and this follow anyway. So, is less than that, but uniform either quantity then also I have I will be learnable learning of with some negative quantity only so, this lambda 1 2 is the guarantee to be some sort of negative normal basically so, that means a is negative definite in R square actually. You know what whatever, the discussion and the quantity and all that this a will going to be a is going to be negative definite actually.

So, this basically why this b of x that means I am done with that this it turns out that on using this karsovskii’s theorem and all that my f which is A plus A transfers turn out to be negative define. And hence, Lyapunov function is given as f transfers f, which I can evaluate and this f transfers f satisfies all the condition that f transfers f if I evaluate at X equal to 0 that means x 1 and x 2 both equal to 0 this b of x is 0 if they are non 0 this is actually. Now, I am mean this is like positive quantity this is some expression whole square plus this some expression whole square so, this is guarantee to be positive number so, this expression is positive definite. And this is also radially unbounded that means, if I increase this negative I mean this norm if I go more and more away from away 0 origin then this expression is going to be more and more it goes to infinity actually.

So, was norm of X goes infinity this expression also goes into infinity basically that means, it is also radially unbounded. So, what is this as is a sense actually I will end it up with some

V of X which is a positive definite and which is radially unbounded and for, which my V dot f X is guarantee to be negative definite. That means, I will I have this globally asymptotically stable condition actually that means, X equal to 0 happens to be globally asymptotically stable equilibrium point. So, that is the have kind of I mean use some of these theorems and all that now, can see some of the further concepts this is like a construction of Lyapunov functions.

Now, how about doing this I mean the another issue that what is what happens when this Lyapunov function we will end up with this V dot is negative semi definite and all that actually. What we do about that and that is where we need this concept of LaSalle's kind of theorems and all that before, which we want to study what this invariant set limit set and think like that actually.

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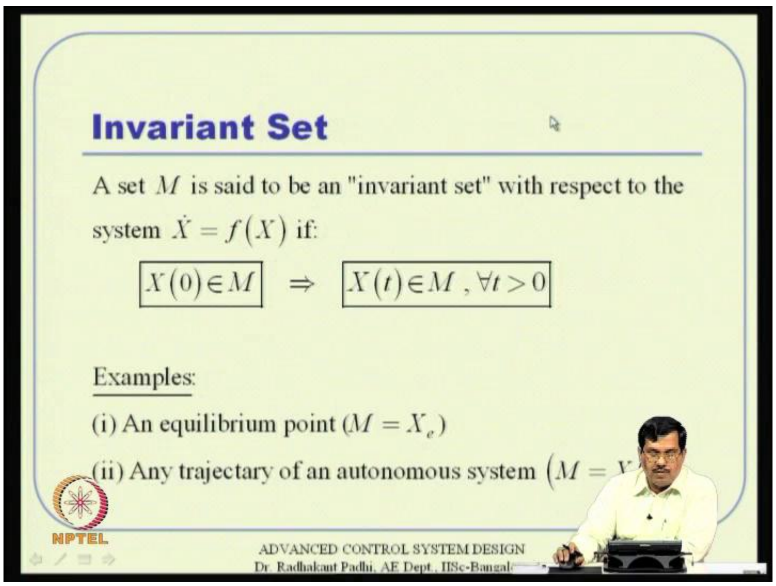

So, let us see what is that thing here so, what he talk about is like a invariant set so set to be invariant set with respect to some system of studies of f of X. If my initial condition set M is a invariant set provided if I my initial condition belongs to that set, then for all time my solution and also belong to that set. So, it obviously invariant it does not go anywhere else actually so what is the example obviously, an equilibrium point is invariant set if I start on equilibrium I will say equilibrium all always. My solution trajectory also in invariant set if I start with any point on the trajectory I will go under trajectory only so if I take all the points, on the trajectory and define a set.

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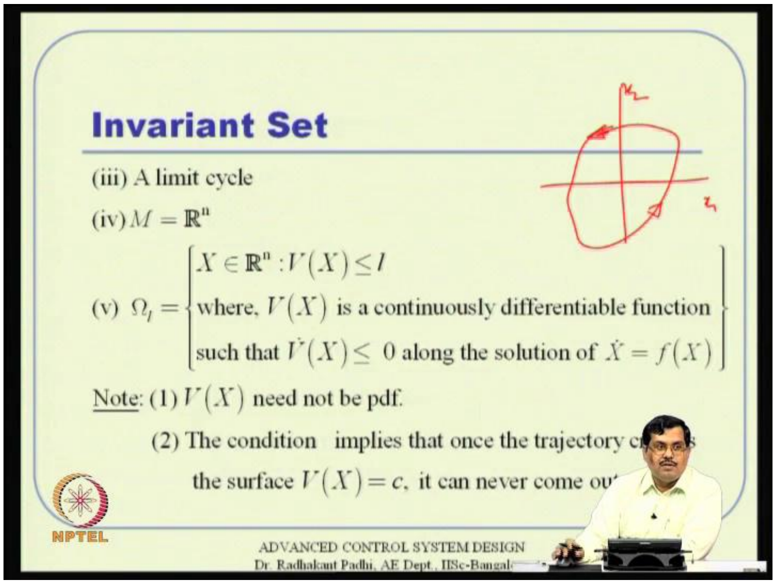

Then obviously, that is where it is invariant actually I start with any point on the trajectory and I will keep on throwing in that trajectory actually. Then obviously I limit cycle is also in variants set and then also limit cycle is cycle in a straight phase in a close curve in the straight phase. So, that means, if I start this is a close curve let say if I start any point on the limit set limit cycle then I will keep on moving on the limit cycle actually. And then there is a this there is an another example which tells us that if I define omega l such way that V of X is less than equal to some number l some positive number l.

Where V of X is continuously differential function such that V dot X is less than equal to 0 along the solution of that then this set is also invariant and why this definition, because this definition is what we are going to use it you in LaSalle's theorem in domain of attraction and all that actually. So, this is where as this examples are very in attractive now, this example is very useful actually so, we define a set to be which is like A X belongs to R n V of X is less than equal to l that means this define some sort of levels set actually.

If V of X is equal to l and that will define a set which is level set what we will define V of X is less than equal to l so, it is a entire domain sort of thing within the domain V of X is a continuously differentiable such that V dot of X is also negative I mean less than equal to 0 along the solution. So, just remember that set is also an invariant set actually. So, condition this condition where it been if you start with that and that V dot is less than equal to 0 that means, I will never be able to come out of this set actually if I define this some sort of number here because V dot is negative anyway. So, I started with some positive number, but we will keep on decreasing so, I will never be able to come out of that surface that is the meaning actually.

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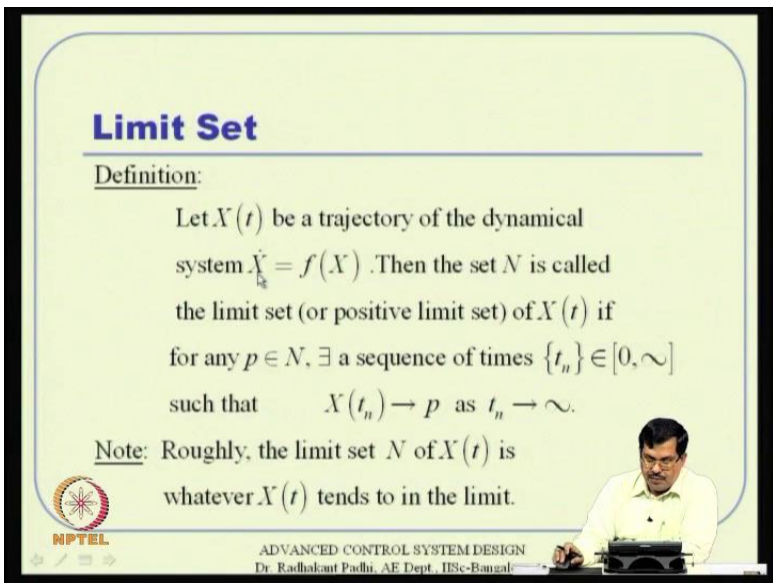

Now, what you limit set? Limit set also shows that into dynamic system X dot f of X is the definition and big definition anyway let X of t be a trajectory of the dynamical system X dot equal to f of X. Then the set N is called the limit set or positive limit set of X of t, if for any p that belongs to be certain there is a sequence of limits t n this semi colon sequence actually and then sequence of time actually t 0 t 1 t 2 t 3 and all that. So, for any point p that belongs to N there in there exist time sequence t n below to infinity such that ultimately my X of t n will approach this point p X actually.

When t n goes to infinity then my X of t should to go this point p actually so, that remains roughly speaking the limits set N of X t is whatever, this X of t was actually set will go of my right obviously. So, wherever, it goes that is actually the limit rate actually I am go to equilibrium point it may go to limit cycle also if limit cycle it is a actually infinite point actually rate on the limit cycle. So, roughly speaking that N I mean the limit set actually is a set of point that wherever, this X of t turns in the limit actually when that t goes to infinity it will go somewhere that is the set actually.

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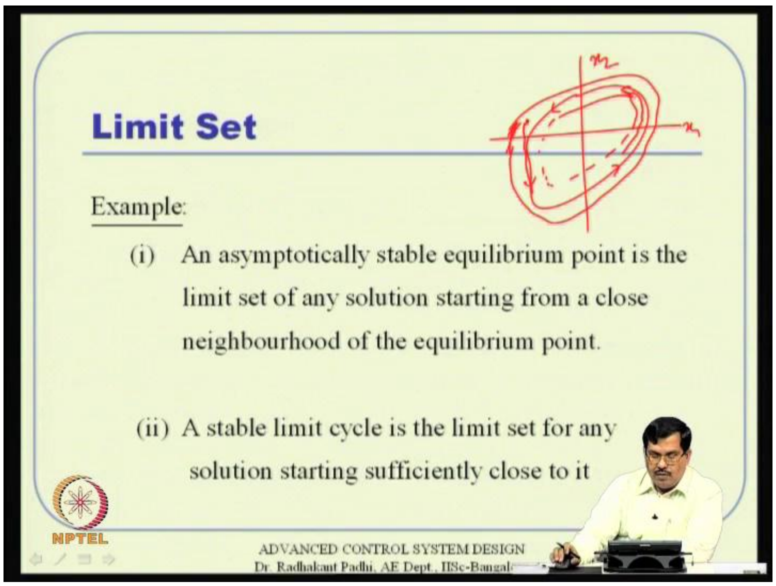

And then example obviously, asymptotically stable equilibrium point is the limit set of any solutions starting from close neighborhood of neighborhood of the equilibrium point the definition. It is asymptotically stable equilibrium point obviously, this trajectory is going to that equilibrium point anyway so, obviously in the limit the solution will convert to the limit point actually how is equilibrium point asymptotical equilibrium point is a limit set actually. But that equilibrium point must be an synthetically stable equilibrium point otherwise you cannot tell that utilize the solution may not convert to that always. And if it unstable equilibrium point it does not make any sense actually, because it will not happening I mean probably if you consider only that equilibrium as I said probably is.

But any domain around that will not satisfy that actually also as you tell it the stable limit cycle is also a limit set, because if we concept of stability of limit cycle is also there that means, if I consider this limit cycle (No audio from 45:31 to 45:40) this is the limit cycle. Let say and this caution moving around that so, if I aim some domain around that limit cycle and in that domain let say if I start with some initial condition in the domain then ultimately my solution converges to that that concept is called stability of limit cycle actually. Whether in some hour some major hood of the limit cycle my solution will converse to that limit cycle actually. So, if that happens that means stable limits cycle is also limit set actually.

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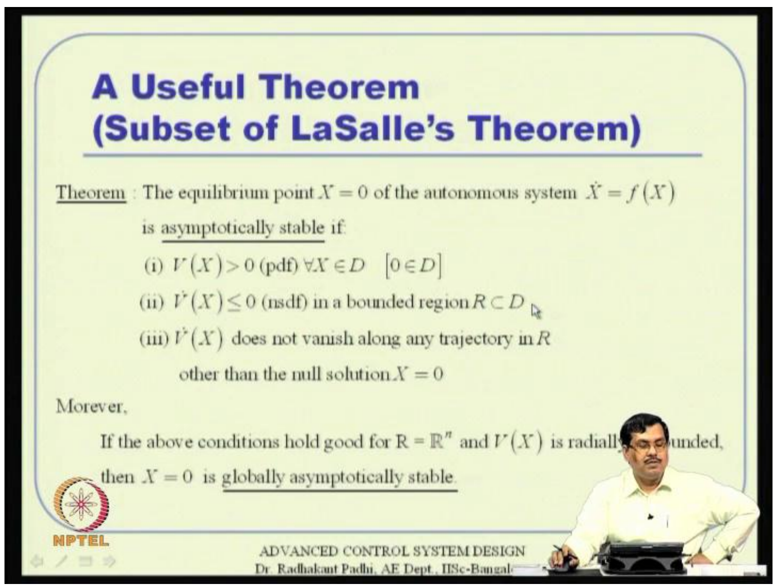

So, this is actually a very useful theorem it is actually like substrate of what we will study as LaSalle's theorem actually this theorem tell us that this V of X is positive definite a regular condition like that. But V dot f X negative semi definite and that is what is the utility comes actually, let V of X is positive definite function in domain D V dot of X is a negative definite negative semi definite function in a boundary region which is substrate of D. And this V dot of X does not vanish that means V dot of X is not equal to 0 along any other trajectory in R other than the non solution X 0 X equal to 0 that is the critical condition actually.

See that see V dot is negative semi definite is fine, but V dot of X is not equal to 0 anywhere else other than X equal to 0. So, using that additional condition we will be able to show asymptotical stability in many case actually in that is where it becomes powerful theorem actually. And if in the above conditions hold good for R equal to R m and for entire surface and all that actually and then it this region R what you are talking about this region R presents to be R n and then V of X is radially unbounded obviously, we will end up with this globally asymptotically stable condition also basically.

So, this theorem is actually very powerful theorem and many times where you struck otherwise that V dot is negative semi definite we will be able to close the chapter and tell really if negative semi definite V dot of X happens to be 0 only at 0 X equal to 0 and hence, using this theorem the system is still asymptotically stable.

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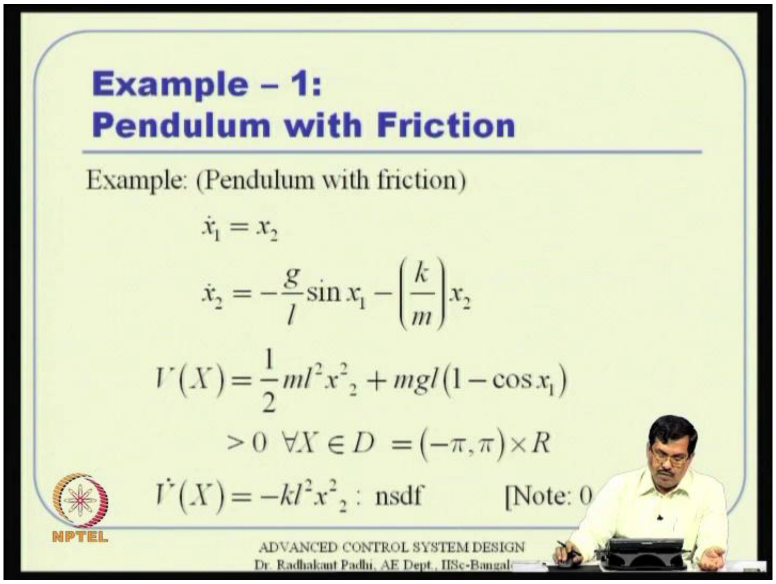

And let see that same example that we discuss last class, we see that pendulum with friction and we started with Lyapunov functions kinetic energy plus potential energy. Landed with the condition of for, which this V dot of X was negative semi definite what about next now, you have to analyze what happens to V dot of X equal to 0 whether it remain 0 only at equilibrium point or it reference anywhere else also.

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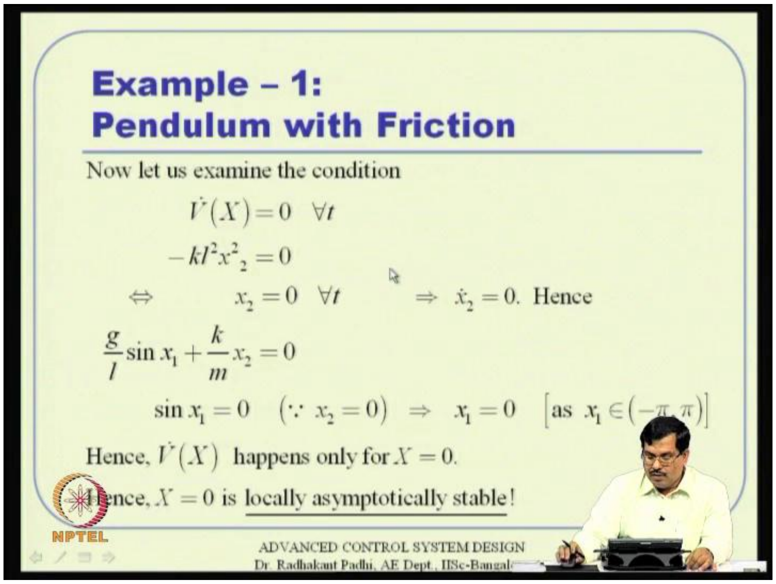

So, now the for that we need to study it so let us study V dot of X equal to 0 for all time that is the condition that we want to study when it will happen actually. Now, V dot of X equal to 0 for all time means, if I see this V dot expression that means X this expression as to be equal to 0 for all time that means x 2 x 2 equal to 0 for all time. Now, if x 2 equal to 0 for all time then x 2 dot is also equal to 0 for all time now, I will go back to X dot expression and x 2 dot expression is this know that means, using this condition here I will be tell I will be able to tell that this happens to be 0 for all time. Now, if this happens to be 0 for all time this expression now x 2 0 already using.

So, if x 2 0 already then sine of x 1 is 0 and sine of x 1 is 0 that means x 1 is 0 in that in the domain actually. Now, only solution is x 1 equal to 0 in that domain so if as long as I start with the domain remember it is an open set the equal to that bound that means the vertical equilibrium condition is rule out actually. If what is that if it close set then I will end up vertical inverted pendulum actually, vertical inverter position equilibrium point that I am not including here. But any other point around that if the entire region the only solution for which V dot is equal to 0 is the equilibrium point that I talked about that means their existence is not for which V dot is actually equal to 0 only on the equilibrium point nowhere else actually.

So, using this theorem that we just talk that means V dot of X does not vanish any along any other trajectory other than the analyze solution X equal to 0 and even this theorem we will able to tell that this system this X is equal to 0 is asymptotically stable. So, that is what you are able to do that actually so, we will end up with this V dot is negative semi definite, but we expected that analyze that little further and tell V dot is equal to 0 where when well there is equal to 0 really. And if it remains there for all time V dot of 0 then it will remain 0 for all time only over the equilibrium point now nowhere else actually.

So, that is the region for which this condition also good and for one hence using this theorem which is a substrate of LaSalle's theorem, we will able to show that these system is asymptotically stable. And now we are happy because we know that is reality in the pendulum with friction we suppose to go to 0 ultimately actually.

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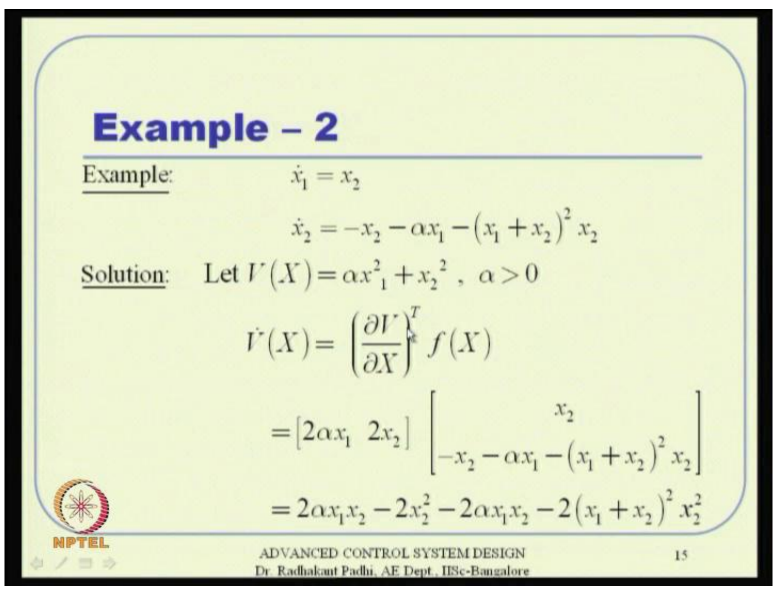

So, that is that is how we will able to show that how about example 2 this let us talk about any example x 1 dot is x 2 x 2 dot is I do not think like that. So, V of X is actually alpha x 1 square cosine alpha dot x 2 is square so alpha is greater than 0, then V dot of X is del V by del X transfers times of prefix and then tell take you through this algebra and all that, because f and f 2 are available we put it there and then we tell them because of this is mass fraction actually.

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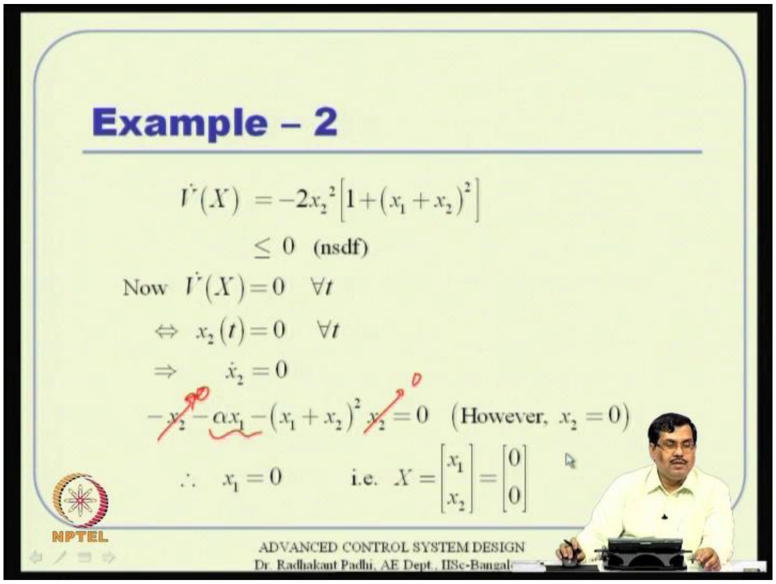

So, V dot if I analyze that happens to be like that again unfortunately this is actually negative semi definite and it is not negative definite and because this expression if it is one more is there this one is not there then I will be able to show that. But if this 1 close expression creates a problem here actually it does not tell that means negative definite what is telling negative semi definite. So, that is what I am telling one you that times we will be able to show that is negative semi definite what semi definite will not be able to now, so what consider the same condition that V dot is equal to 0 for all time then it tells me that V dot this expression anyway.

So, this is nowhere 0 because 1 plus some expression never 0 so that means x 2 dot I mean x 2 x 0 for all time. That means, x 2 dot is also 0 all time that x 2 dot is 0 means this is x 2 dot expression so, this expression has to be 0, but x 2 is already 0. So, wherever x 2 is there I will take out and that will give me this is 0 x 2 already so that is that is 0 and this is also 0 so, what is left out the left out is only that so that means x 1 is also 0. So, that means x equal to 0 actually where able to show that V dot is 0 V dot of X is 0 only on the equilibrium point nowhere else actually. So, that is how we are able to conclude in using this theorem that this is the equilibrium point that we are talking is still an asymptotically stable equilibrium point.

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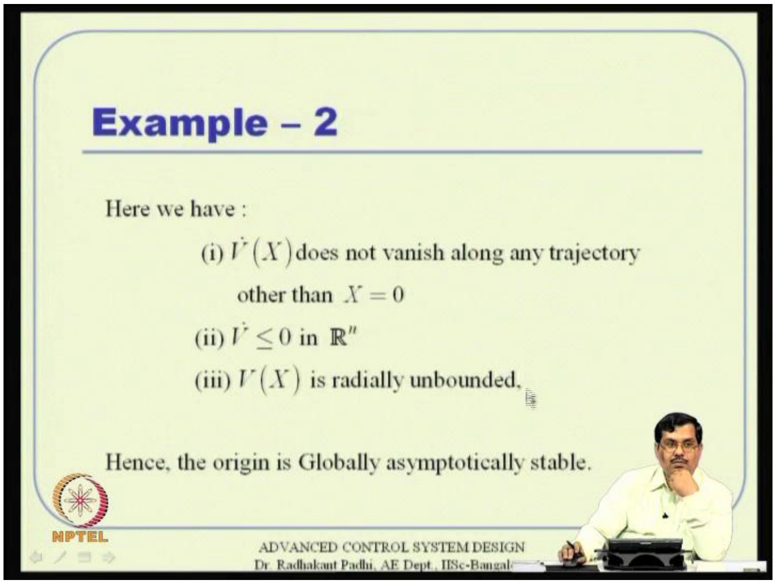

So, what it what we shown that V dot of X does not really when is along any trajectory other than X is equal to 0 V dot is negative semi definite and V of X happens, to radially unbounded also actually this V of X what you are talking is the radially Lyapunov function of problem for alpha is equal to 0 was radially unbounded.

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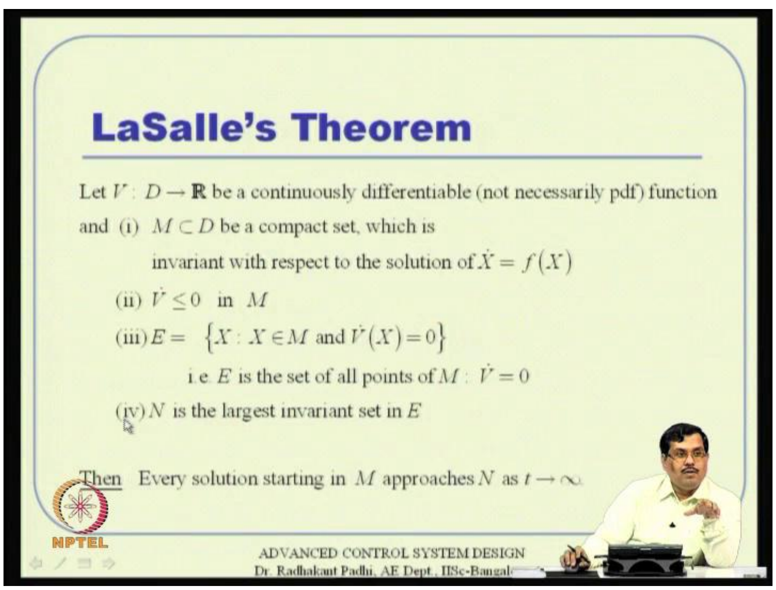

So, sensual the region is globally asymptotically stable basically now, finally the full the LaSalle's theorem, because the full LaSalle's theorem the original LaSalle's theorem is much more than that. And it the different some sort of this set theorem this deforms another set actually we say that in brief that let say m I am means, this b which is define from data to R which continuously differentiable. So, all that remember this V of X that your demanding here in LaSalle's theorem in that all that you needs initial continuously differentiable first definite function reference to be continuously differentiable motor cycle. What it all that in the V of X is unstable I mean what you are talking here this V is did not really Lyapunov function it is a function which is continuously differentiable.

And this three four conditions are really that means, one which is substitute of D is compact set and compact set where definition is closed at bounded actually. Set as to be closed as well as bounded so, that is what the M is a compact set which is invariant set with respect to the solution actually. So, that means either it will have something like equilibrium point something like limit cycle something like a like a semi center trajectory also whatever, can thing we discuss before it as to be one of that. Then V dot is negative semi definite in that particular set and then we will define a set E for which this V dot is equal to 0 all these things we have whatever, examples we studied all that will satisfy that basically.

So, if you want to see examples you can go back and see these examples carefully and try to make have meaningless self actually. So, that is what it tells us actually so, D belongs to I mean the I am mean this V dot the negative semi definite and then this define another's set E will that way that X belongs to M such that V dot S is equal to 0. And then this that means, this V is set of all points of M for which V dot is equal to 0 that is we mean then who tell another set which is actually largest invariants set in E. So, we have to talked about several definitions and notations here actually so, we define like M k is substrate of D is a compact set invariant with respect to solution the system V dot is less than equal to in that set a less than equal to 0 in that set m.

Then we define as a set E for which E dot is equal to 0 not less than equal to 0, but equal to 0 so obviously, that E is substrate of M. And then you define another set M which is larger set in E that means M is substrate of E. So, ultimately we will be able to tell that every solution that starts in the domain M starting in set M will ultimately converts to N that is the theorem tells actually as on N approaches to infinity.

So, that means actually is if I start with any set term which is define something like that way means, then I will be able to go to N basically so, it is says substrate like that and ultimately I have some substrate for which the solution is eventually go actually. And all these theorem that is what we discuss as substrate of this theorem I mean this substrate of loss of invariance theorem and all that is we will satisfy all that actually. But this is with respect to equilibrium point only that is with respect to invariant set and N set is much more general.

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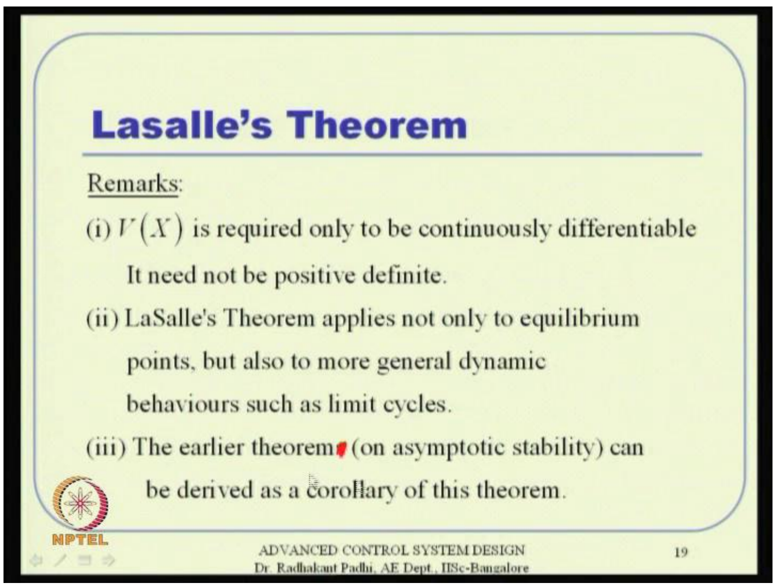

And this generalities written I mean remarks or like that that V of X is required to be only continuously, differentiable in positive definite and LaSalle's theorem applies not only to equilibrium point, but it also is very much general and it is it can be study for something like stability gradient of limit cycle as well. So, the earlier theorem so I mean that the on asymptotic stability can be derived as a corollary of this theorem actually this is the theorem what we discussed. It is actually like a earlier theorem that we discussed and we derived as a corollary very straight forward corollary rather actually.

If you simple define the sets and N happens to equilibrium point ultimately actually this it will happen to be like that, anyway with this thing. I think we will stop here in this class and we will see this study this theorem once again in the see some example and then proceed for that we explain this in something called domain of attraction with examples and all that later in next class. Thanks a lot.