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Lecture No. # 31
Lyapunov Theory-1

Hello everyone, this particular lecture in this series we will see Lyapunov theory, and it will be followed by extensions of this theory I mean by within the same concepts in the next lecture as well actually. These are heavily used in non-linear system analysis as well as system design that means, control system design also. We will see some of those applications as we go along, let us first understand the theory part of it.

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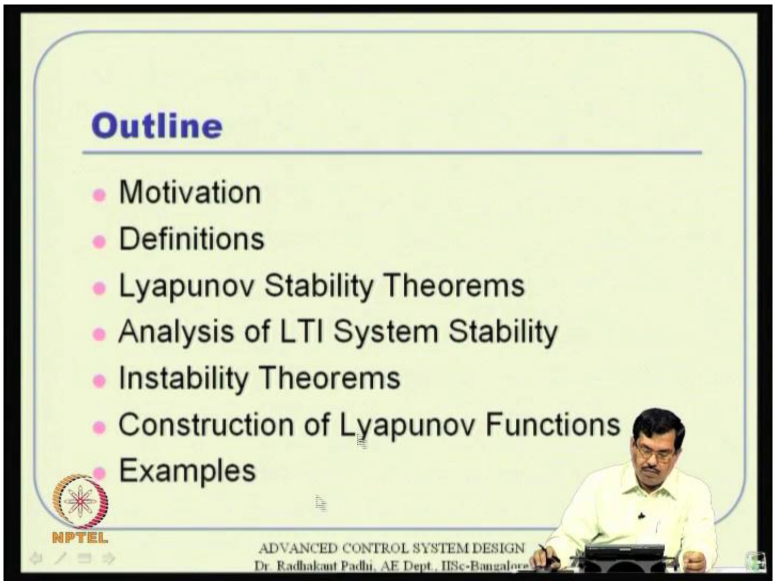

So, outline of this particular lecture will be something like first is motivation, then followed up with followed by some sort of definitions some little bit conceptual things. Then, we will go to stability theorems, and most of the theorems we will not worry about proofs part in this lecture. We will continue with the conceptual I mean concepts only, and then we will continue with this applications of this theory I mean this Lyapunov concepts for linear time invariant systems stability, there are reasons for doing that also. And then we will see one or two theorems for instability that if you are not successful in applying for like stability analysis, then obviously there may be (( )) that the system is unstable.

So, we need to have some conceptual things for instability analysis as well, then towards the end of this lecture we will also see some sort of construction of Lyapunov functions, some ideas are there that how do you construct Lyapunov functions and all. On the way we will see lot of examples as well actually, references I have largely taken this material from first two, but all the three books are very good books, and there are some revised editions available as well for Khalil, so maybe you can think of buying some of that if you are interested actually.

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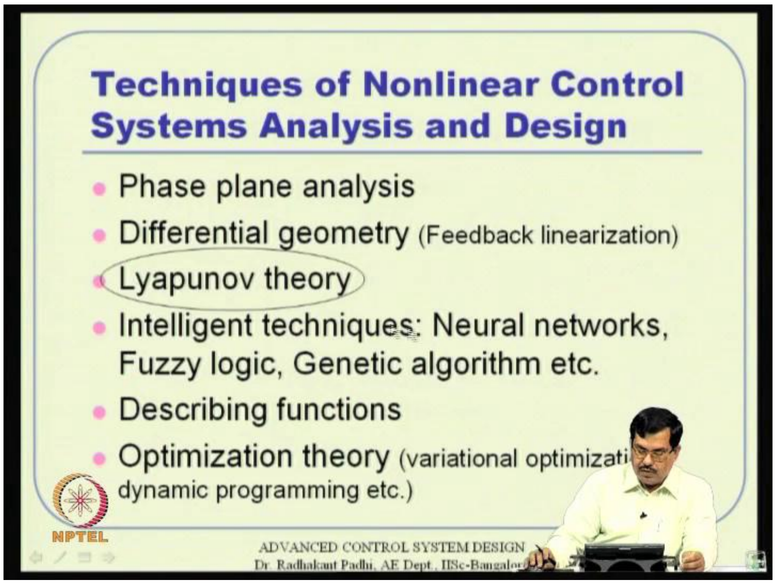

When you talk about non-linear systems analysis and design there are variety of ways that you can do that there are lot of techniques first is phase plane analysis is valid for small systems like second order system and think like that and can be extended on mostly up to third order but beyond that graphical interpretation is difficult and hence we do not use that and then there is in differential geometry based concepts some of that we have seen that already that is feedback linearization based or dynamic inversion concept. Then Lyapunov theory that is what we are going to see this class and next class but other than that there are techniques like neural network fuzzy logic genetic algorithm all these fall under the intelligent control technique and then there are concepts like describing function analysis and there are optimization theories like optimal control theory based on all that actually.

So, there are variety of techniques that are available to us to deal with non-linear systems so techniques are also getting developed but I mean, theorem I mean, the conceptual things are already available actually anyway we will talk about Lyapunov theory here.

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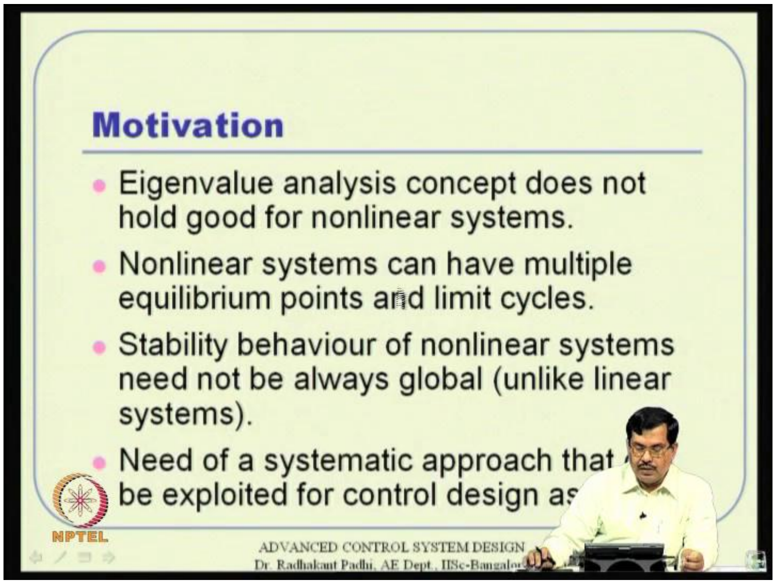

So, motivation for Lyapunov theory is something like this first of all the Eigen value analysis concept does not hold good for non-linear systems, because Eigen value concept is related to matrix and matrices means you should describe the system in the form of a kind of AX dot equal to X plus V where, the w here the matrices come in actually. If you do not have matrices you cannot talk about Eigen value concept at all actually.

Still these Eigen value concepts are useful in an indirect way sort of thing in this non-linear system as well. But in general they are not available and hence we will not worry so much on that actually. Second thing, non-linear systems can have multiple equilibrium points as well as limit cycles these are these are typically not concerns in linear systems actually. And stability behavior of non-linear systems need not always be global so when you when you discuss about linear systems they are typically global once the linear systems do no t worry I mean, the stability behavior of linear systems are I mean, initial condition independent actually that means is global theorems and all that.

But, here we will we will be seeing that most of the time we will able to conclude only local stability, global stability is difficult to I mean, kind of infer actually and need of a systematic approach that can be exploited for control design as well. And that is what we are mostly worried about not only want to have a understanding or analysis technique but we are also interested in exploiting the same to design control systems actually for practical uses that is another motivation.

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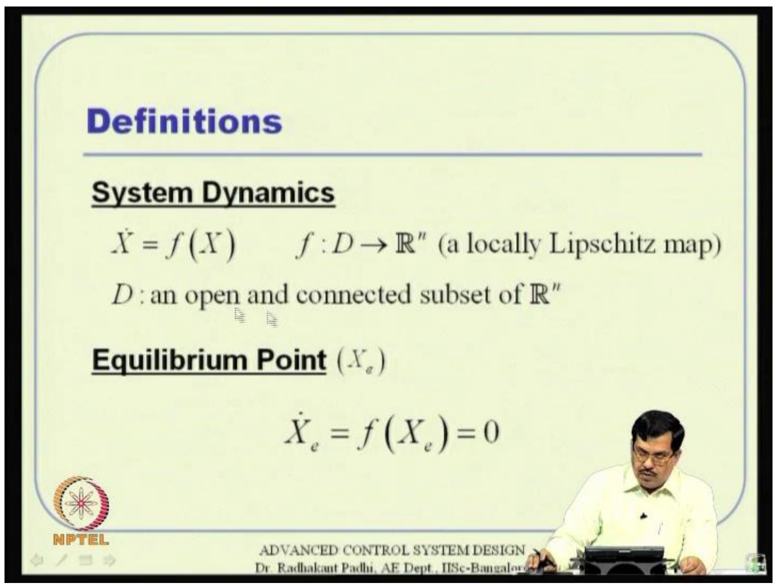

So all these motivation intake to we will study about some definition concepts and all that and then see how we I mean, exploit this. So, when you are in this particular lecture when you discuss stability theorem I mean, definitions and all that we are mostly worried about homogeneous systems X dot equal to f of X there is no u component part of it. But if u if u is designed as a feedback system like state feedback form, then this also kind of in includes that and if you if you put u equal to phi of X then f of X u is nothing but f of X ultimately basically. So, that is why it is X dot equal f of X. So, we talk about subsect of homogeneous system dynamics.

That is what Lyapunov theory is all about actually. So, the conceptually Lyapunov theory can be I mean, there are extensions of this to talk about non autonomous systems as well. That means f of X is not only a function of X but explicitly function of time as well but here we are not talking about that also we are simply talking about X dot is f of X.

So, that means it is autonomous as well as homogenous system and this f that you are discussing this function f is defined in a sub set of like R end which is D and it takes the I mean, the domain D the function is defined and it is range is in R n actually. So, it can start from D and then the mapping can be on the round R n space, where D is suppose to be an kind of open set, as well as connected set. We will see that this what you mean by open set and connector set in the next slide. But before that what are you interested is, in this Lyapunov theory is that this particular system X dot is f of X contains some sort of a equilibrium points actually.

So equilibrium points can be easily seen by enforcing X dot is 0 basically right. So, when you when you do that in when you enforce this X dot equal to 0 then whatever solutions you will get for that particular thing that will give you that equilibrium point and that is what you are doing here. So, X e dot is f of X equal to 0 solution of that will give you X e and remember this X e solution need not be unique and this a non-linear function and then this equation solution may have multiple solution actually.

So, you will be left out with a number of choices for X e and for about each of this equilibrium point we are interested in analyzing the stability behavior actually so if you remember what is open and connected set.

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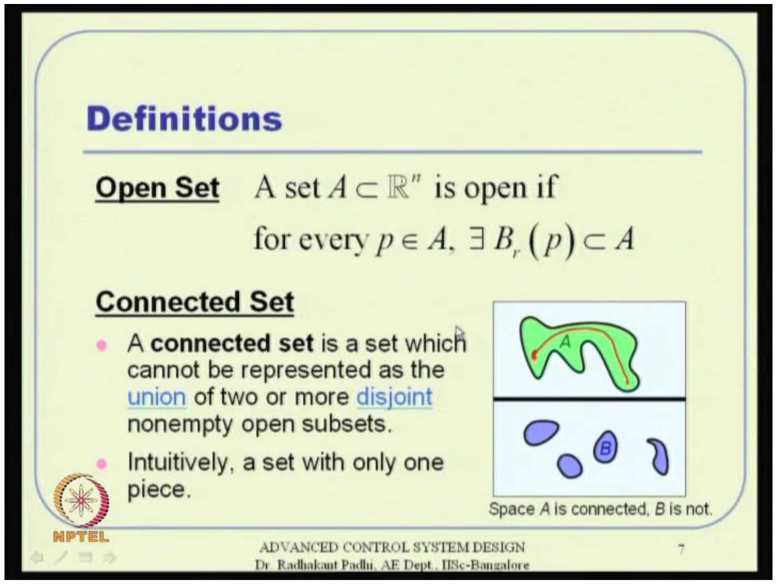

They are the definition when you talk about open set a which is A sub set of R n is open if for every p that belongs to a there exist a wall of radius R centered at V which is also a sub set of a that means I mean, the in a simplistic language the boundaries are not included actually. As long as you do not include the boundaries then they will also be some sort of neighborhood both sides which will belong to that set actually.

So, one that is what it means open set when you when a set is closed that means you also include the boundary points or boundary lines and think like that actually. So, that is what we mean by open set. And what is connected set, connected set is a set which cannot be represented as the union of two or more design non f t opens of sub sets actually. That means if you construct a like any set A you cannot represent by union of a number of designed sub sets actually and intuitively it is a set with only one phase actually that is what the picture tells you. So, for example, like a like set A what you see here is like a connected set and set B what you see I mean, this bottom side what you see here is certainly a kind of a non connected set actually.

If you concentrate on a is a connected set B is a non connected set sort of thing actually. So, why that had happened because if I start with any point a in a somewhere here where wherever actually then any point in a any 2 points I can always join them without leaving the set actually I can connect them, connect these two points without leaving the set. And that is the convenience of a connected set and we are interested in that because the trajectory of the system that you are talking about has to, has to evolve within the set itself. If it goes out of the set then, obviously we are not interested in that kind of, I mean, kind of concepts actually.

So, what you are demanding here is D has to be an open set as well as a connected set actually and it lead to some equilibrium points and about each of those equilibrium points we want to know whether the system is stable unstable or I mean, is locally stable or globally stable or there are various concepts actually and when you talk about non-linear systems so the definitions has to be like carefully remembered actually. So, that is what it take us little more definition concepts first you first is what you mean by stable equilibrium.

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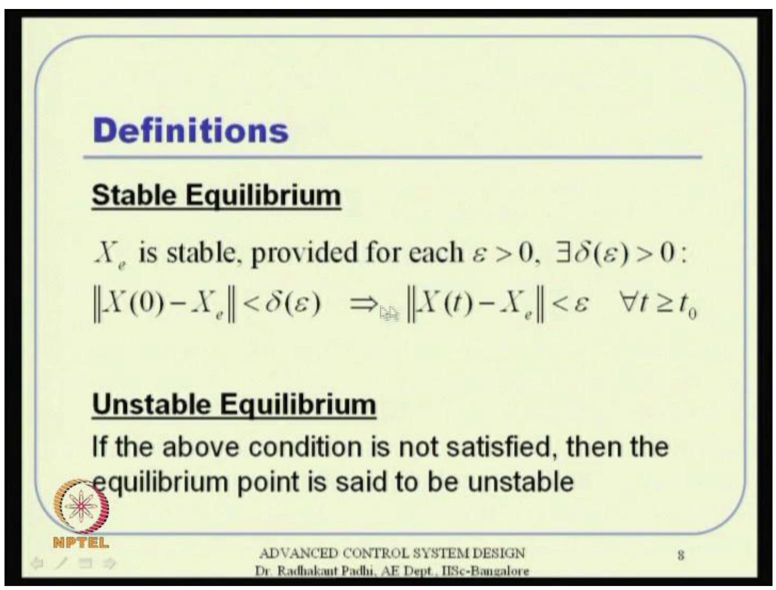

By stability we do not mean that it non like linear system so we do not mean universal stability I mean, it is not just a unique definition actually there are various definitions and first definition what you see here is what is the weakest notion actually. That means X e that equilibrium point X e is stable provided for each epsilon greater than 0 there exist a delta of which can be a function of epsilon which is also greater than 0 such that if I start within this delta bound initial condition X of 0 is initial condition initial condition is close to the X e equilibrium point within a delta bound ball then for any time greater than equal to t 0 initial time I will never leave this epsilon ball actually, that is the concept.

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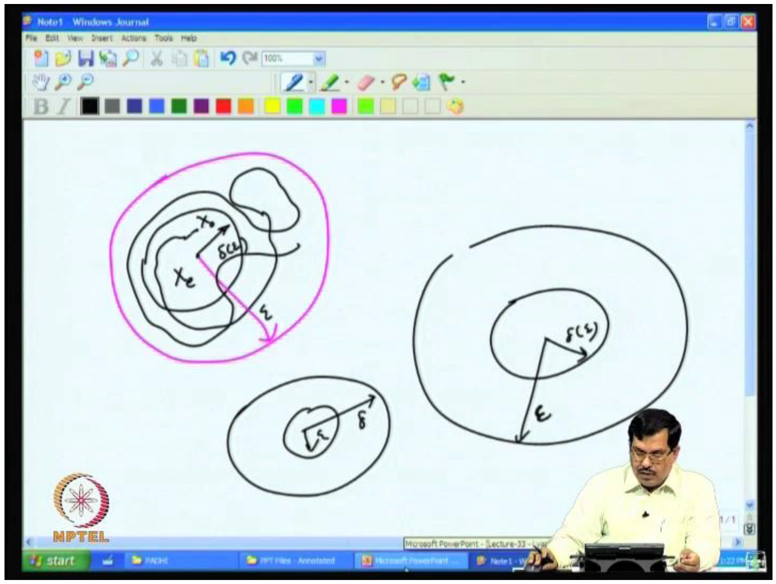

So that means, suppose any point there's AX X e here I start with some sort of a ball of a let us say delta and then if I my initial condition is somewhere then it I leave ball somewhere. So, that ultimately this there is some epsilon wall somewhere this is the it I start with that that is my X 0 X if I start with wall of a radius delta I will keep on moving around that but I will never kind of I can go wherever I want but I will never leave this space actually. So, that is the (( )) and this epsilon what you are seeing here has to be finite of course, actually.

This what it happens what it turns out is that this delta that you are talking about can be a function of epsilon actually. It need not be an independent thing so if it is independent that is again a stronger notion and all that but if I talk about a larger wall then I can start with a kind of a larger delta or smaller dimension just what you are telling it is a function of epsilon basically. So, it need not be an independently given boundary actually it can whatever drips whatever epsilon bound we are talking about delta can be this is epsilon bound delta can be a function of epsilon and it may so happen that you can also talk about like delta being larger and epsilon being smaller this is also possible.

So, it does not tell you that ep silon has to be smaller than delta or delta has to be smaller than epsilon nothing like that so even if you start with a smaller bound and eventually stay in a larger bound then in then also the system is called I mean, the equilibrium point is stable . However, as it is very clear that it is weakest notion of stability we need more than that to understand I mean, to infer good behavior of the system actually I mean, after all this stability notion is nothing but defining what is a good or bad system actually really.

Obviously, this weakest notion of stability is also not satisfied then only we call that the system is unstable actually. So I mean, that is the kind of a contradictory note I mean, kind of definition if the system is stable it has to be stable at least in the weakest notion otherwise there is no meaning for that.

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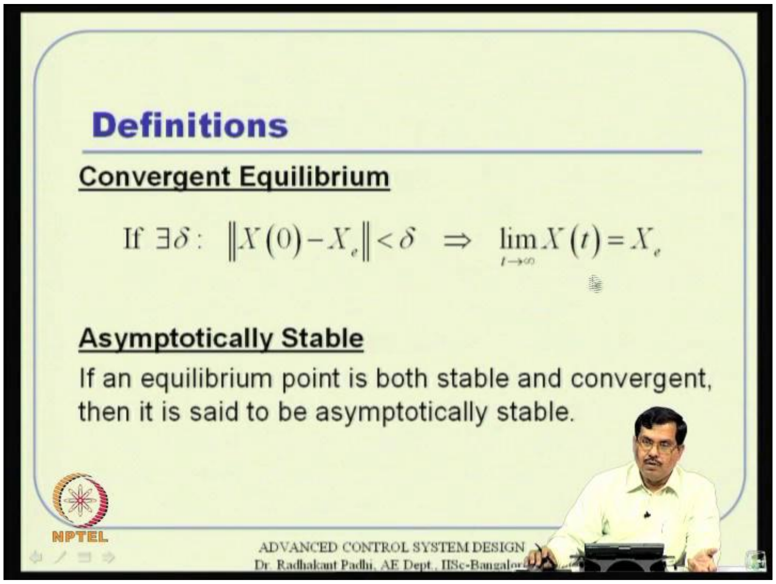

Then, we also need some sort of a convergent condition I mean, this definition is what is called as convergent equilibrium that is noting to be stability.

See, remember stability is for all time t greater than t 0 it has to remain within that epsilon wall actually but convergent notion tells you that it in between it can go anywhere it wants. But ultimately when t goes to infinity my trajectory will converge to equilibrium point.

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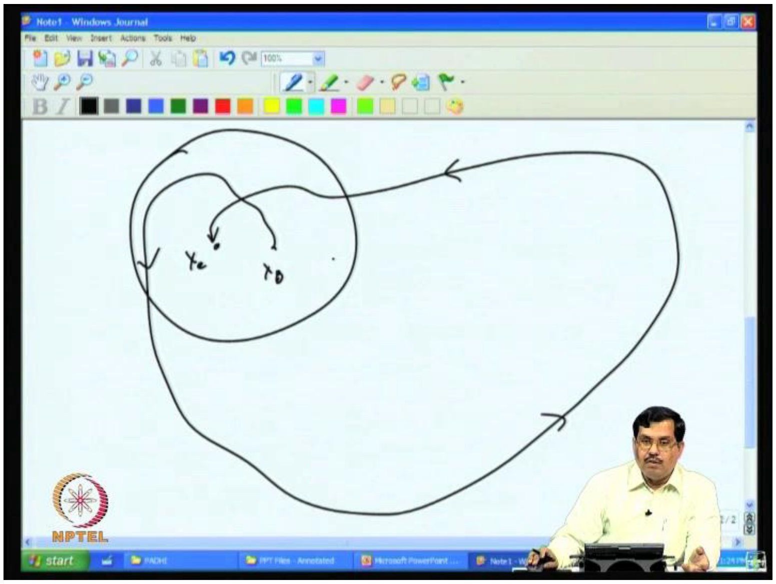

Again if you see that conceptually, conceptually like if I have got equilibrium point something like X e here I have some initial condition X 0 then in between I can go large deviation. But ultimately, I will convert there so that there can be epsilon bound can be briefed actually in between that means if I go here and ultimately I come back and converge that X e then X e is actually convergent equilibrium condition.

That is the notion of convergent equilibrium actually. So obviously we need standard notions of stability so how about combining both the definition. So, if you combine both the definition that leads to something called asymptotically stable equilibrium point. That means, when the equilibrium point is asymptotically stable provided it is both stable as well as convergent that means it will not, it will never leave that epsilon bound but ultimately it will converge to the equilibrium point. And see both the things happen then only you call that as equilibrium asymptotic stable, like stable behavior.

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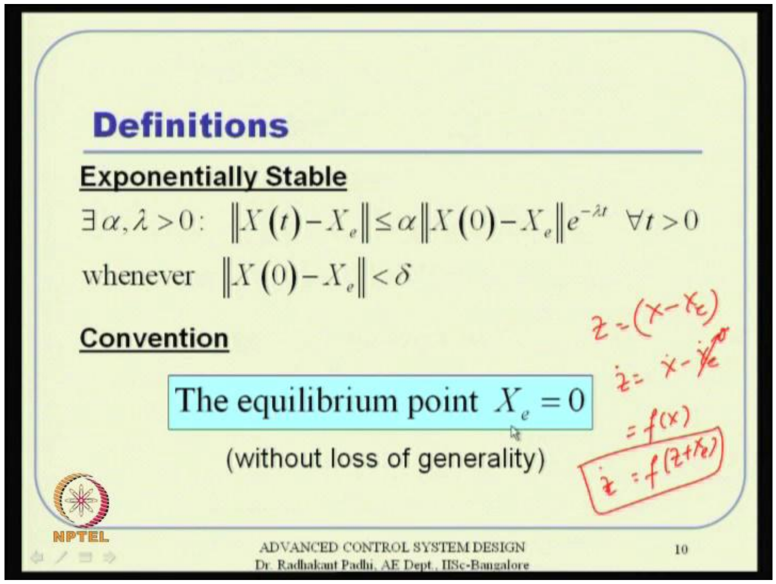

Then there is a stronger notion that is called exponentially stable so here we worried about we are worried about not only it will go to 0 but it will go to 0 in an exponential decay rate. Basically exponential rate of decay basically what does that mean that that means there will exist some alpha lambda greater than 0 such that this particular relationship will always be satisfied and remember this one is the deviation of your initial condition from the equilibrium point this all these norms I mean, whatever you see double bars and all these are norms actually norms are nothing but distances also actually.

So, if I start this equilibrium this initial condition I mean, this whatever X of 0 minus X of e norm actually gives us some sort of a distance measure of the initial point, point from the equilibrium point actually. So this at any point of time X of t this is my distance from the equilibrium point and at initial time t 0 that is my distance from the equilibrium point and these two are related by this relationship where alpha and lambda are positive constants actually that means whenever I start with a I mean, this delta (( )) that has to basically within the double of course, for local stability behavior and all if I start with this delta (( )) then this relationship will always be satisfied in… The rate of shrinking is actually exponential because this this factor is there actually u to the power minus lambda t so that the rate of shrinking is exponential decay.

For further these are the definition first three things we are worried about first is what is stable equilibrium point and if it does not satisfy that that is unstable but if it satisfies that then we will worried about I mean, we are worried about convergent equilibrium condition. And if this two are both there that means if this I mean, equilibrium point is both stable as well as convergent then we call asymptotic stable I mean, the equilibrium point is asymptotic stable and then even stronger notation is exponential stable.

Obviously, the nicest thing to have in any problem is obviously exponentially stable but also remember the more and more you demand or more and more powerful things that you want to show it will also demand more and more difficulty actually that means it may or may not able to show that. In general, so most of the time we are more I mean, very happy to show asymptotic stability condition as in if a equilibrium point is asymptotically stable that is most of the time it is satisfactory actually. And also remember that this exponential stability behavior is invariably satisfied in linear systems. Now, linear system solution is given e to the power eighty anyway already basically iterated by the that kind of a formula basically.

If linear system is stable it is globally as a global asymptotically stable, global exponentially stable. But non-linear systems we cannot talk about that you have to very careful about each of the equilibrium point what kind of stability we have here, that your that you are having actually. Now, for a further discussions we will, when without loss of generality we will discuss that. I mean, we will take it that X e equal to 0 and if X e is not equal to 0 then you do this coordinate transfer that means you define some sort of a new variable that something like z equal to X minus X e and then convert z dot is nothing but X dot minus X e dot sort of thing and if it is a equilibrium then that is 0 and then you left out with X X dot is f of X u or sorry f of X and then this is f of X is nothing but z plus X e actually and X e is a constant number.

If you see this equation z e z dot is again X z actually. So, with that change of notation basically we will be able to proceed further so we will not and as far as X e is concerned it will considered as something like a constant number here. So, with this change of coordinate we will able to proceed further so further discussion we will invariably assume that X e is 0.

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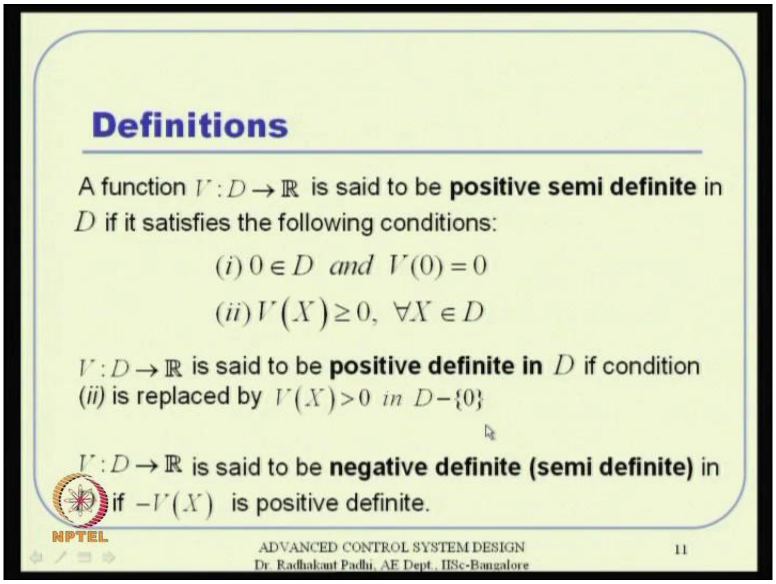

Now, we need further definitions and all to discuss any concepts here and these are concepts of positive definite, semi definite, negative definite like that these functions actually.

Now, a function V which is defined from domain D to R the same domain D but the output is a scalar variable I mean, scalar value,that is the difference here. So, we define a function V from D to R it said to be positive semi definite. if it satisfies these two conditions actually. That means this 0 belongs to D remember 0 is an equilibrium condition now so when you talk about 0 and not just its origin but it is the equilibrium point after change of coordinate actually.

So, equilibrium point it belongs to D that is why the connected set comes into picture. We want to start from that side, so, that, equilibrium point is contained within that set basically, then only you can move around and probably go there actually. S o 0 is part of D and V of 0 is 0 that is a requirement actually. And for any other point that means V of X has to be greater than equal to 0 V of 0 is 0 anyway, but any other point X belongs to D this V of X has to be greater than equal to 0 that is called positive semi definite. And if it is so, that V of X is greater than, strictly greater than 0 in D minus 0 that means, just take out the equilibrium condition point and then, for every other point we have a that strictly positive then it is positive definite function.

Similarly, we can define negative definite or semi definite with respect to the same definition. if minus V X satisfies this condition actually or alternatively you can change this sign, greater than equal to becomes less than equal to and this greater than strictly greater than become strictly less than actually. So we know what is positive semi definite function, positive definite function and negative semi definite function and negative definite function actually. These are all required as part of Lyapunov theorems.

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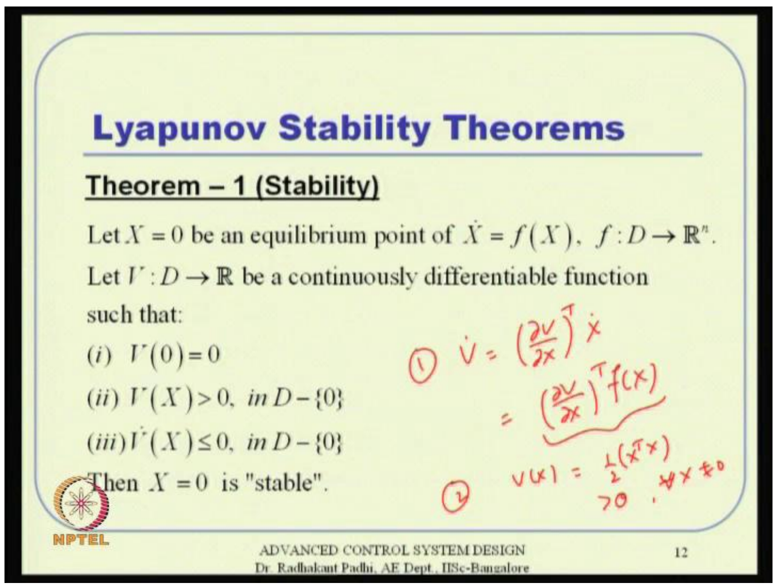

Now, here we are ready almost I mean, now we are ready to talk about the stability theorems and their meaning actually. What you are talking the theorem for one and all these theorems what you are going to discuss in Lyapunov theory these are remember these are all sufficiency conditions, they are not necessary at all actually. That means, if this conditions are satisfied you are guaranteed to get the get the stability I mean, stability behavior. But if this conditions are not satisfied with respect to that V of X that you have selected then, there is no guarantee. In other words, you cannot infer that this system is unstable you may need to try out you may have to try out with a different V X or keep on trying out or you have to ultimately show that the system is unstable using instability theorem and all that.

But, all these theorems whether, it is stability theorem or instability theorem these are all sufficiency conditions alright. The theorem reads like this, let X is equal to 0 be an equilibrium point of this system regular thing that we started with f is defined from D to R n and let V this D to R be a continuously differentiable function such that these conditions hold good. That means V of 0 has to be 0 you have to select V of X which satisfies this V of X is greater than zeros strictly greater than 0 in d minus 0. That means conditions one and two taken together all that you are telling is V of X has to be strictly a positive definite function you start with positive definite function define in this domain D same domain D.

So, V of X is a scalar function defined in the same domain D and positive definite function and with respect to that selection the third condition has to has to be satisfied. In other words, V dot of X has to be less than equal to 0 in D minus 0 the same domain what you are talking about. V of X is positive definite whereas, D dot of X has to be negative definite negative semi definite then, X of 0 is stable actually. Now, there was couple of comments there were. Here, first of all this V dot of X Y how do you compute V dot so if you talk about the V dot if you talk about V dot here V dot is nothing but.. See remember V is a function of X only so we will compute del V by del X this is this vector now because V scalar X is vectors then transpose times X dot that is why chain rule actually.

This is nothing but del V by del X transpose times X dot is nothing but f of X so we will end up again with a function which is like purely a function of X so that function we have to analyze actually. So and remember this V of X is something that we choose so, that means this satisfying this first two conditions statistically not the difficult but V dot of X is related to I mean, it is something to do with our selection coupled with the system dynamic equation that we have f of X is comes into picture actually.

So, V dot what you want to want to analyze later is a function of both our selection as well as what system dynamics we have, that is where the difficulty comes actually because our selection in system dynamics may not match exactly actually that is the that is the first comment. The second comment, the second comment is, if nothing is given then, V of X is something like we typically like to start with something like transpose X let us say, that is certainly a positive definite function because if you take X equal to 0 it is 0 any other X it is a positive right you may it is strictly positive, strictly positive for all X not equal to 0 basically.

So, if nothing is given to us then this is a certainly a what we call as Lyapunov function candidate. That means, any selection that is positive definite is certainly a Lyapunov function candidate. But once the third condition is also satisfied then, we call that as Lyapunov function actually. That means it has some meaning to the system dynamics that we call, that we have actually. So, if nothing is given that is a choice, if this selection fails then, this the other selection is like kinetic energy plus potential energy. So, we have, if we have the mechanical system elliptical circuit something like that, then if you take total energy total energy of the system is kinetic plus potential energy which is certainly guaranteed to be a positive quantity basically.

So, with respect to that expression you analyze V dot and think like that proceed from there actually. Again they may or may not succeed and you may be able to see give only limited inferences and all that actually. So, all those things may not be good for you and in the other words it may it may not be sufficient for inferring this stronger conditions and all that. We will see some couple of examples later actually.

Anyway the summary of the story is, you select a positive definite function so that V dot is negative definite sorry negative semi definite in the same domain. Then you can only infer that, X dot that X equal to 0 that equilibrium that we are talking is stable. Remember, this is the weakest notation of stability that is just discussed if you can only show this, one then you are just talking the system is stable. That means you will you are getting some sort of a bounded solution that is all you are talking nothing more than that actually. Now, what about little more than that we are mostly interested in this asymptotic stability anyway then that is actually can be a theorem 2.

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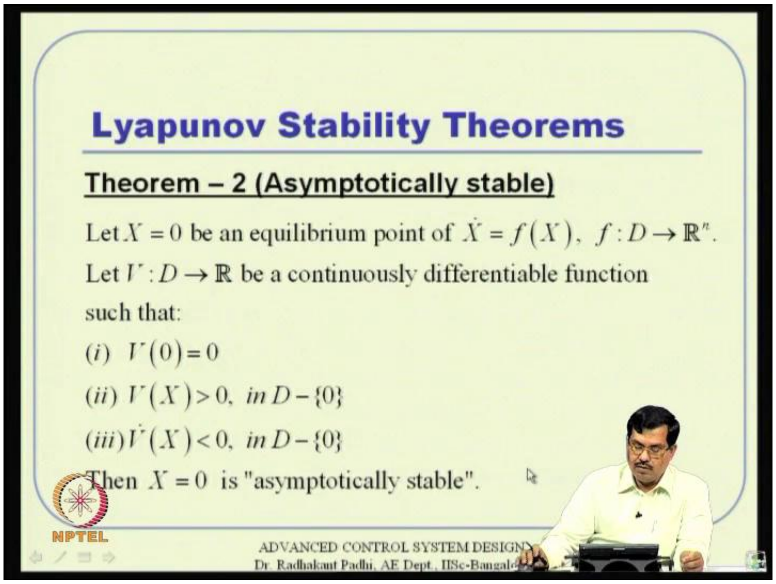

Carefully observe theorem two that is if this gives you asymptotically stable, that is the condition, everything else remains intake. But the condition that changes here is only this, one this negative I mean, less than equal to 0 becomes strictly less than 0 that means V dot has to be negative definite this time and if you I mean, intuitively it fills of course, what is the big deal actually. I mean, if it is negative semi definite I can show that but it turns out it is order of magnitude difficult actually. To see any problem if you really want to show V dot is strictly negative definite it is an order of magnitude complexity actually higher.

Many times you will be able to show that, from here to there it will demand lot of work we may not be able to show that also. And that is why we have to take help of some other theorem actually which is a LaSalle's invariance principle and all. We will also talks about that actually but anyway let us understand that is this the basic Lyapunov theorems first. First thing tells you is start with the positive definite function so that V dot is negative semi definite so that equilibrium is stable, the next one tells you that V dot is negative definite so that you can conclude asymptotically the equilibrium point is asymptotically stable.

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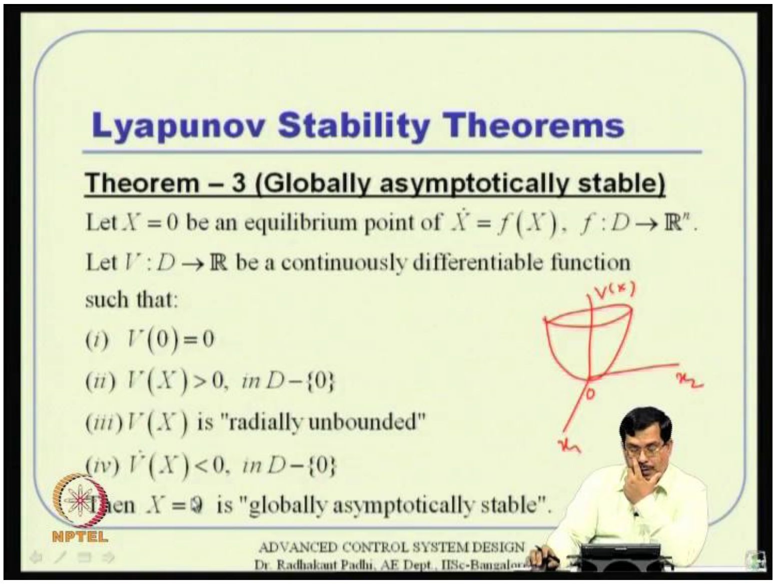

Then, there is a third definition these are all for what we saw is local, that means around a neighborhood only my solution will converge and things like that. What about global stability? And global stability is I mean, is valid, provided there is a one additional condition which tells you that V of X is not only positive definite but V of X also needs to be radially unbounded. Actually, I have to select a V of X in such a way that V of X needs to be radially unbounded. That means, if the… more and more I go away from equilibrium point, my V of X is going to increase more and more is never going to be decreasing and all that. What is the what is an some example it is a very easy that X transpose X that we talked about in two dimensional space you can see that the function takes a shape of this something like this.

This is x 1, this is x 2 let us say and this is V of X if you go more and more away in any direction from 0, then obviously the function keeps on increasing actually. So that those are functions called radially unbounded actually. Other mind This is a necessity condition, otherwise you will you will not able to show the global behavior of the equilibrium, I mean, the stability condition basically. So, it in other words it this additional definition kind of truncates our freedom actually that we cannot take anything that we want we have select something which is also which also has this radially unbounded properties basically.

They are necessary when if you really want to think a little that is, something comes from levels at (( )) we are not going to discuss so, much on that. But level set is something, that that is a set for which the V V of X has one particular value basically. So let us say V of X is c 1 that defines a set, V of X is c 2 that defines a set. And if all the time you are interested in some behavior for which, this my V dot negative definite does not mean, does not necessarily mean that my level sets will keep on decreasing in a way actually. But these are all different concepts we will not worry so much in this particular course actually.

But, if you are interested you can see some of those books. books have some nice examples which will clarify your concept, why this radially unboundedness is necessary. Actually, there are counter examples also; if the radially unboundedness is not there then V of X can keep on decrease decreasing and it will never converge to origin actually that is also there actually so that is a necessary condition I mean, necessity here actually.

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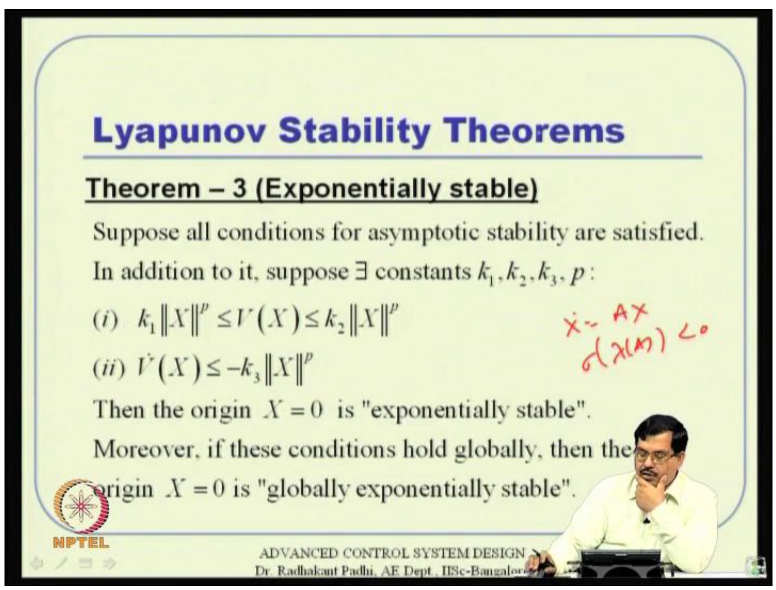

So, if your V of X is positive definite, in addition V of X is radially unbounded and V dot is negative definite, then X of 0 is globally asymptotically stable actually. What about another condition? What about exponential stability? This theorem tells us that suppose, all conditions of asymptotic stability are satisfied in addition to that, there exist concerned constants k 1 k 2 k 3 and p such that this V of X is bounded both sides and V dot of X is also bounded from above actually. That what are that mean actually like V of X see this asymptotic. I mean, this these asymptotically stable conditions are already there with us. Actually, we want to see whether the exponential decay is there or not in that. If you want to conclude then, this is a condition that you have to I mean, that restricts our selection further that V of X has to be bounded between these values and V dot of X should always be less than equal to this minus k three times X of e actually.

If you really want to see where is the application and can I show that example and things like that, you can do that for LTI systems LTI systems. If you… that linear time invariant systems if you take X dot equal to a times X and you know that these Eigen values of s are actually stable. So, if I take lambda a and then sigma part of it is less than this, the condition necessary. Actually, this one is a real part of the Eigen values are all less than 0, we know for sure that the system is stable and in those conditions you can actually show this theorems getting satisfied using the (( )) equality actually.

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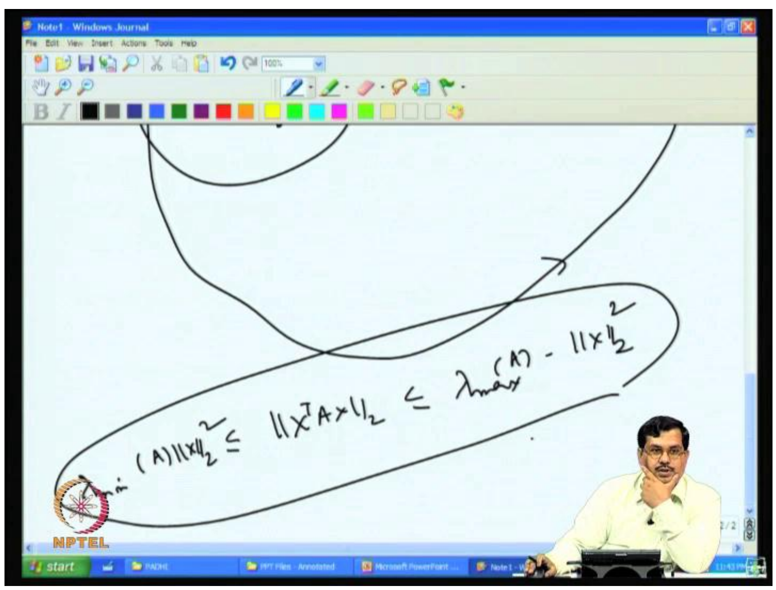

That means, if I talk about… well let me a we have to excite a condition that it tells you that this X transpose X norm of that in second norm, especially is bounded both above and below using lambda minimum of a, then norm of X square and with lambda max of a into norm X square. So, if you use this theorem we will be able to show that these conditions are satisfied for coordinative linear time invariant systems actually.

Alright we will not regress this too much we will keep getting focused here and these are the necessary conditions for exponential stability actually. And obviously if these conditions hold globally that means D is R n actually really, then origin is globally exponentially stable actually and obviously these conditions are more and more difficult to show. But for linear time variant, time invariant systems they are satisfied anyway actually but other conditions are… suppose you are lucky in a particular non-linear system then probably you can if you are able to show this nothing like that actually.

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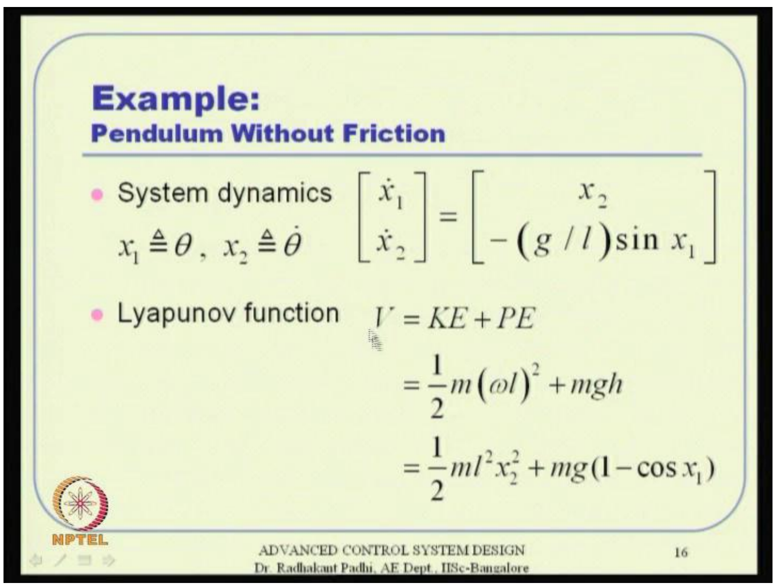

Now, time for some example to understand this thing better and let us start with this example, with this pendulum without friction. So let us say there is a there is a regular pendulum, not inverted pendulum and it actually oscillates about its vertical equilibrium point, where the bob is down actually then what do you I mean, if you define this x 1 theta and X this x 2 s theta dot then this x 1 dot and x 2 do,t then is theta dot and theta double dot if you start with that and all that you will end up with this kind of equation in state phase. And it is easy to derive that also, it only oscillates under the influence of only gravity; actually there is no friction remember that.

So, obviously intuition tells us that because there is no friction it will keep on oscillating actually. So obviously we will able to show only stability not asymptotically stable it is a stable behavior because they know the energy dissipation actually, nothing will go down so it will keep on oscillating, so can we show that. Let us start with this Lyapunov function candidate, very naturally we will take kinetic energy plus potential energy and we expand this term and tell this is my expression to start with. Certainly it will satisfy the property that if X is equal to 0, that means x 1 and x 2 are 0 then this term is 0 1 minus 1 is 0 this is also 0 so V of 0 is 0. Any other point any other condition I have got some some kinetic energy plus potential energy anyway, so total energy is never going to be I mean, total energy is always going to give positive actually. So, that I got a right candidate for Lyapunov function.

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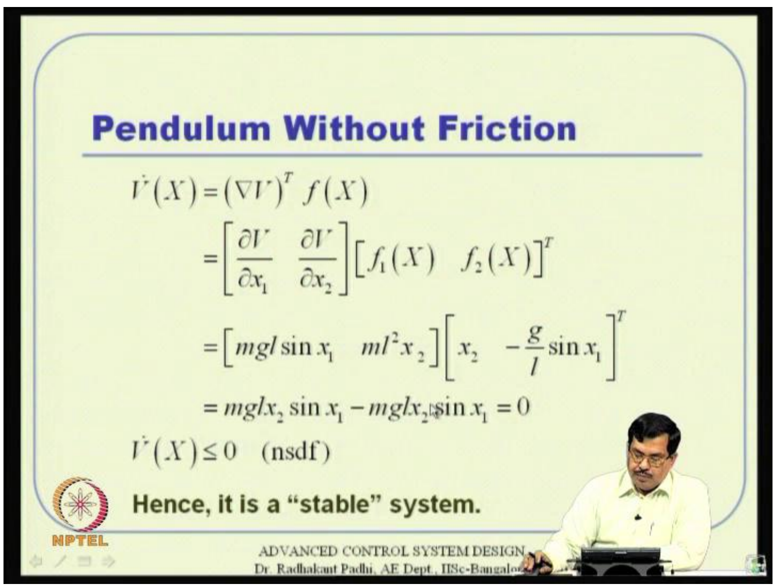

Then, what about my V dot V dot is as we discussed del V by del X for gradient of V transpose times f of X so that is my gradient. V transpose del V by del X, del V by del x 1, del V by del x 2 as a row vector and then f of X now f 1 of X and f 2 of X is a column vector this is a column vector with a transpose remember that.

So, if I do carry out this algebra I have got a V expression so with respect to that I carry out del V by del x 1 and put that, and del V by del x 2 and put that and f 1 and f 2 are already available to me from here, that first one is f 1 and this is f 2. So, I substitute that and then see that these two expressions cancel out nicely. That means, I have got a V dot which is equal to 0 so obviously this V dot is less than equal to 0. In other words it is a negative semi definite function.

So we started with a positive definite function and showed that V dot is actually a negative definite function and hence it is a stable system. All that you are telling here is a system is stable we are not claiming anything more than that and as long as the system wall oscillates, keeps on oscillating actually then obviously the system is stable, is not going to increase its I mean, it is not never that theta and theta dot it is never going to go infinity that is all you are claiming here actually.

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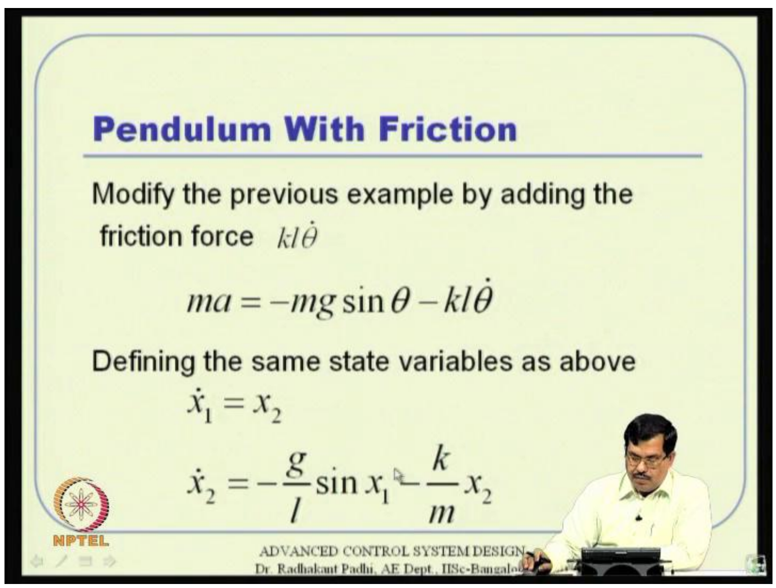

What about with friction because with friction we know for sure that the pendulum is going to come to 0. Now can we I mean, are we able to show that. So, we have to modify the system dynamics and then tell I have got a k l theta dot term now here and then if you if you carry out the same analysis and all that you will have an additional term which is coming here as a friction term and all that minus k by m x 2.

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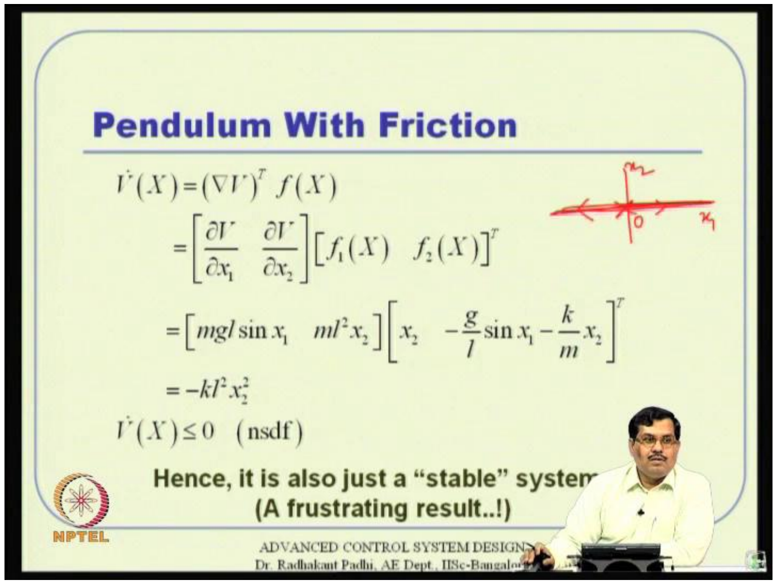

Start with the same analysis same algebra and you are left out with this expression actually. That means V dot of X turns out to be minus k l square X t square and that is the difficulty and in other words this expression is actually we can only claim that it is negative semi definite it is never negative definite because it does not contain an expression in terms of x 1.

And this expression has to contain all variables actually you cannot leave out one component of in the state phase and tell it is negative definite. Why is that, remember if I talk about this expression let us say this expression feels like its negative all the time but really not so, let us let us understand that part of it because if I talk about x 1 and x 2. This is x 1 and x 2 all that it guarantees is that, this expression will always remain negative actually that that is fine but what about x 2 equal to 0 when I put X to equal to 0. That means I am moving along this x 1 line the function is equal to 0 that means I have got a infinite number of points other than equilibrium point for which the function V dot is equal to 0 or the but that is not the definition of negative definite, negative definiteness negative definiteness tells that the moment I go away from 0 anywhere in any direction. I should be able to get a negative value, but that is not happening here. The moment I travel in any direction in a I mean, direction along x 1 axis I my x 2 value is still 0 and my V dot is still 0 actually.

So all that I am able to show is that V dot of X is negative semi definite and that is the frustration really and if it is negative semi definite we are still able to tell that system is stable and we are not able to tell system is asymptotically stable and that is the frustration. We know for sure that the system is asymptotically stable but we are unable to conclude that using this particular candidate of Lyapunov function. And that is the motivation to why we need to study LaSalle's theorem and all that. Actually, using LaSalle's theorem we will be able to show where the system is actually asymptotically stable and that is what we will study later actually.

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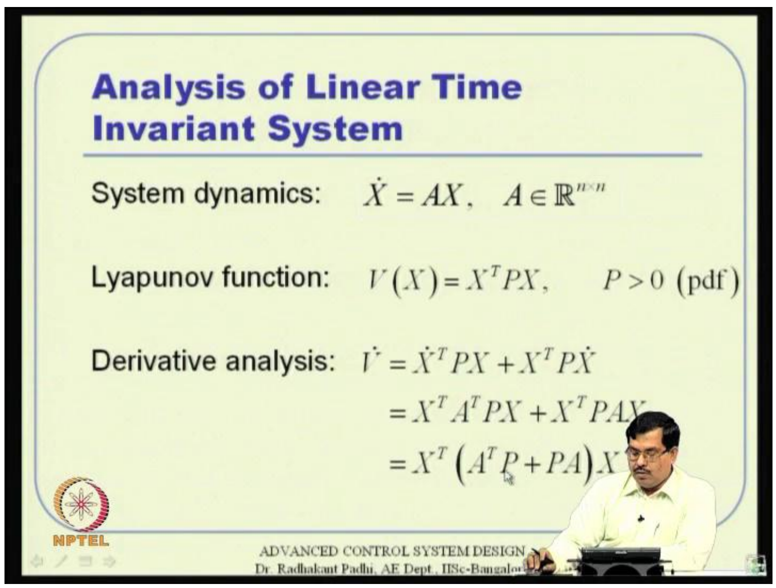

Anyway, before moving further, what about this utility for linear time invariant systems? Because that is something that we understand well and then, this particular analysis will also give us a tool to exploit it further in many control design application actually. So, let us study this little bit here and we are interested in not only any non-linear system but we are interested in applying the same thing for this particular linear system X dot is AX where A is just a constant matrix. Remember, the non-linear systems by definition are not necessarily linear system that means anything that is valid for non-linear system is also valid for linear system actually. I want to verify whether my concepts are right or wrong number one and number two, it will give me some of the tools to analyze and some other tools to exploit in control design later.

So, let us start with a Lyapunov function candidate and X transpose PX where P is a positive definite matrix is certainly a positive definite function. We have studied that in matrix theory also, if I start with X transpose PX remember X transpose X means P is identity and identity is certainly a positive definite matrix basically. But in general if I put a positive definite matrix here X transpose PX then V of X is certainly positive definite function. Now, how about how about a kind of analyzing V dot? Now, if you start from here and V dot remember P is constant matrix. So, V dot is X dot transpose PX plus X transpose PX dot and X dot you substitute A of X; that means, X dot transpose is A transpose X transpose and X dot is X. So, you take X transpose to the left and X to the right, you are left out with this expression that means V dot is nothing but X dot I mean, X transpose whole into A transpose P plus PA into X actually.

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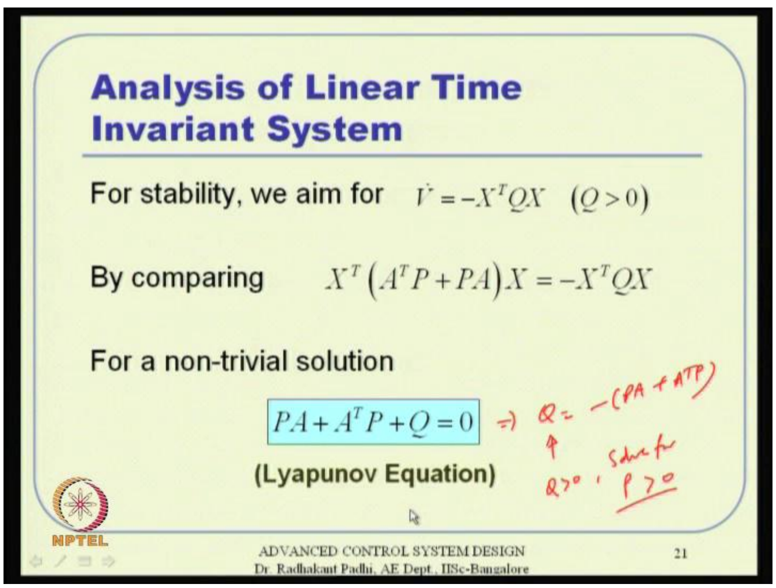

So, for stability we aim for something like a negative definite V dot that means V dot has something to be like we aim that V dot should be something like this where, Q Q is positive definite that means V dot is negative definite if it happens. So, that means if you compare this expression with this one this is the result that you are getting so if you really want a non trivial solution that means this equation is valid for all X which is not equal to 0 then all that we have is I mean, with this equation has to be satisfied and what is called as this is this equation is famously known as Lyapunov equation actually.

So if you get a P and Q solution from here we satisfy this equation and if I select a P and select a Q which will satisfy this Lyapunov equation I have done actually. So, obviously it gives us a gives us thing I mean, kind of a idea to tell that I got Q Q is equal to negative of that anyway if I solve for Q this expression gives me something like Q Q is equal to negative of PA plus A transpose P. So if I start with a positive definite P I can simply evaluate my Q and I see whether this is positive I mean, whether Q is negative definite or not and sorry Q is positive definite or not and then I am done actually.

But, it turns out that this is not a very smart way of doing actually because then you can keep on doing that; there is no harm in doing that. But we really want one shot results actually. That means, we just want to try out once actually that way and if you really want to do that it turns out that you have to start within a little bit reverse direction that means you start from Q do not start from P you start with start with Q so select Q positive definite and then, solve for P. If you solve for P and then show that P is also positive definite then the result is your sort actually; that means either you succeed or you fail that that way.

That is why the procedure is always like that. You start with the positive definite Q and solve this Lyapunov equation to get a P solution and see whether the P is positive definite or not actually. And this is remember, this is a linear equation. So, the solution does not take computational intensive way and things like that is a just a unique solution actually.

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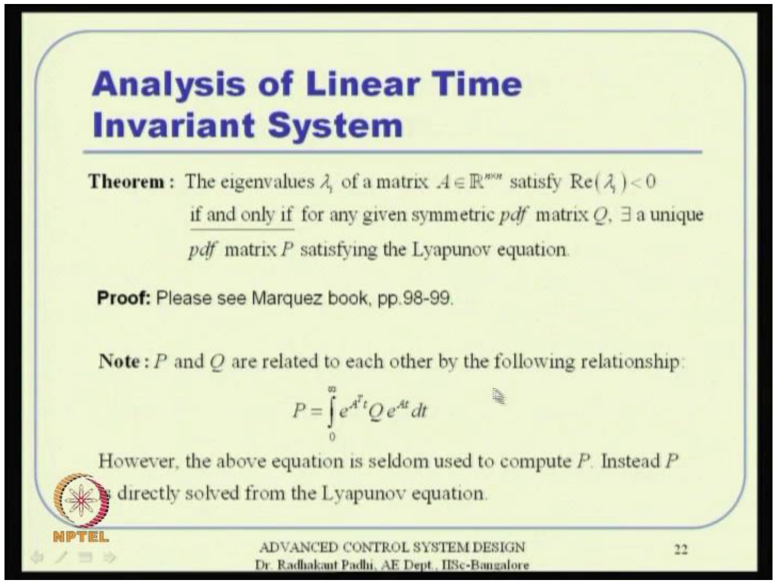

There is a theorem which winds up this property and all that actually. So, this theorem tells us that the Eigen values lambda e of a matrix A which satisfies this real part of lambda e less than 0. You find only for any given symmetric positive definite matrix Q there exist a unique positive definite matrix P satisfying the Lyapunov equation. That is exactly what I told now. You start with a Q positive definite and then solve for P. Then, this theorem minds of that this if and only if conditions are nice to see because if they are both necessary and sufficient actually.

So, you just try out one time and then you have done actually that way and if you are interested in seeing the proof part of it you can see this book page 98-99. And this P and Q are also related by this expression that is P k and Q. I mean, if you know Q you can evaluate P but it is seldom done that way actually never we never bother to evaluate P that way we simply solve this equation to get P. I mean, if you solve this equation you will get P anyway so but this P and Q whatever solution you are getting they are also there, this relationship will be satisfied. So, that is probably you can say that is verification tool or whatever actually.

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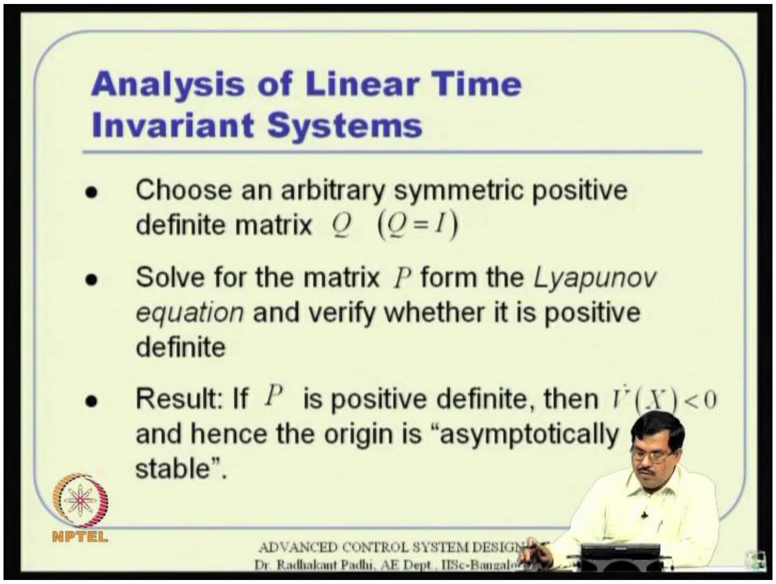

See, this is what I have already told; so choose an arbitrarily symmetric positive definite matrix Q. You can select any positive definite matrix Q by here just solve the Lyapunov equation once and see whether the P matrix is positive definite or not, that is it. Linear system theories are nice in that sense actually the results are one sort (( )) just one time. We say check out and all that so if I am if I have a liberty of selecting any positive definite matrix Q then, obviously my natural choice will be identity. Identity is a positive definite matrix. Simplest form of positive definite matrix is, identity. It is all diagonal elements are one one one all are positive and since it is positive definite matrix and algebra we cannot simplify for me actually so I just select Q equal to identity.

Then, solve for P matrix using Lyapunov equation and verify whether it is positive definite or not. And if P is positive definite then V dot is guaranteed to be negative definite and hence the origin is asymptotically stable actually. And as I told before, you can, looking at the Eigen value properties and all we can also show invoking relaying in inequality and all you can show that system is exponentially stable actually.

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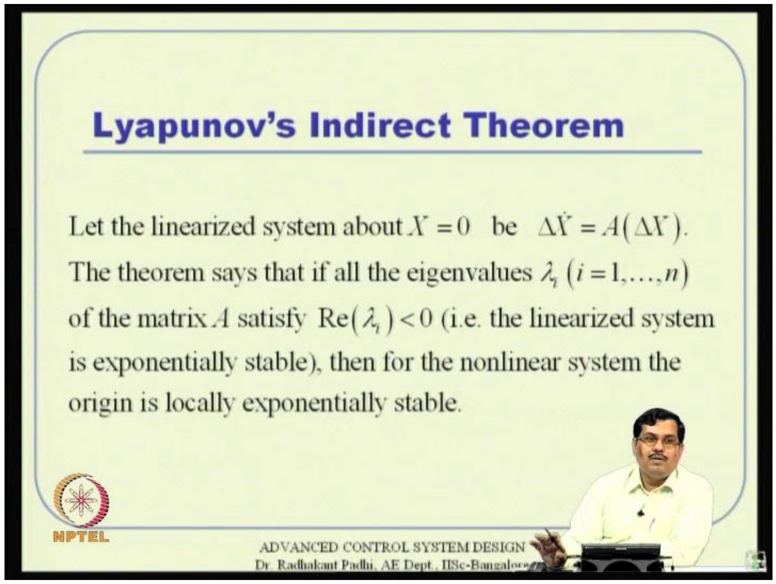

Now, those theorems that you discussed here is all like Lyapunov's direct theorem and there's a Lyapunov's indirect theorem which we heavily used in the. I mean, in engineering problems and all that. So, what it tells us, it tells us I mean, that is why this Lyapunov's indirect theorem is the theorem in my view for which the linear systems theory is very popular. Otherwise it will not it will never have been that much popular actually.

What is that theorem which tells you that this theorem tells us that let u s linearise the system about this equilibrium point whatever non-linear system we have and we get this delta X dot equal to 8 times X. By linearizing the system, original non-linear system above the equilibrium point and then this theorem says that if all the Eigen values of this matrix A lambda e they satisfy this left hand side property basically. That means real part of all lambda is should be less than 0 that means the linearized system is exponentially stable. Then the non-linear system that then for the non-linear system the origin is locally exponentially stabilized well.

So, all that we are doing is there is a equilibrium point we linearise the system and study that linear that linear system. The linear system happens to be stable it certainly going to be exponentially stable and it also tells you that the non-linear system is also locally exponentially stable. Remember, the linear system is globally exponentially stable but using that property you can always say that non-linear system that I started with is also exponentially stable. So, that is the powerful theorem actually in that sense actually now that is why this linear systems are heavily studied and they work and all I mean, they work wonderfully and all that actually.

Anyway, so this is the Lyapunov's indirect theorem and easy way to do that also. Just linearise the system and study the Eigen values and all the Eigen values are in left hand side. That is what we know first under linear systems theory then not only this, the linear system is exponentially stable but the non-linear system is also locally exponentially stable. The domain of stability can be less that is another issue actually so the global is the linear system the domain of stability can be large but the non-linear system the domain of stability can be very less that is where you need to study non-linear system but never the less we will be able to claim that my equilibrium point is at least locally stable basically.

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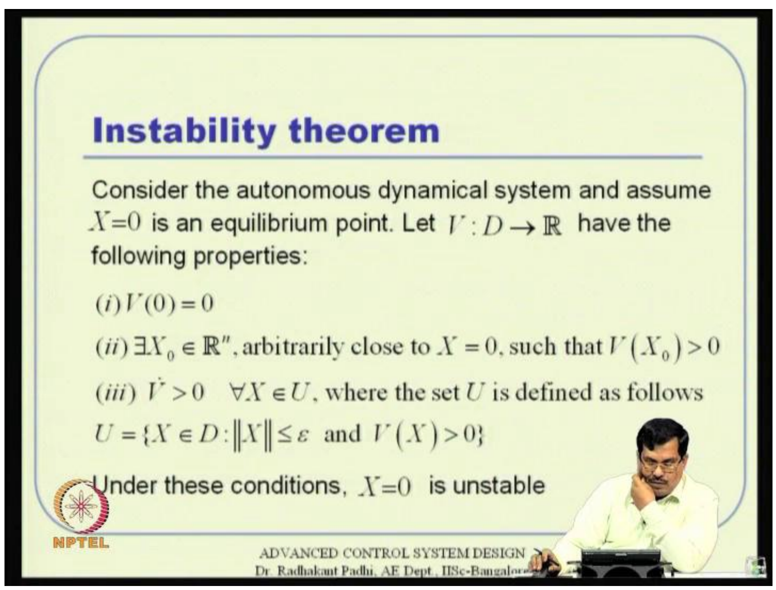

Anyway, so what about instability theorems? See if you let us say we are somehow unable to conclude anything. Like we tried out many Lyapunov functions and think like that started with let us say X transpose P is they did not work out then we started with this the kinetic energy plus potential energy did not work out, then it had we something else also did not work out then how many times you will start actually that means that gives us some sort of a hint that probably who are working in a wrong problem actually. That means may be the system is unstable and I am unnecessarily trying my I mean, wasting my waste in trying to prove that system is stable.

What about exploiting the possibility that system can actually be unstable? And, that is where you will see a number of indirect, number of instability theorems. They are not really in Lyapunov theorem. But Lyapunov like theorems actually, we will see some of this at least one condition which will search like this actually. Consider the autonomous dynamical system and assume that this X is equal to 0 is an equilibrium point and V D to R have the following properties and V of 0 is 0 and the second condition remember, this V that were studying here is not necessarily positive definite that is why this conditions are necessary.

The V D to R should have this property that means V of 0 is 0 and there exist some X 0 arbitrarily close to X of X equal to 0 such that V of X 0 is positive. That means, if it happens to all such X 0 then only it is positive definite if it is arbitrarily close whatever happens but all X 0 like that. V of X 0 happens to be positive then it is positive definite but here the condition is much much much lesser all that you are telling is that is only one point somewhere which is close to X X e for which V of X is positive that is sufficient to show instability.

Then, we are telling that V dot is positive definite but actually is not positive definite per se but V dot is positive for all X in that U set U where U is defined as follows actually. That means U is X belongs to D such that norm of X will be less than epsilon and V of X is greater than 0. All these things I mean, kept aside if in a simplistic language if you see if I start with the positive definite function and so that, V dot is positive definite also then obviously system is unstable.

Why does it happen and in other words a positive definite function as you see that it also happens some sort of energy content behavior. That means any positive definite function I can associate with some sort of a energy of the system may not be kinetic plus potential but some sort of energy content there and if the energy content of the system keeps on increasing V dot is positive the what are that mean actually keeps on increasing increasing then obviously my system is unstable actually. But what condition tells you? The theorem tells you, really do not have to show that V dot is positive definite all that you have to show is V dot is positive in this set and this V of X 0 is positive for one X 0. So, the conditions are very I mean, kind of weak conditions and is hence it is easier to show probably. But intuitively speaking if you I do not have to really worry about so much about that because if V of X happens to be positive definite and V dot is also positive definite then certainly it will it will satisfy these condition.

So, that it is what we are worried about but V dot positive definite is difficult to show. So what you want to show is V dot is greater than 0 in this particular set. So X belongs to this domain D such that norm of X will be less than equal less then equal to epsilon and V of X is positive actually. So in this particular set if you show that V dot is positive it is sufficient to tell that the system is actually un stable actually so under this conditions it turns out that X of 0 is unstable.

Now, let us let us review that quickly what we studied in this particular lecture. So we started with some sort of like motivation and tell then tell why this Lyapunov theories are important explained and stability behavior of non-linear system is not necessarily global unlike linear systems and all. Then, we will study we particularly are interested in X dot is f of X and if it is X of f of X U and U is also a function of X that means tested feedback design then close loop since it is Taylor homogenous system.

So, this motivation intake to studied further X e turns to be an equilibrium point but we change a variable and all that we will be able to tell that X e 0 and all that actually. We will I mean, we change a variable what we discussed here in the next further theorems and all X X e 0. So, if that original system equilibrium point is non 0 then you have to do this coordinate transformation first and then interpret the system in terms up variable z; then, proceed further to show the Lyapunov theorems and all that.

Then, we studied some sort of this definitions it is a open set connected set think like that. Then, what is called a stable? What do you mean by stability equilibrium? What you mean by unstable equilibrium? What you mean by convergent equilibrium? That means, the system trajectory comes come ultimately comes back to the origin in between it can deviate large and think like that and it happens to be both stable as well as convergent then, you call that is asymptotically stable equilibrium point.

Then, there is a notion of exponential stability that means not only it goes to 0 and it remains bounded on the way also but the rate of decay is kind of exponential decay basically starting from this difference that my original distance and any point of time I will be distanced from origin. These two distances are the kind of related by this in equality condition and then, there are definitions like positive semi definite negative semi definite all sort of things this is the definition we have to select. I mean, some particular V of X scalar function of course, which satisfies that I mean, this domain D contains origin equilibrium point as well as these conditions satisfies then it is positive definite semi definite. And, then hence positive definite if it is less than I mean, greater than equal to becomes greater than means positive definite and if you replace that by less than equal to and strictly less than and then it becomes negative definite thing.

Then, there were couple of theorems that we studied we told this is a like stability theorem first weakest notion of stability all that we have to do is start with a positive definite for a Lyapunov function and show that V dot is negative semi definite. And then there are couple of candidates that we discussed V of X we can start with like X transpose X or you can start with kinetic energy plus potential energy and then V dot is something that you compute like is partly selection and partly system dynamics that will come into picture.

That is V dot is something that we need to analyze here ultimately and all these conditions that we discussed are all sufficiency conditions actually. So, if you want to tighten up the condition and all from stability to asymptotic stability then, V dot has to be negative definite negative semi definite is not good enough and if it is global behavior you want to study then V of X has to radially unbounded. Remember, we this is still V of X that means we can select a V of X which will satisfy that.

With respect to that selection V dot of X has to be negative definite it then only it can talk about global asymptotically stable behavior actually. Then exponential stability these two additional condition in addition to asymptotic stability condition then we start with these example kinetic energy plus potential energy for the pendulum and pendulum what we discussed here is a regular pendulum that means about this vertical equilibrium point I mean, this things like that this is the thing that. We are talking this is my theta dot all actually .

So this the pendulum problem and we are able to show that the system is stable but the frustration is even if it is with friction; we are only stable to show that it is still stable only I mean, is we are not able to show it is exponentially stable I mean, sorry it is asymptotically stable. But we know that is asymptotically stable so we need further conditions in then that we will study later then this application of LTI system we started with this kind of this X dot equal to X which is a linear time invariant homogenous system and then it tell this has to satisfy Lyapunov equation that means and the procedure is I select a Q which is positive definite then solve this P and then see whether P is positive definite or not actually and that is one sort procedure just one time I have to do. And, if it satisfies the system is X X stable and hence, it is because the linear system if it is stable it is asymptotically stable and it is globally exponentially stable also; that all I can show there. If it does not satisfy then I am done also I mean, one time if it fails I mean, that system is unstable actually then this theorem is all about that what we discussed and this relationship is there but we hardly worry about that you can think about like some sort of availability some check actually.

Then, they are there is indirect theorem and indirect theorem tells us that if the linearized system is stable about the equilibrium point then the non-linear system is also stable globally I mean, also stable in exponential manner about the same equilibrium point but the stability domain can be small actually. Then instability theorem we in addition to viewing positive definite V dot has to be positive definite in a rough way but the conditions much milder and these are the… this the one condition; that is to satisfy remember, instability theorems. There are many theorems you can if it is this theorem this condition you are not able to show; you can show another theorem and all that you can. So, these are not, we are not talking here with that I will probably stop this lecture; thank you.