Advanced Control System Design
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Lecture No. # 25
Optimal Control Formulation Using Calculus of Variations

Hello every one, we have discussed calculus of variations last class, we have reviewed the concept as well as some of this fundamental theorems and all that primarily Euler Lagrange's equations associated with boundary conditions things like that. Now, using those concepts we will be able to formulate optimal control problems. And we will see the applications and rather directly here actually. So, this particular lecture, we will talk about optimal control formulation using calculus of variations ideas specific.

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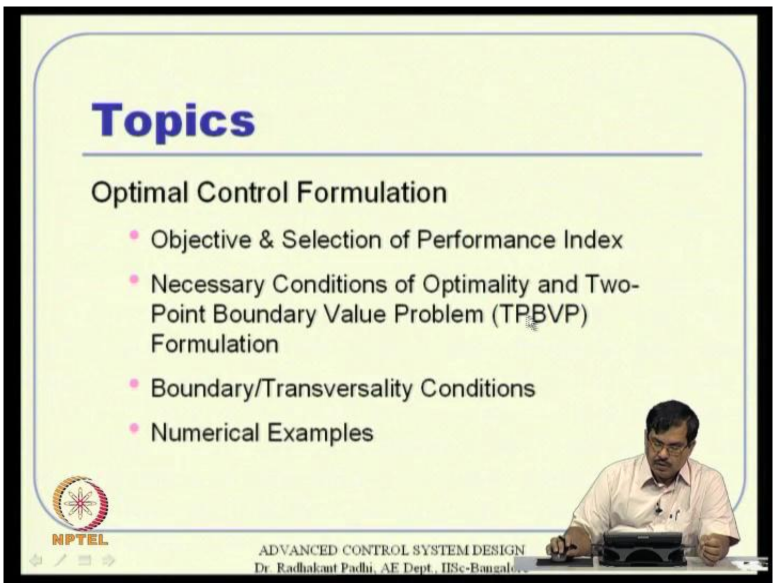

So, the topics to be covered in this lecture is something like this, first we will have an idea of what is an optimal control problem, what do you mean by that. And what is the objective of this particular formulation, and how do you select these performance index actually that which is a critical component actually. And then how it leads to this two point boundary value problem formulation, that is a important thing, because that is the most critical point in why this problems are computationally difficult in general. We will also have some associated boundary and transversality conditions of followed by numerical examples actually. So, let us see what this optimal control formulation means, what is objective, and how do you select this performance index and things like that.

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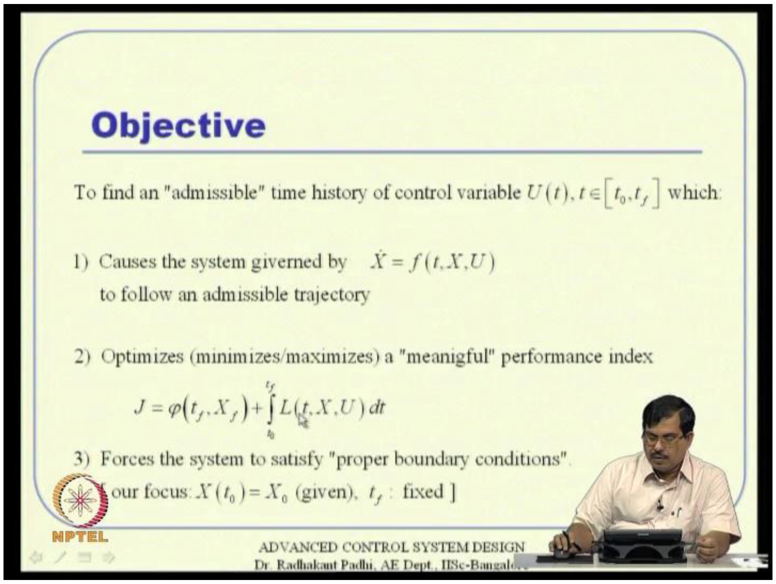

The objective will (( )) is something like this, we need to find an admissible time history of the control variable U (t) from t 0 to t f, which satisfies all these conditions, number one, two, three. And one is the first condition is it needs to satisfy the system dynamics, that means this control is to U should cross. The system governed by this differential equation, which essentially based on system dynamics to follow an admissible trajectory.

And while it does that, it also needs to optimize certain cost function and that is to be meaningful actually. And many times it is called as performance index, some people call it as cross functional also basically. So either you want to minimize it or maximize it to why, I mean while finding out this admissible trajectory, we need to also keep in mind that this cross function needs to be optimized. Associated with it needs to satisfy certain boundary conditions, in the boundary condition in our case is initially condition is like this, like t equal to t 0, your initial condition is X 0, which is specified. And at t equal f X, f is typically free and we will consider rather t f is fixed.

Remember that, this is the boundary condition since many things can be flexible out here, I mean in other words X 0 need not be fixed t f need not be fixed you can have see essentially, we have t 0 X 0 and t f X f combination. So, any such combinations (( )) and here typically interested in this type of a condition, because this is practically more relevant.

We also see that this the typical form of the cross function, that we take here is also kind of practically relevant from many, many example, problems actually we will see that next.

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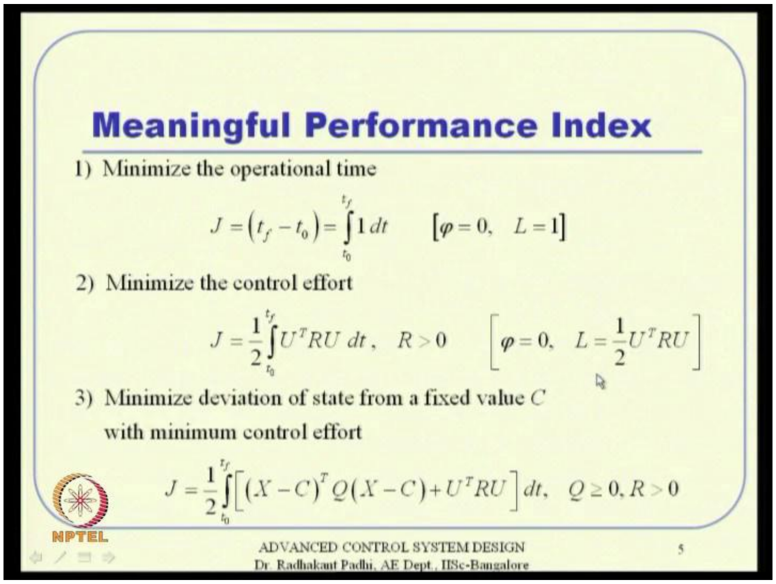

So, without I mean if you see this cross function, if this is this entire formulation, this state equation or this system dynamic equation is something that is given to us, where we do not have to much also as in that, but while formulating the problem number two and three will typically play a role. And this lecture, we are fixing three also that means we are interested in this particular case, where t 0 is fixed, and initial condition x 0 is also fixed t f is also fixed, the only thing that is free is x f. So, but still we have the flexibility or freedom of selecting this in a meaningful way, how do we select that? So, let us give some iterative examples, which will type of make our radius clear. So, let us say we want to minimize like the operational time actually, so this some particular application, we are interested in minimization of operational time.

That means, we want to minimize this t f minus t 0, which essentially I can write it that way integral t 0 to t f 1 d t, if you go by you can see then, this phi in this context turns out to be 0, and in this in L turns out to be 1. Similarly, if you want to have minimum control effort, that minimize the control effort then essentially, we want to minimize this kind of a cross function, where R is equal to difference matrix, remember this is like a quadratic cross function that means we are minimizing this something like half of R 1 R 1 U 1 square plus R 2 U 2 square plus R 3 U 3 square like that actually.

If you take R as diagonal matrix with most of the time this R is taken as diagonal matrix, any way. So, to minimize the control effort, we want to minimize such a quadrative term and similarly, if we that means if you select this cross functions that way. And if you again go by it then phi turns out to be 0 and L turns out to be this one half of U transfers R U. Similarly, if you want to have a different problem, where you will very essentially, want to minimize the deviation of state from a fixed value C with minimum control efforts this is typically, you can related this to like let us say helicopter robbery.

That means, the helicopter needs to stay at a fixed positions C with that needs to be done with a minimum control efforts, then you know the position C. So, essentially you want to formulate a cross function that way. Where you can talk about the deviation of X from C needs to be minimum actually. So, that means I need to have a this quadratic term for that particular job, as a and this particular term makes, the that we have minimum control effort also basically. And whenever, you see this quadratic term containing a state in control, typically the conditions like, conditions of like Q s p positive senario that is say, it is if you go back to this particular cross function form then obviously, in this case also phi is 0 as well as this L turns out to be everything inside that with half actually.

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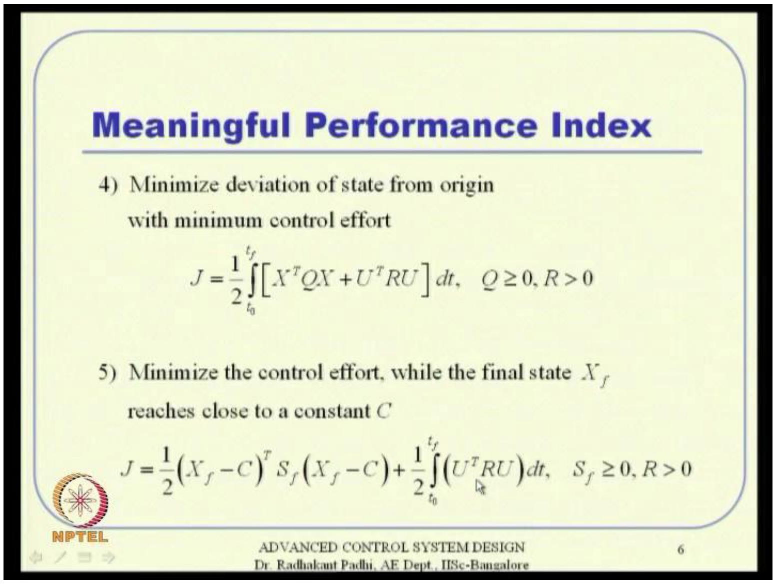

Now, another one suppose, you can talk about minimize minimum of deviation from state, I mean minimize the deviation of state from origin with minimum control effort, like kind of a special case, for this and this is essentially the regulator problem. So, later we will see that, this linear quadratic regulator to non typically assume, this such a cross function actually. And here also again phi is 0 and only is the center thing including half actually. So, that is kind of a thing suppose, we want to another problem where you do not want this path dependent control, I mean minimization of state, but we want to minimize the state at the end when t equal to t f, we want to minimize the same on the way we do not care as long as the control is minimum.

So, then we know the t equal to t f. The let us say like aero specifical, then you take and think about something like a position of the destination airport basically probably. So, if you really want that t 0 to t f at t equal to t f then, your position of the aircraft from the destination airport should be as minimum as possible that way. So, then this turns out to be phi in this situation, because outside the integral and the L is a is half of that actually, if you correlate to this is anything that is outside is t f dependent on the final time dependent, anything that is inside is the path dependent is the t f actually. So, again in this situation this turns out to be phi and this turns out to be L.

Now, optimum control is in calculus of variation that is that 7 10 f formulation. So, that is I mean these are all this examples, as given some idea that this cross function that, we had discussing here has sufficient general iterative talk many different class of problems. However, also remember that this is also not the exhaustive state actually, we can you can in a even think beyond this particular form of cross function. And that is at a (( )) valid as long as we got this solution actually. So, those things will not discuss in this lecture.

So, let us go back and see how do you use this calculus of variation ideas and things like that. To come up with some solutions to this class of problems actually, remember we talk of optimum control formulation depending on the problem, you need to select J is nobody is giving also, we need to select ourselves. And phi also, I mean this phi L also nothing and as long as, I mean along with boundary conditions and everything we need to formulate it properly. But once you formulate it properly, then the solution procedure and all how do you find out that is the thing that we had going to talk next actually.

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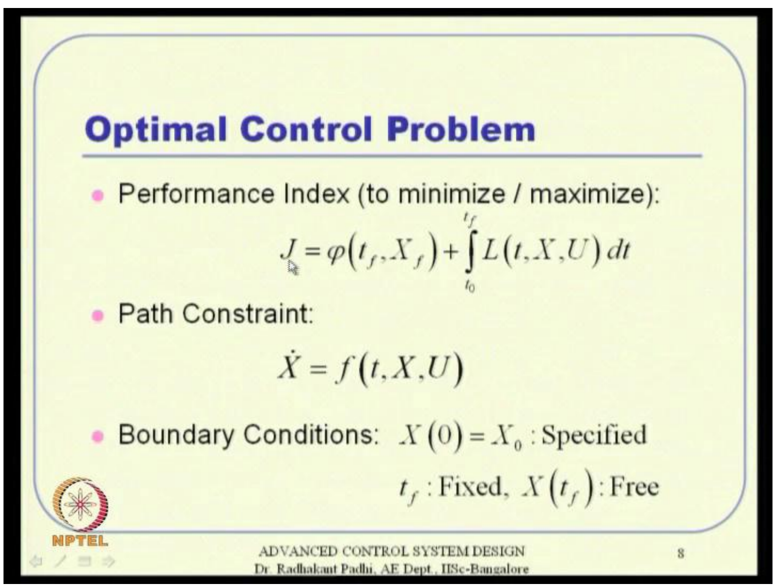

So, let us see go to that and then tell optimum control problem is something like this, we want to minimize or maximize a certain performance index or cross function, which is of this form subject to this path constraint, which is system dynamic equation X dot equal to this f of t X U systems in general. And the boundary conditions are needs to be kind of well formulated. So, for example, here we had, we were talking about t 0 is fixed so at 0 and X X of 0 is X 0, which is also specified t f is fixed except t of x actually.

So, how do you do that, now if you remember this starting optimizer and all this whenever, you have a equality constraint, we actually had something like a augumental performance index, they were with which is like lambda transpose times this, I mean this f of t X U minus x dot equal to 0 that way. Now, here also we will proceed with the similar idea, but here we will tell this is a path sculpture, which is valid from t 0 to t f everywhere. So, this path constraint this augmented cross function, J bar needs to have this particular term inside the integral actually, that is what you are doing there?

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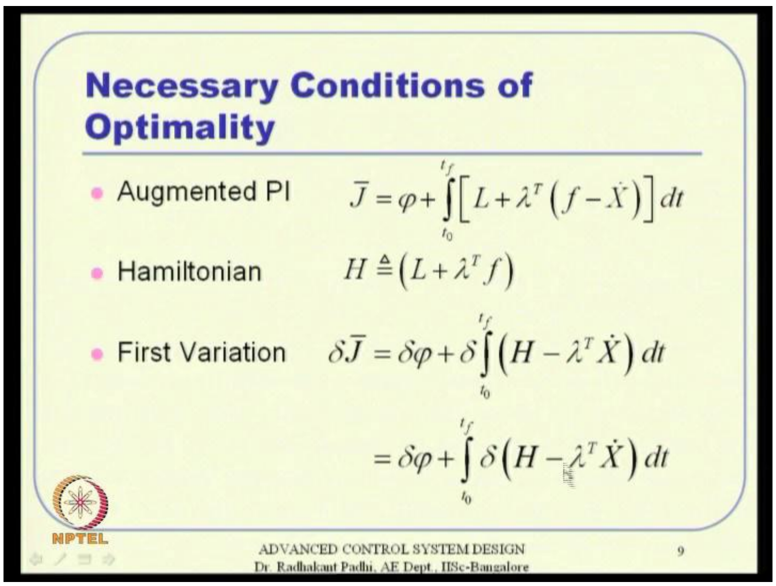

And for the sake of simplicity am dropping out this augments and all that actually. So, J bar is nothing but phi plus L plus this lambda transpose times f minus f X dot remember, f minus x dot is equal to 0 that is the path constraint actually. So, that has gone their actually. Now, for further simplicity what you do here is this derivative independent or what you say or L plus lambda transpose f is actually, independent of derivative. So, we are just defined that simultaneous this is some of standard definition, which helps us in algebra simplicity basically. And whatever, is derivative dependent to we will live it as it is actually.

So, essentially, J bar is nothing but phi plus integral t 0 to t f then, H minus lambda transpose X dot. And hence, the first variation of that will turn out to be like these variations and all remember, they are like linear operator that is that definition. So, this variation of J bar is variation of phi plus variation of this one. And now, remember this the theorem in the calculus of variations, which talks about variation of integral is integral of variation this. So, will within that we will do take this variation, inside the integral and then again this variation of this particular term, will take a make, I mean will take advantage of this linear operator property of this variation.

So, this is dell phi plus integral t 0 to t f variation of all this term, where you will expand this like linear terms and all that actually, like variation being linear operator like that. So, let us expand that inside the bracket and then tell there is a first, I mean the first variation of which minus the first variation of this with this quantity. And again this first variation of multiplication will satisfy, this derivative sort of property their and using that we will be able to expand that. So, this is the first variation of dell J bar is nothing but dell phi plus integral this dell H. So, we will keep this term as it is minus dell lambda times X dot minus lambda transpose times dell X dot. So, that is that is coming from this very first variation of this expanded form actually, this is what you have.

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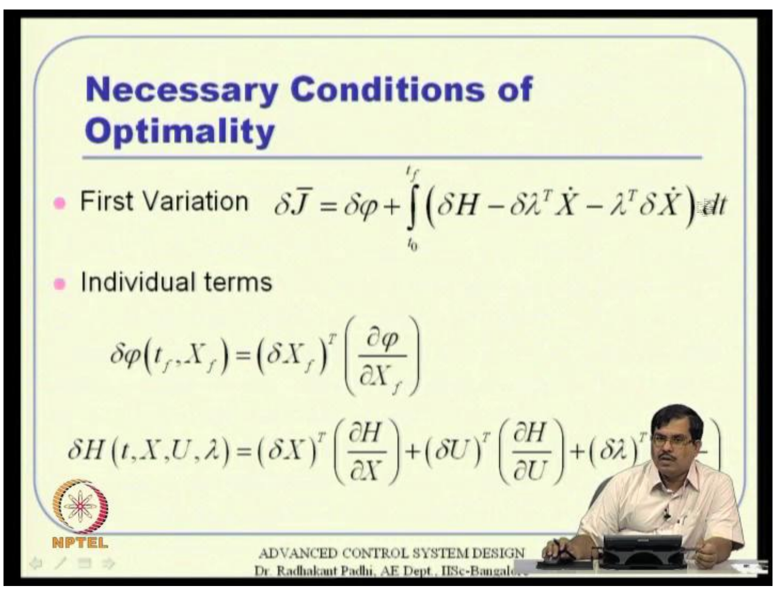

Now, let us analyze this term an individually, term by term sort of expansion and all that. Now, this is dell phi for example, so phi is a function of t f X f and remember this, we are talking about t f is fixed. So, there is no variation on t f s actually so the variation of phi comes through the variation X f only basically. So, enhance this first variation of dell phi turns out to be like this dell X f transpose times dell f or dell X f. And the first variation of dell H is something like so remember H is a function of t X U lambda everything basically, because H is nothing but L plus lambda transpose f L m lambda are function L is t X U lambda, I mean this lambda a function of time and then it is a function of everything actually.

So, first variation of H will come from variations of X U and lambda remember time is an independent quantity in calculus of variation, we can where the flexibility of starting some where we want. And then stopping somewhere, you want in other words t 0 and t f can have some variation, but on the way once you fix the t 0 and t f there is no variation on t this is an independent quantity actually. So, because of that this first variation of Hamiltonian takes this form actually. So, we will consider that as variations coming from variation of X and variation coming from variation of U and variation coming from variation of lambda. And hence, this expression this expansion holds good actually.

Now, what about the next term actually, now next terms contain remember, these are like derivative terms and all. And this particular term, we will keep as it is but this particular term or some quadrate talks about something like variation of derivative so, which is not kind of so much comfortable. So, we will let us try to examine that and we will examine in the context of integral directly, because ultimately we want to put integral back actually.

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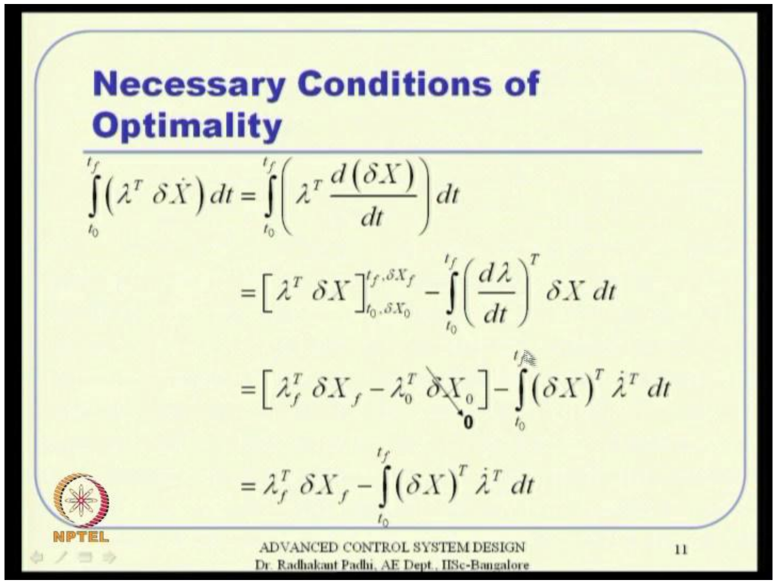

So, this integral this lambda transpose dell X dell X dot that is what, we want to analyze and by definition this is d by d t of d X, because variation of X dot is d by d t variation of X. And then we said that, I mean this integral by parts actually, so this is the first function, that is the second function, we put that integral wipers. And hence, this is this first term is something like this, evaluated at this values t 0 and L X 0 into t f dell x f minus the derivative of first term into the integral of second actually. So, that way it turns out to be like that actually.

And hence, if you put them together, this turns out to be like lambda transpose dell X f minus lambda 0 transpose dell X 0 this particular term, but remember this there is no variation of X 0, we are interested in only fixed initial condition cases. So, we obviously this case goes to 0 actually. And hence, when the they are left out with only this term and because of that, we in a simplified way we can write it that way actually. Remember you can exchange this term, based to that and this is primarily, because whenever we have a two vectors multiplying each other of the same dimension, one is row and one is column.

So, the product is scalar then the multiplication is interchangeable, in other words x transpose y is equal to y transpose x, because both are the same quantity, which is nothing but like x on y 1 plus x 2 y two like that, which is equal to y 1 x 1 plus y 2 x 2 like that actually. So, because of that this x transpose y is equal to y transpose x and that is what we have done here actually, because they are two vectors of same dimension multiplication of that both the vectors ultimately a scalar so taking advantage of that here. Now, we are ready with all these terms, because this is something we do want to do anything, we just live it this is something, we have expanded this is something we have expanded. And this integral sense also, we have expanded. So, let us as go back and try to put everything and then try to see what delta J bar takes actually.

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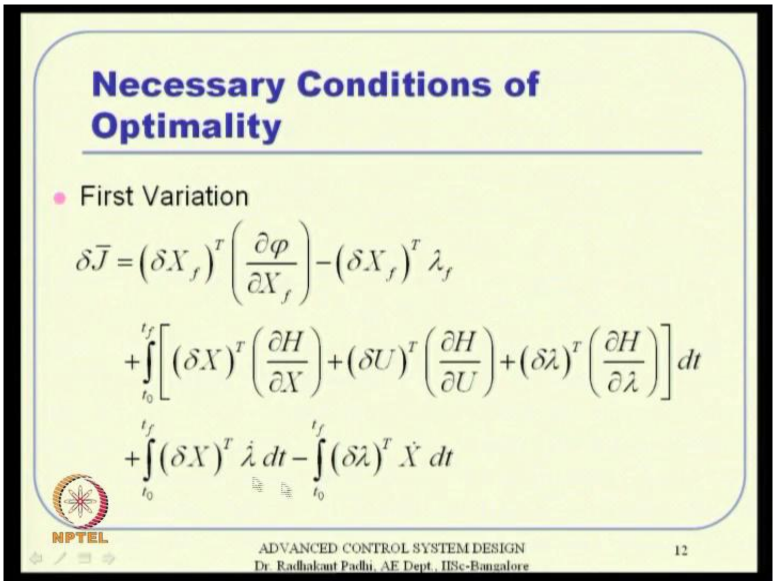

So, if you do that this then delta J bar takes that form. So, first term is a is coming, because of that this term you just keep it, that way minus this dell X, I mean dell X of transpose is because of that remember, ultimately this is a minus term here. So, we have this (( )) and there this is minus term coming here, which is like that again we can exchange this quantity, because here again same two vectors of same dimension are ultimately the product is a scalar. So, we can except that and write you that way. And the other terms, we just keep with that this is various, first variation of dell H this one. So, this term is nothing but that so we keep in there whatever, we have. And then the other term is like that. And the left out term is like that it is a remember this term will become positive later, I mean because of this sign change and all.

So, once you put that together all these things and then try to combine terms as much as possible. For example, I have dell X f transpose here and dell f dell X f transpose here and both are acting to the left actually. So, I have a liberty of talking common to each other. So, I mean just I take that common similarly, if you see all this terms are integral quantities, all are at the same integration limits t 0 to t f. So, I can put one integral in I can combine terms, within that as well. For example, if I see these two terms this dell X transpose is multiplying to the left, I can combine this two term and similarly, this is dell lambda transpose, if we are into the left I can combine these two terms.

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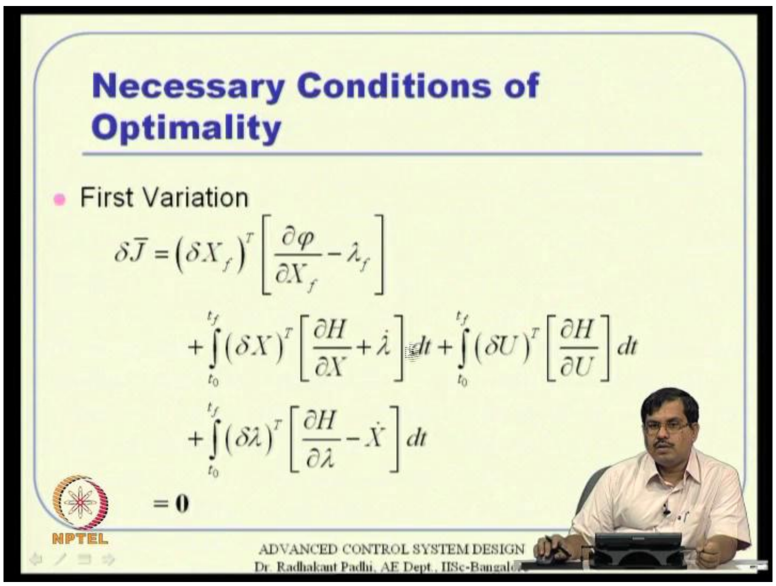

So, that is what am doing here, I have combined this all the terms here where only thing that is left out is this one, because that dell U transpose acts only here actually. Now, because of this I mean ultimately, we were left out with this and remember the one principle of this optimization, necessary condition from calculus of variation is the first variation of dell J bar that is delta J bar needs to be equal to 0. So, that is what we want ultimately. And then, if this condition happens for all such a all sort of variations that is that is the another theorem, that we outline last in the last lecture, we talks about if we (( )) all possible variations, then only where that it can happen is the coefficients, needs to be 0 actually.

So, because of that, we collect coefficients various coefficient whatever, happens here. And then we put then together and make it equal to 0 here for example, this coefficient has to be equal to 0, this coefficient has to be equal to 0, this one has to be 0 this one has to be 0 like that. So, once you do that, the first equation pops up in something here X dot is dell X by dell lambda and dell H. And then, if you go back to the H definition, dell X by dell lambda is nothing but simply f actually.

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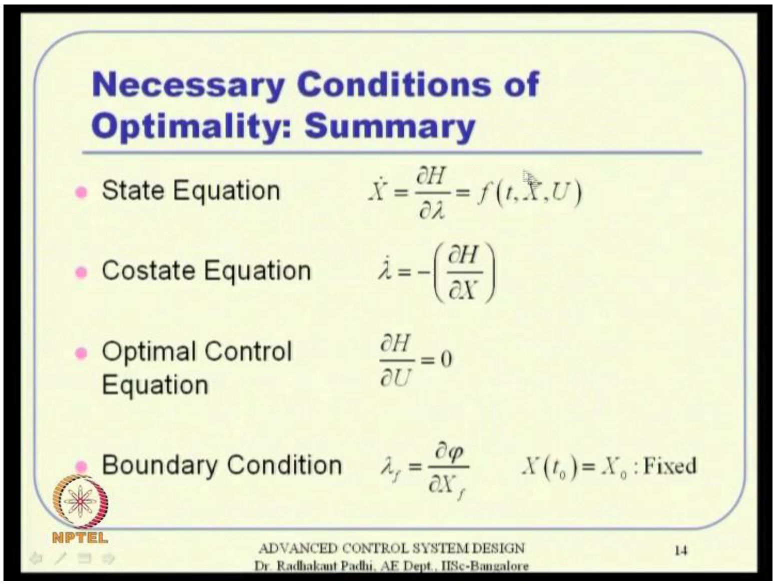

So, that is what we are trying to see here so X dot is dell H by dell lambda, which is nothing but f. So, that is f and X dot is f of t X U is nothing but this state equation that we started with. So, the system dynamic equation or state equation becomes part of the necessary condition actually, that automatically happens that way. And then the first equation will pop up from this one. So, this coefficient if you make it 0 then lambda dot equal to minus dell H by dell X. So, that is what happens, here and then optimal control equations, there is nothing dell U transpose is only getting here so dell X by dell U has to be equal to 0. And in the boundary condition, we have this term equal to 0 so lambda f is dell phi by dell X f.

And obviously X of t 0 is X 0, which is known to us fixed. So, what is happening here if you see this these quantities, there are couple of things to observe first a fall all this conditions are necessary conditions, because they use first variation only second variations and all never explode. So, the it can happen to be either maximum or minimum, if you satisfy all these equations, but from the nature of the cross function that you we select, we can tell of either, (( )) minimization or maximization problem iteratively, because cross function is something that we are selecting. So, that way we do not need to much of further, mathematical testing, and all we can (( )) it can only minimum or it can be only maximum, then one is good enough any way.

So, that is in how we use this results for engineering problems actually. Now, second point we have a bunch of equations here and, if any one of this equations is not satisfied, then the entire solution is non optimal. And the solution can be very far away from optimal solution also actually. So, just by changing the boundary condition for example, we can have a different solution from by putting out the boundary condition that way. So, just makes you at that all these conditions, as it is (( )) this one is left out then, the entire solution is non optimal actually.

Third point it, if you see this X n lambda the dimension of X n lambda turns out to be say simultaneously, scale quantity dell H by dell X will give you the same dimensional vector is X actually. So, lambda and X are of same quantity, I mean lambda and X are of same dimension, in other words this is also n by n n by 1 and x is also n by 1. So, essentially to solve the n by 1 problem, we have to actually, solve this 2 n by 1 dimensional problem through lambda actually, using lambda basically. Now, if you also remember you will observe these equations carefully, the first two equations are actually dynamic equations that contains a derivative term, but the third equation actually, a exotic equation there is algebraic equation actually.

But all these three equations are valid from t 0 to t f everywhere actually. So, from t 0 t f you satisfy, all these equations, out of which two are dynamic equation and one is static equation. In the boundary condition since, it is that is the most difficult thing by the way it turns out to be that this part of the initial condition, part of the conditions have given at t 0, which is initial condition. But part of the condition are given at t f, which is final condition, remember the entire problem lies with two particular variables, X n lambda n two n plus I mean two n by 1 dimension actually.

Out of this two n variables n variables are given at t 0 and n variables are given at t f really. And because of which this sets the condition, what is called as two point boundary problem. And hence, it is difficult to solve also, because you cannot directly use this numerical integration techniques, to propagate this equations either in forward time or in vapored in time. So, you have to do this (( )) calculations several times repeatedly in an in iterative sense to get the solution.

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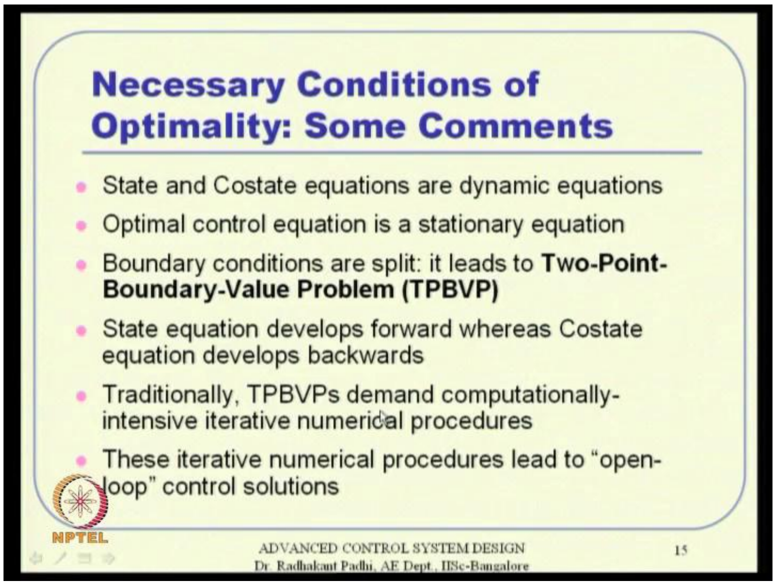

So, all these points are summarized here so first thing to note is state and costate equations are dynamic equations, optimal control equations, which is this equation sometimes it is also called as stationary equation (( )). And actually a stationary that means it is actually, a algebraic equation. And, if you solve this algebraic equation, attempt to solve then U becomes a function of X n lambda actually. So, for getting this solution from here, we really need the need to have an idea of both X as well as lambda, I mean that is the difficulty. And third point, because the boundary conditions has split is essentially leads to two point boundary value problem.

And that is the point, I have explained already so that the problem is in two n dimension and n conditions are given at one point of time is initial time and (( )) are given at the final time actually. So, that (( )) the case of two point boundary problem, which is which are usually difficult to solve. And also, remember the state equation is give in this form, but the initial condition is given, that means the state equation double of (( )) in time, where the costate equation if you see in the boundary condition is at t f the equation is like that. So, it is actually by quadratic. So, that is the state equation develops forward, where as the costate equation develops really backward.

And traditionally, this two point boundary value problem, you want computational intensive iterative numerical procedures. And some of that we will see in the next class and these iterative numerical procedures even, if you do it not only computational intensive, but essentially you are getting a solution for that particular initial condition ultimately. So, if your initial condition is somewhere else, is then you have to really excite that solution loop one more term. In other words, say because it Is initial condition dependent the solution is really not in not in close loop.

So, essentially what you are getting here is ultimately, even after this iterative numerical procedures, we are getting (( )) open loop solution actually, which is typically not good in practical implementation of course, there are ideas how to augment this control solution, with a little deviation killing controller to make it operate on a close loop actually. So, at we will probably see in next class.

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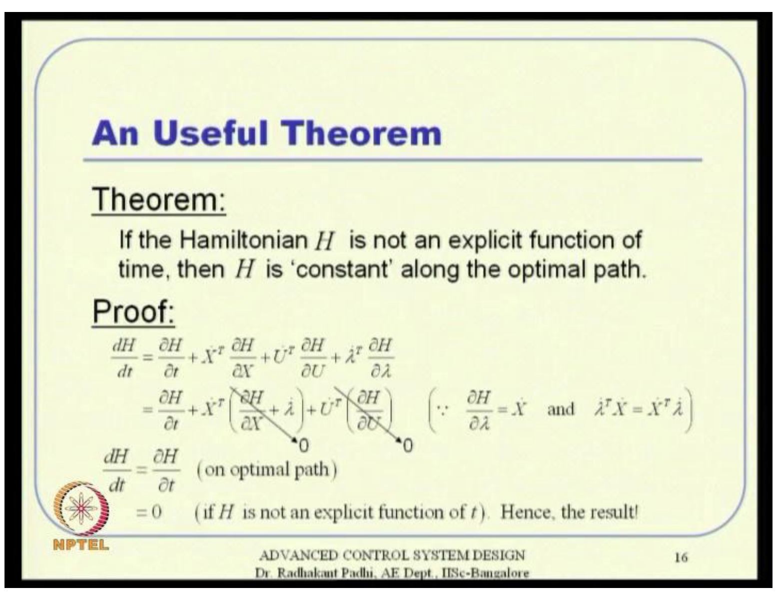

Now, before proceeding further let us observes this couple of things, first useful theorem is something like this, which tells us that the Hamiltonian H, if it is if Hamiltonian H is not an explicit function of time, then Hamiltonian H is constant all along the optimal path. So, how do you do that, how do you prove let us say there is a very state forward proof rather, which tells about if you take total derivative of Hamiltonian d by dt of H, they are nothing but partial derivative of that plus this terms actually, if the X dot transpose times dell H by dell X plus like U transpose, I think like that.

And then vertical to here is like a you can combine this two term, you get dell H by dell lambda is nothing but X dot this is one of the conditions. And then you can also except this x transpose y equal y transpose x equal sort of ideas there. So, using all that, you will be able to combine all these two terms. In other words this one, this term and this term can be combined and then written in that form actually. Remember dell H by dell lambda equal to X dot is nothing but your state equation, equal to X dot equal to f of t X C. So, by combining while combining that, we have already used the state equation really.

Now, after combining you can also see that this particular thing satisfies costate equation, because lambda dot is minus dell H dell X. And dell H by dell U equal to 0 since that satisfies, stationary equation at the optimal control equation really. So, using this state equation costate equation and optimal control equation, which are any way true along the optimal path, we are left out with the fact that, the total derivative of Hamiltonian is nothing but partial derivative of the Hamiltonian on the optimal path of course.

And hence, if the Hamiltonian is not an explicit function of time and most of the time it is. So, because the basically that is the truth, I mean the cost function is selected, but most of time you talk about non autonomous system state equation also, the Hamiltonian really is not a function at time, then essentially what happens is the total derivative of Hamiltonian, with respect to time is 0. And hence, if h is not an explicit function of time then this is true and hence what you are telling is H needs to be constant along the optimal path actually, the only condition that is necessary is Hamiltonian should not be an explicit function of time, it is a free variable.

Alright that is easy for useful theorem, sometimes it helps us to validate our results whatever, results we get if you claim that Hamiltonian is really not an explicit function of time, then probably you can evaluate this two derivatives, when this partial derivative Hamiltonian with respect to time and due to. So that they essentially that is 0 basically. So, this essentially is a useful theorem, which helps us in validating sometimes validating our results, Sometimes it may also help us in finding out little new numerical efficiency, I mean numerical efficient of I mean this computational procedure to solve this problems. So, that is the theorem.

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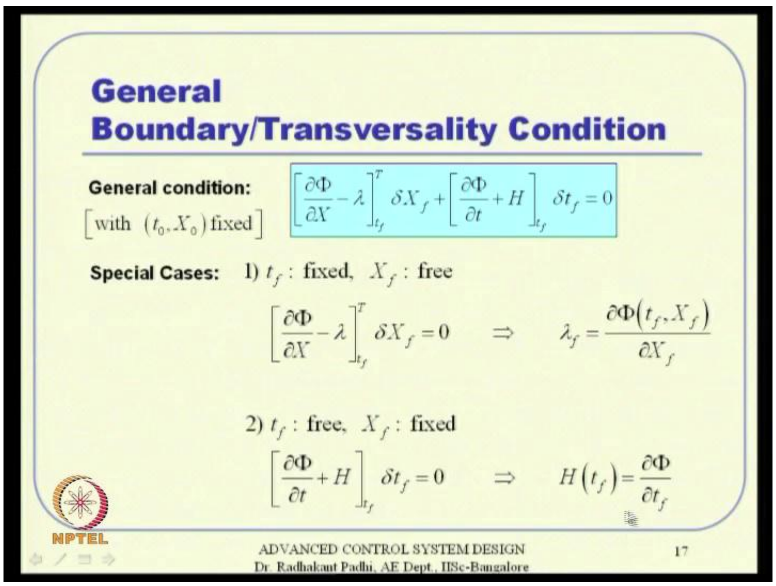

Now, before proceeding further we also notice that there is a general transversality condition or general boundary condition, with which t 0 X 0 fixed if t 0 X 0 is not fixed then also it is possible. And what here we are talking it t 0 X 0 fixed. So, the general condition turns out to be something like this where this final time variation of final time is also included. So, as a special case, if you see if t f is also fixed what f s is free, which we have discussed as. Now, then obviously this t f is fixed variation of t f is not there. So, that means that is 0, so you are left out with this coefficient equal to 0. And hence, lambda f is equal to lambda t f is lambda f is equal to dell phi by dell X at t f, which is already, which you have already seen.

Now, what if t f is free, but f X is fixed, those that can also happen that is a part of the problem from the formulation. Suppose, you want to do that way, then the condition turns out that this is 0 any way, because both X 0 is fixed, that is we no variation on X 0 any way already. And X X f is fixed so the variation of X f is also 0 delta X f is also 0. So, that part is also totally 0 and you are left out with only this term, where delta t X delta t f is not 0 that means, the coefficient needs to be 0. So, what your telling is if t f is free, but X f is fixed that is a kind of a alteration in problem, then Hamiltonian at t f has to be like this dell X dell phi by dell t f actually.

So, this is another boundary condition, because we need one more condition to proceed further, because t f is actually a free variable. Now, so this gives us kind of that one particular condition, which will help us adjusting the t f. Alright so this is all about transversality condition and all that. So, what is the summary, if you given a problem you try to formulate a proper cross function. And you select Hamiltonian is that L plus lambda transpose f and all these conditions are known to you state equation course, costate equation, optimal control equation as well as boundary condition. And all these things, needs to be utilized for solving any particular problem.

And let us say demonstrate our ideas using one or two example problems. Now, we will see hopefully make our ideas clear. So, the first example is essentially, a small problem which is a class room demonstrative sort of a problem, which we will now typically call as try problems also many times.

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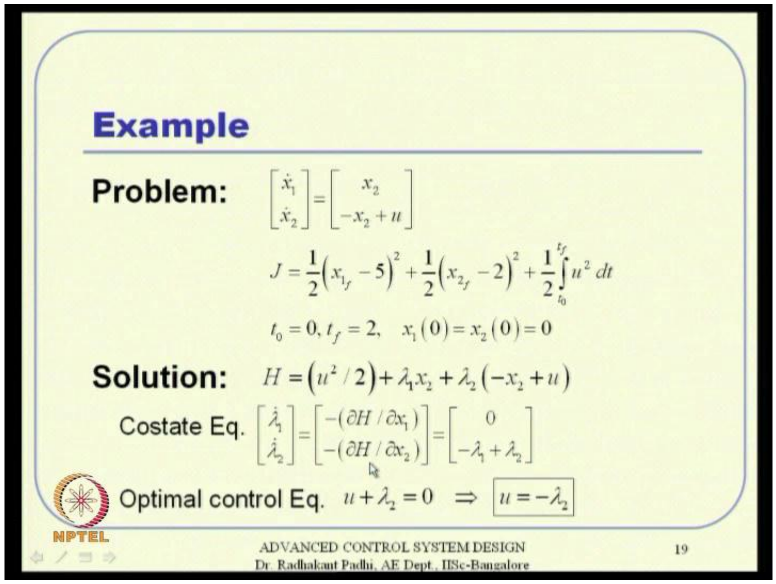

So, problem is like this x 1 dot and x 2 dot is like this x 1 dot is x 2 and x 2 do is minus x 2 plus u. So, and the cost function that, we need to optimize is something like this. So, J is like that and then t 0 is 0 and then t f is 2 that is what we select, I mean if we select something else also, will be possible to solve. And then we are putting that x 1 of 0 and x 2 0 is 0 that means the initial condition for four states are 0. So, what is objective how do you actually formulate this problem. Now, t 0 is 0 t f is to is the kind of obvious you can change it depending on control operation duration actually. Now, this is a typical initial condition, what you are seeming that the initial condition, sense it starts from the origin that means, if the origin is somewhere else, we will see do the coordinate transfer to put the initial condition, at 0 actually basically.

So, that is also not a problem, now how these terms come because ultimately, what we want is the solution should go to 5 and 2 s s close to 5 and 2 h possibly that means, x 1 of a x 1 at t x if it go x close to 5 as possible and x 2 of t f should go x close to 2 as possible, that is why this these are deviation in terms. And that is what you are telling that error square has to be minimum and this error square has to be minimum. And along the way, I also want to mention that the control effort is also minimum. So, that is how this u square term pops out here in the cost function actually.

So, this is how we formulate the problem and let us try to see the solution part of it actually. So, the first thing to start the solution process is defining Hamiltonian you see. So, if you define Hamiltonian properly, I mean then the results are ready any way. So, Hamiltonian definition is something like L plus lambda transpose f. So, L is like this u square by 2, which is coming from here plus lambda transpose f means lambda 1 times f 1, f 1 is this one x 2 plus lambda 2 times f 2, f 2 is that one. So, lambda 1 times x 2 plus lambda 2 times f two basically, so that is how it is there.

Now, state equation is already there with us so, we what we need to try to find out is costate and optimal control equations actually. So, the costate equation tells us that lambda dot is equal to minus dell H by dell x that means lambda 1 dot is minus dell h by dell x 1 lambda 2 dot is minus dell H by dell x 2. Now, here is the Hamiltonian so if you take partial derivative with respect to x 1 there is no x 1 anywhere. So, the partial derivative with respect to x 1 is 0. So, the lambda 1 dot is 0 means naturally, lambda 1 is a constant actually.

Now, similarly, lambda 2 dot is minus dell H by dell x 2 and now x 2 terms are there here so first term is here so that will give us minus lambda 1 remember it is minus dell H by x 2 that is minus lambda 1. And here it is because of minus n is already there so, if you take partial derivative (( )) and then the sign and turnout to be plus lambda 2. So, what is happening here lambda 1 dot is 0 lambda 2 dot is like this. Now, optimal control equation sense here this Hamiltonian here dell H by dell u it needs to be equal to 0. So, H is like this so dell H by dell u is something like first term is u, remember that 2 u you cancelled it to. So, the first term is u and then the second term is partial derivative with respect to u is lambda 2. So, u plus lambda 2 is equal to 0 and hence u equal to minus of lambda 2 basically.

So, ultimately if you get lambda 2 somehow then out optimal control solution is there with us actually. So, and we will see the, see now that getting that lambda 2 solution is actually, can be existive actually. In other words, you may require lot of algebra to get this lambda 2 really.

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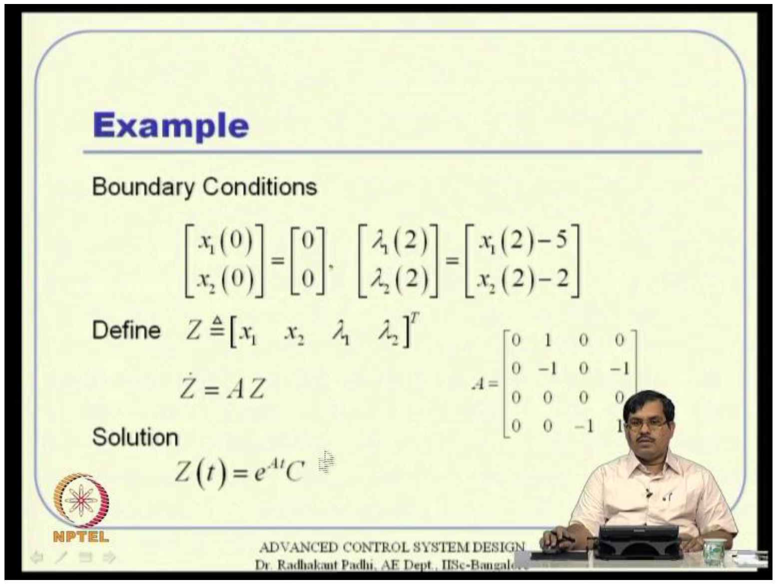

So, let us see that how do you do that and boundary conditions sense, we already know the initial condition that is origin 0 0, but the final boundary condition, at t f equal to two (( )) t f equal to two here, they are nothing but dell phi by dell x f alright. So, your five term is something like this. So, dell 5 by dell x 1 actually turns out to be like that x 1 f minus 5 so x 1 of 2 minus 5 that means lambda 1 of 2 needs to satisfy this equation and lambda 2 of two need to satisfy similar this equation actually. So, how do you do that that is the question? Now, remember, if I put u equal to minus lambda 2 back here then, I can think that this x 1 dot x 2 dot along with lambda 1 dot lambda 2 dot form a complete set of differential equation, they are functions for themselves actually.

So, that is what I do here by defining this new state vectors Z, which is which contains both states as well as costate, first elements are states the next elements are costates. And as I told, if I proceed this equation back in here I can actually, see that this is a linear equation. And this is also a linear equation here that is why these problems are kind of a simpler otherwise, we will call them as toy problems anyway. So, this is actually the linear system of equation in x 1 x 2 and lambda 2 and here it is a function linear system, both in lambda 1 lambda 2. So, I can actually put them together and then talk about this particular system, Z dot is a Z, where a is given like this. So, essentially am left out with a set of differential equations that are linear.

And I need to have this only solve this homogenous differential equation, Z dot is a Z that is what we need to solve actually. Now, so obviously what we have here is a linear homogenous differential equation, in state phase form. So, the solution is obviously this way Z of t is actually e to the power A t times C. Now, typically when a study linear systems solution and all what we what we have before, if you know Z 0 Z of t 0 then the solution is C is also Z of I mean, Z of t 0 or Z 0 and you have done actually, that is that is the solution, but unfortunately you have we do not have that, because the lambda 1 0 and lambda 2 0 are not known to us what is known to us is lambda 1 of two and lambda 2 of two, which are like conditions at the final time.

If everything, where known at the initial time itself, then you could have been done by now really. Now, because of that let us proceed what we can do here what, we have to use this conditions any way. So, let us start using these conditions first condition is that t equal to 0 this first two conditions x 1 of 0 and x 2 of 0 and x to be 0 0 actually. So, if I put that at t equal to 0 then e to the power a t is e to the power 0, which is identity and hence Z of t 0 or Z of 0 is actually C and Z of 0 first two elements will contain 0 0 any way. So, that 0 0 comes here and the right hand side it is C vector that means, the first c 1 c 2 comes in the right hand side.

So, this expression tells us that e one and c 2 needs to be both 0 and 0. So, this out of this flexibility that, we had C contain four dimensional vector right, I mean that is c 1 c 2 c 3 c 4. And out of that, we say that c 1 and c 2 are already 0 0. So, we are left out with c 3 and c 4 really.

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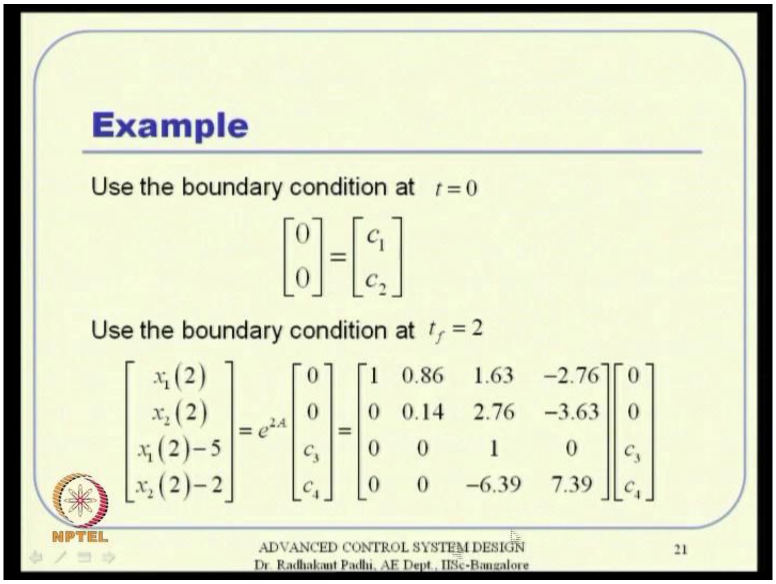

And that way to find out for using, even this condition this lambda 1 of two and lambda 2 of two using that condition, we need to find out actually alright. So, what do we do now write t f equal to 2, we put that condition. And then, we go back to this solution what we had so, we put x 1 of 2 x 2 of 2. And lambda 1 of 2 and lambda 2 of 2 is like this so, we put that here. And now, it has no more identity it is e to the power 2 A, because t is 2 really. So, but a matrix is something known to us, so e to the power 2 A can be evaluated, we have seen that before. So, if we evaluate this refer two a turns out to be like this remember c 1 c 2 is already 0 0. So, I do not have to care of them, I just simply put that 0 0 and simplify the algebra already basically.

So, am left out with something like this so, if you ignore this term in between, I mean this metal term, if you can ignore a little bit see the first and last term, what you see here is this is a system of four equations with four unknowns two unknowns are c 3 c 4. And two more unknowns are x 1 2 x 2 of 2. So, it has two unknowns here and two unknowns there so, it is simply possible to solve. And these are all we are lucky, because we have all these equations are linear equations anyway. So, how do you do that we now have to solve these set of equations to get c 3 c 3, which is possible any way. So, for solving that in a vector matrix sense, what you need to do is expand these equations, all these equations all these equations. And then, we were in these terms are little bit actually.

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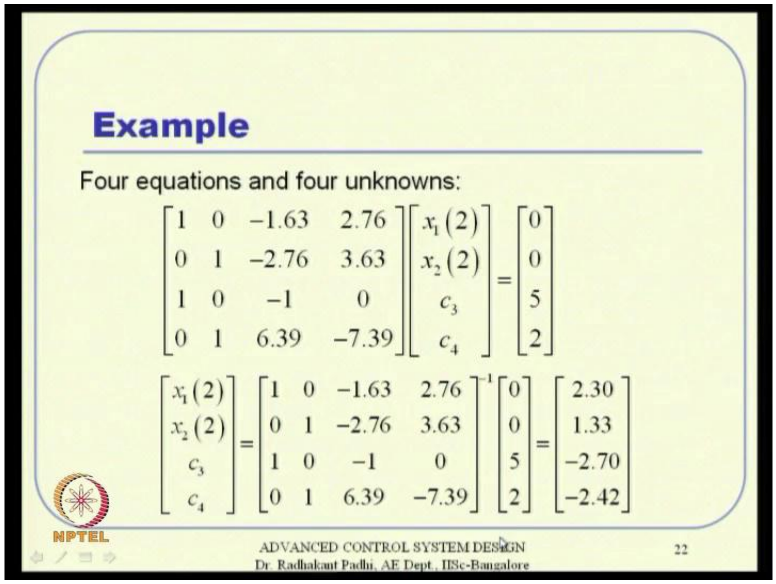

So, once you do that, you can rewrite this same equation in the, in this form where we have all these unknown variables are here in this vector actually. And hence, the solution of this vector is this one, which is inverse of this matrix times this vector, which is nothing but like that right. So, ultimately we got what is c 3 and c 4 actually, c 1 c 2 is already 0 0 any way. So, that is why the solution that, we are looking for e to the power A t times C is now available. And because that is available our lambda 2 that is our ultimate objective, that is also available now.

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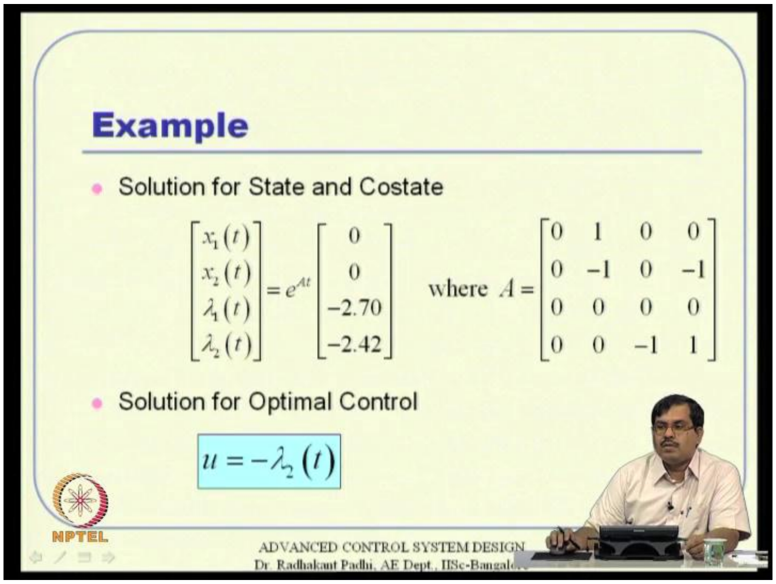

So, that is how we see the problem so this x 1 of t x 2 of t lambda 2 of t are all given like that, where this is the power A t can be evaluated with a something that is known to us actually. And ultimately, this is the solution of optimal control turns out to be like that, where lambda 2 t is the last row of this equation. And remember this is this was a very simple problem to start with, because all we had a linear system of equations, we had all I mean when x 2 dot equation which is de couple from x 1, x 1 dot was not a function of x 1 itself. And the cross function was a state forward quadrative cross function, even under those situation, we really had to do this much algebra to get some solution actually, for the optimal control.

So, that is the typical situation or when, if you are able to solve a optimal control ultimately in the close form, we have to a along algebra to get there actually, but ultimately the solution is. Beautiful, because essentially you can do many things, now once the solutions ready, you can let us say I can put some let us say I put some x 1 here x 2 here to begin with. And resolve the and solve the entire problem, in terms of x 1 x 2 then, I can simply plug more I mean different values of x 1 and x 2 here. And then see the solution is already there with us. Now, similarly, if I do not put the value two here I simply talked f now retain the generality that way. And then ultimately, I put t f equal to two three five and whatever, I can do and I still get keep on getting solutions directly basically.

So, that is how after getting the solution, the optimal control solution certifically, very nice alright. So, that is was first problem, let us move on to the second problem actually, here is a another slightly difficult problem. And probably, the dynamic sense it feels simpler, because this is the dynamics x double dot is u, whereas the dynamics, there was something like this. So, in that dynamic sense it feels slightly simpler, but we will see white is little more complicated problem actually. And here we typically talk about double integrator problem as a bench mark problem, because several problems can be thought of an approximation to this double integrator problem.

For example, if we have a satellite attitude control problem in aerospace engineering. And probably, you have linearized the attitude dynamics that means, you talk about theta double dot is equal to 1 by I times tau basically, about each of those axis principle axis basically. So, instead of this 1 by I times tau that, tau by I is something that am defining as u angular (( )). So, using this dynamics, I can actually solve I can actually design u for controlling the satellite attitude, if I assume linearize dynamics small deviation angles say now that actually. So, that is a kind of a practical relevance, why this problem are why we are talking about this particular problem.

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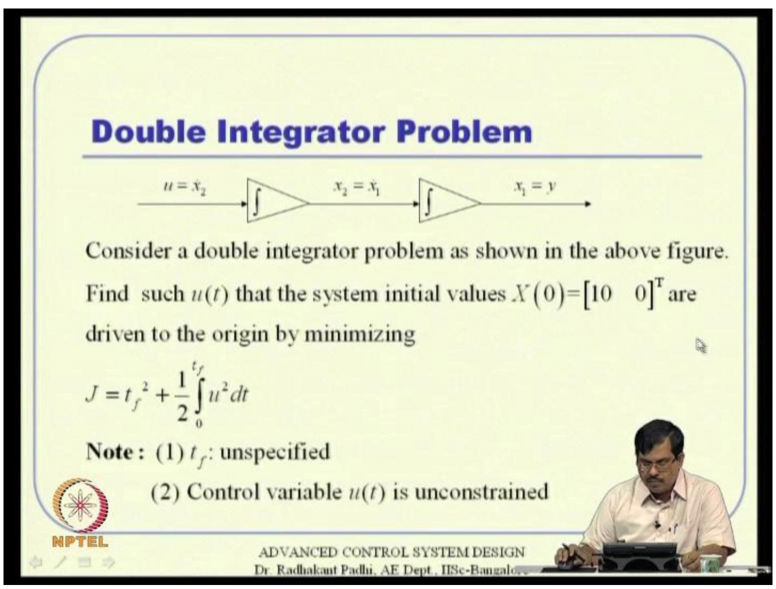

So, the double integrator problem, we can pictorially represent like this I mean x 2 dot integrated gets x 2 and that is nothing but x 1 dot and then, see one more time if you integrate you get x 1 and typically the output y is also x 1. So, why this problem is slightly more complicated, because we want to minimize the final time actually, you have not talking about a kind of a fixed final time, we are talking about some sort of a free final time. And we want to minimize that time and which is not specified, this t f is not specified a priory, because we want let us say it is attitude anywhere, you want to go from one position or other position we better go as soon as possible actually.

So, that way we want to formulate a problem where, they not only the control magnitude is minimum, but the final time is also minimum. By the way this control magnitude minimum is also a significant property of a satellite control, because the control availability is limited there, either you talk about r c control or you talk about reaction wheel things, like that. The control amount is not infinite this a kind of severally constrained by control availability. So, you want to minimize the control effort and we want to do your job as quick as possible actually. So, that is why I selected the cost function that way.

So, what I told is that t f is unspecified and the control variable, what you are assuming here is unconstrained that means, there is no hard bound for the controlled magnitude actually. So, I mean still we are minimizing this cross function so that is indirect way of doing that, using a soft constraint approach actually. So, that is what we are doing here. So, let us see how do we solve this so the dynamics that we are having is this x double dot is u, where the cost function that we are having something like this, where I want to minimize my control effort throughout, as well as I want to minimize my final time actually, the I how do I solve this problem. Now, solution first a fall we need to put this dynamics in state face form, because the entire thing we know it is in state phase form only.

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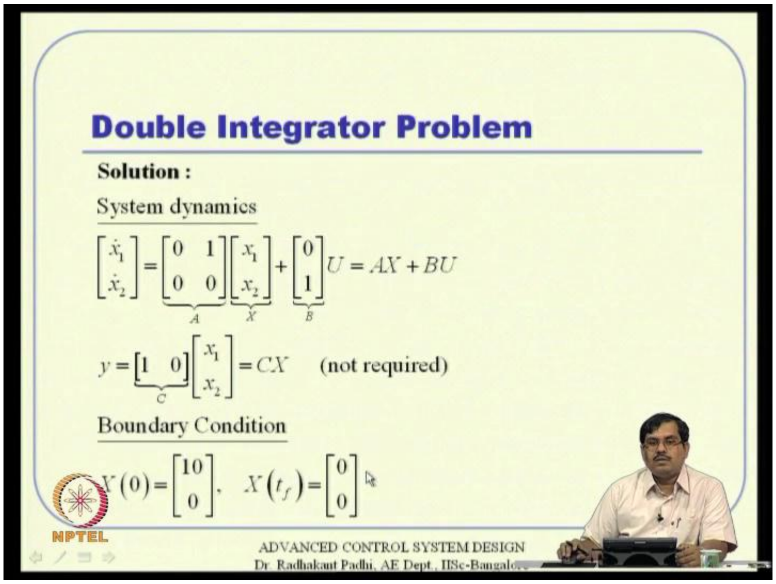

So, state phase form are well integrator is straight forward it is something like that x 1 dot is x 2 and x 2 dot is U basically. So, we have again got a linear system of equation, when you talk about a and b matrices are defined that way actually. And why is we typically define x 1 suppose, we tale position of the angle value only, then y is actually something given like that, but c is like that typically, it is not required here because we are not talking about estimation design or output field by control things like that actually. Alright so, we have this is the thing and let us say lets impose a boundary condition, where you take initial perturbation for something like this initial angle for example, I have whatever, is my desired position will take the talk that is 0 0, that is the reference point actually.

And with respect to that my initial condition was perturbed something like 10 degree deflection, I mean deviation I had actually, we can visualize this problem as something like rest to rest (( )) actually. The ultimately, the velocities initial velocity was 0 final velocity that you want is also 0. So, initially the satellite was kind of stabilized at one angle which is of course, not the desired angle, you want to make it desired, but at that point also you do want velocity build up to happen. So, it is like something for rest to rest anywhere in satellite attitude control problem, initially the it was rest the final thing that, we want is also rest, but at a different angular position actually. So, that is these are the problem that is the problem that, we have talking in a linearize setting actually, remember when is when you talk about the non-linear system dynamics for the satellite attitude, the it will have a additional component before you basically.

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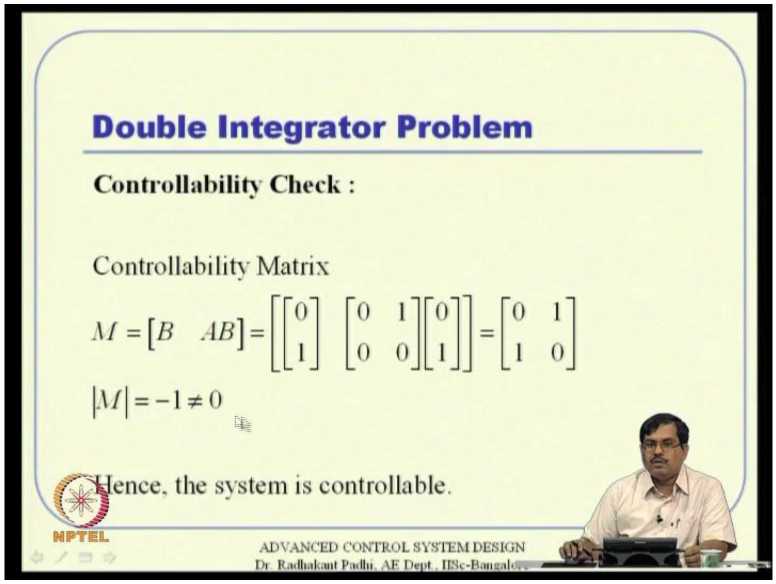

Anyway so this is how the problem, formulation this is a b c and then these are the boundary conditions, that we have first thing to proceed further, even though we did not talk that in the previous example, is to have control ability check that is standard in any control design. Because that is a great technique, that we have for linear systems actually and remember this in, this is a already an L t A system. So, why not making use of that which is know very clearly. So, the control ability matrix is A and A B so that is what it is you that is a control ability matrix obviously, the data when end of this one in minus 1 which is not 0 and hence, the system is certainly controllable actually.

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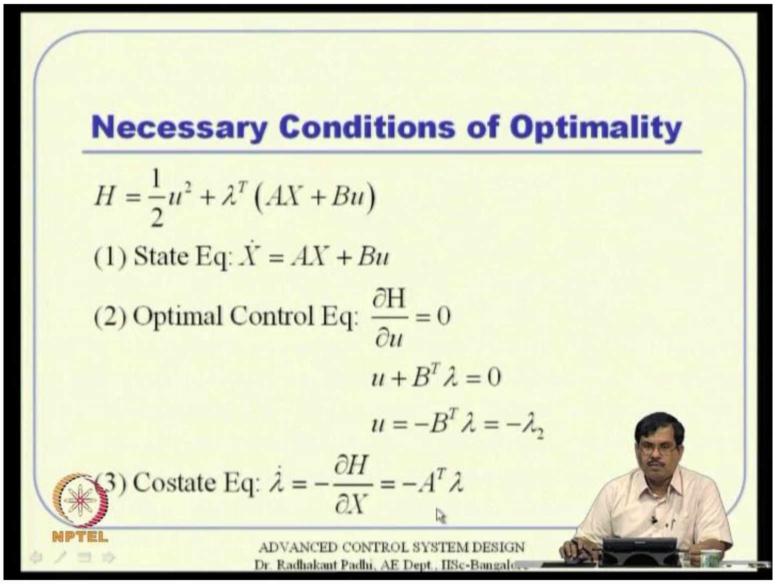

Now, necessary condition of optimality first thing is Hamiltonian, Hamiltonian is remember that half u square is coming from here L plus lambda transpose f is that. So, that is L plus lambda transpose f, the A X plus B u the necessary conditions optimality is essentially three path related things, one is state equation optimal control equation. And costate equation, costate equation already we know, optimal control equation is dell H by dell u equal to 0 that means, if you apply this one here first term is u plus the second term is B transpose lambda from this term. So, u plus B transpose lambda is 0 so u equal to minus v transpose lambda or again, if you do that B transpose, because B is in this form. So, B transpose is 0 1, 0 1 into lambda that means, it is essentially lambda 2. So, again if you know lambda 2 the negative of that is the optimal control actually.

In the costate equation sense it is lambda dot equal to minus dell H by dell X, again you go back to this minus dell H by dell X, the first term will pop up from here, that is dell H by dell X first term is a transpose lambda. So, that is minus a transpose lambda in the nothing there actually anywhere else. So, lambda dot is equal to minus A transpose lambda here so that needs to be solve actually. And very clearly, you can see this lambda dot turns out to be a kind of homogenous system, it does not depend on X in this particular problem actually.

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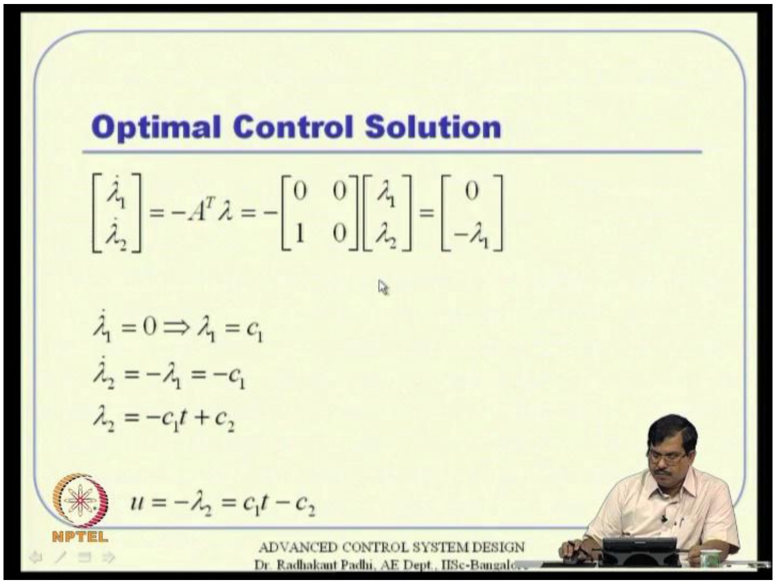

So, how do you do that with a lambda dot is A transpose lambda it turns out to be like that, because a is also a simplified matrix now. So, lambda 1 dot is 0 and lambda 2 dot is minus lambda 1, because lambda 1 dot is 0 lambda 1 is constraint c 1 now lambda 2 dot is minus lambda 1 that is minus c 1 and hence lambda 2 is minus c 1 t plus c 2 actually. So, because the lambda 2 is like this u equal to minus lambda 2 which is that way, but also remember that remember c 1 c 2 is something that, we have not got it yet we have to find it out actually. Now, optimal state solution it will go back, because here we have a utilize optimal control equation and write and in costate equation as well actually. So, state equation will go back and then try to use this costate equation what you had here, sorry this one and then we I mean, because we already know this form.

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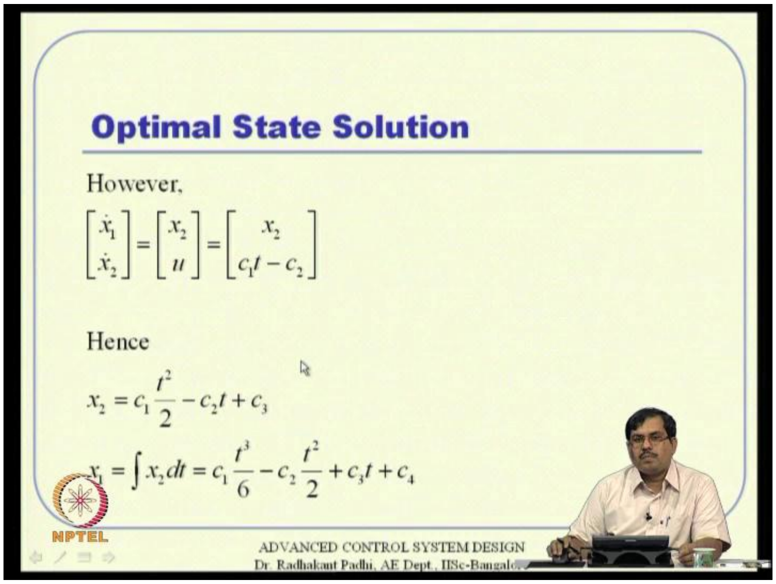

Now, state equation is x 1 dot is x 2 and x 2 dot is u so x 1 dot is x 2 so let us keep it that way, but x 2 dot is u u is nothing but this one so we will put u that way. Now, because x 2 dot is that so x 2 is obviously one integration so that means this form actually, c 1 t square by 2 minus c 2 t plus c 3. Now, we go back to that one and tell x 1 dot is x 2 so x 2 is available to be now this form so x 1 x one of t is nothing but integral of that particular function. And hence, it turns out to be that way.

So, your having essentially having c 1 c 2 c 3 c 4, which is make sense because we have a two dimensional problem x 1 and x 2 except I mean in a state space it is two dimensional problem, but state and costate space together we have a four dimensional problem two n dimensional problem. So, we have this four constraints to find out actually. Now, this is where our boundary condition, will help and dell x of 0 is there two boundary condition. And except t f is also there two more boundary condition, so let us plug in those.

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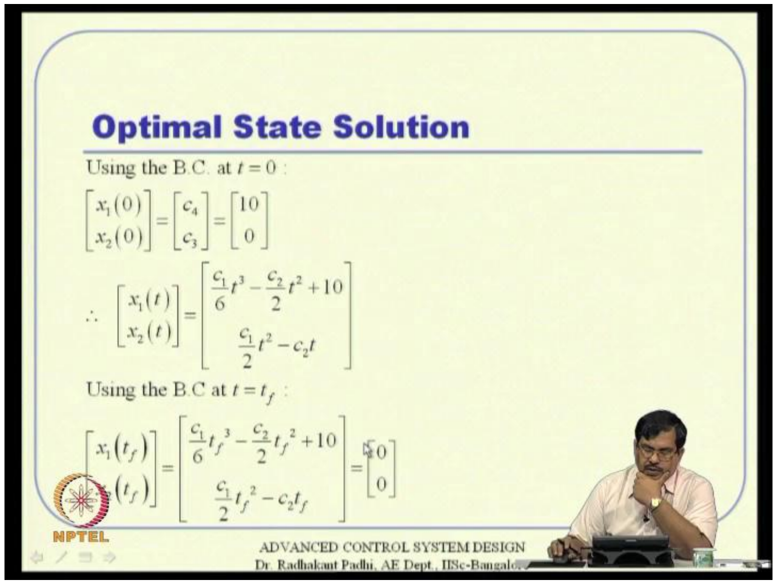

So, try to put it back and then once you put x 1 of 0 and x 2 of 0 the x 1 x 2 like that, once you put 0 0 we are left out with c 3 and c 4 all other terms are 0 that means, c 3 and c 4 turns out to be 10 and 0, because that is what in initial condition given to you basically 10 n 0. So, c 3 and c 4 values, we got already, so after using this values this x 1 and x 2 turns out to be like that, where c 1 and c 2 are still left out unknowns actually. Now, we will use the boundary condition at t equal to t f so x 1 of t f is turns out to be like that, but the problem is here t f is an unknown quantity, if it is known then we could have actually kind of got the values. And we could have got the values for c 1 c 2 actually. So, essentially we have two equations, but three variables c 1, c 2 and t f as well.

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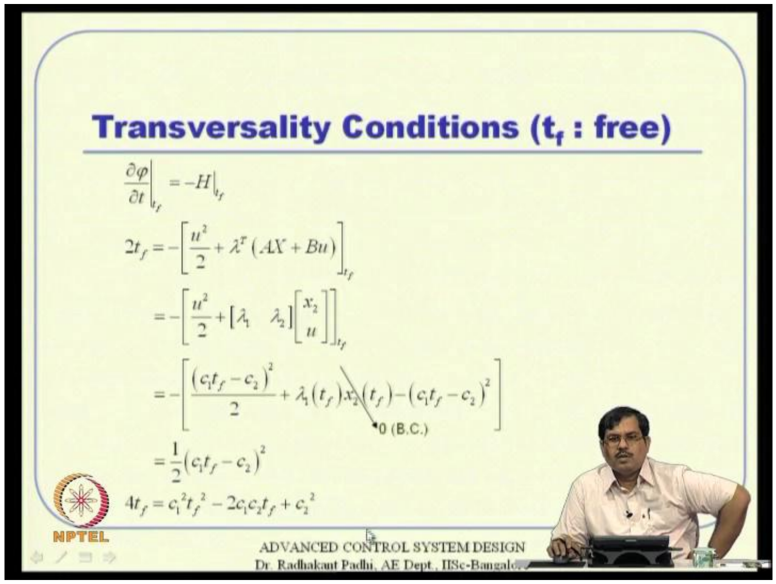

So, now how do you solve this that is where we bring the other boundary condition and tell dell phi by dell t at t f has to be negative of Hamiltonian at t f. So, if you put that condition, with now and then phi is an explicit function of t f. Now, remember that, because that is what we have formulated this is out phi. So, I put it back there and then tell this is what kind of a this is additional condition. So, put it back this is two by 2 t f and this is the this side of the story. Now, we have everything known in terms of c 1 c 2 c 3 c 4 already we found out. So, we put it back every condition and then use whatever, boundary condition we know x 2 of t f is 0. So, then we left out with something like that. So, this is one more equation that we need to satisfy. Now, we have got two equation from here and one more equation from here.

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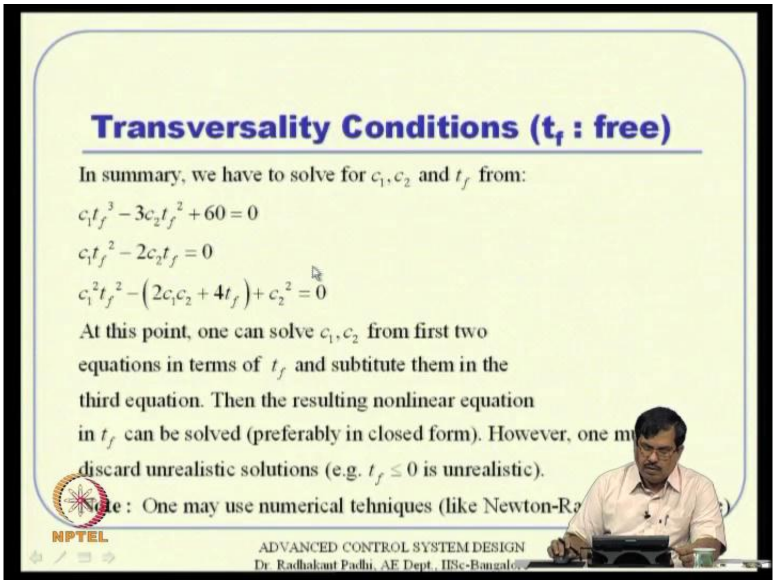

So, put them together now we visualize a system having three equations and three m quantity. So, essentially we can solve for that and preferably we need to solve for solve it in close form that is what you are looking for. So, we can receive to numerical algorithm like Newton's laws and technique if nothing is visible. And you can do that what essentially, you can think of solving it in a close form with, the where to go is the probably first two equations, if you observe these are linear equations in c 1 and c 2. So, you represent c 1 and c 2 in terms of t f from we solve first c 1 c 2 in terms of t f.

And then substitute for that here and then you will get a bigger polynomial equation, for in terms of t f. And this polynomial equation, you can try to factor out and try to solve it discarding this kind of unwanted solution, which are practically non realistic. For example, if it turns out that t f is less than equal to 0 certainly that is unrealistic, we can kind of discard that so that is the way to solve. So, the ultimately will be able to solve.

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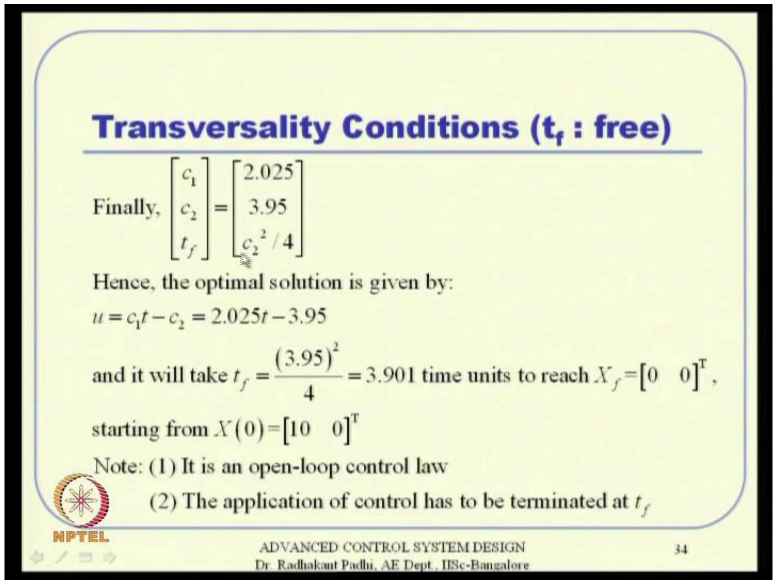

And then, we will tell you want c 1 c 2 t f is something like this, in this particular problem hence, u is something like this, c 1 t minus c 2 which is like this ultimately, we got what is our u optimal control actually. So finally, you remember this is actually even though we got the solution, we essentially learn with an open loop control. And the application of control has to be terminated at t f, which is needs to be decided a priori it is I mean beyond that the solution is not valid, we do not have to define t f, because t f is something that you are evaluating here, which is minimum t f, but the control lies is relay not valid after that actually.

Anyway so this is where this is what I wanted to discuss here, we discussed about optimal control formulation necessary conditions as well as other things actually, many things the references are also given here, you can see some of these references to get more information from there actually. Thank you.