Advanced Control System Design

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Lecture No. # 24
Calculus of Variations: An Overview

Hello everyone, we will continue our lecture series here with this time. I mean last class, we discussed about static optimization, and how that I mean how what are the necessary conditions associated with that and think like that, that only gives us kind of a background to understand this dynamic optimization, which is our primary goal here.

So, that dynamic optimization realize some something called calculus of variations, and this particular lecture I will give some sort of an over view of what are the concepts of calculus of variations actually. I say no way it is a complete lecture obviously, like previous class it is an overview of a static optimization and this lecture will be an overview of dynamic optimization which is it foundations on calculus of variations.

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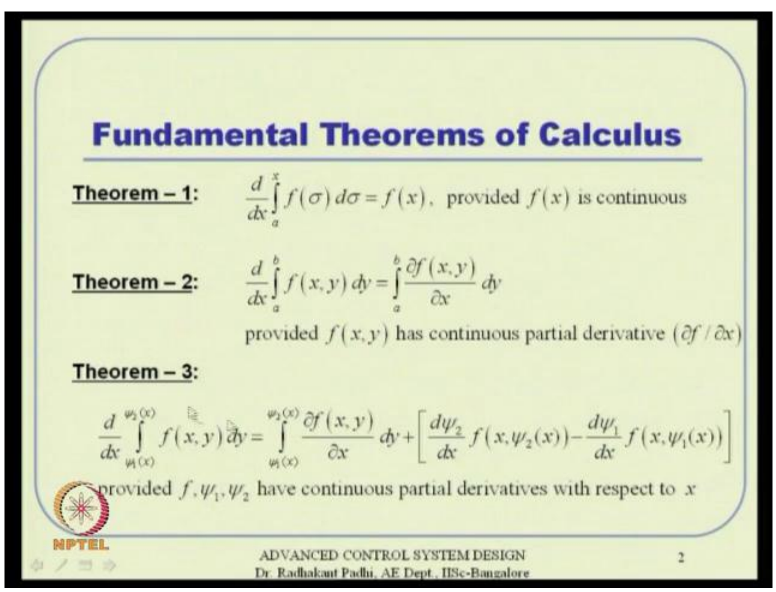

So, let us see some of these concepts here. Before we move into calculus of variations. Let us talk about some fundamental theorems of calculus which essentially deals with a derivative of integrals. So, if you have an integral and then you take a derivative remember the integrand variable is sigma, which is again integrated out toward then there is a variable in the upper element x actually. So, that means the entire the integral will be a function of x.

So, if you take derivative of that. What is the question? I mean the question, what is that and nicely the uncertain have to be f of x provided f of x is continuous. I mean that is the only requirement actually analyzing. It little further, we can tell what if this f as a variable x also inside that what then the integral variables. I mean this lower limit and upper limit at just these constants actually even then, because it is integrated with respect to y the left out thing is x so we can always take a derivative of that and it turns out to be the answer turns out to be as if you take the derivative inside the integral, but then this is a function of two variables. So, you talk about partial integration partial differentiation. Actually where again the condition here is f of x y has continuous partial derivative for x actually.

Similarly generalizing that generalizing theorem to a little more that means you were talking about not only these function has a like, this f of f f of x y that means the function is a variable the function contains a variable x, but in addition to that we have this limits of integration also as functions of a x. Actually that is I one in the lower limits I two in the upper limit. So even in that case that the uncertain have to be like this away part of the answer is like theorem too and this rest of the thing comes from this to boundary conditions, I mean these two boundary functions actually.

So, these are the fundamental theorems that we should remember if you are going to calculus of variation actually.

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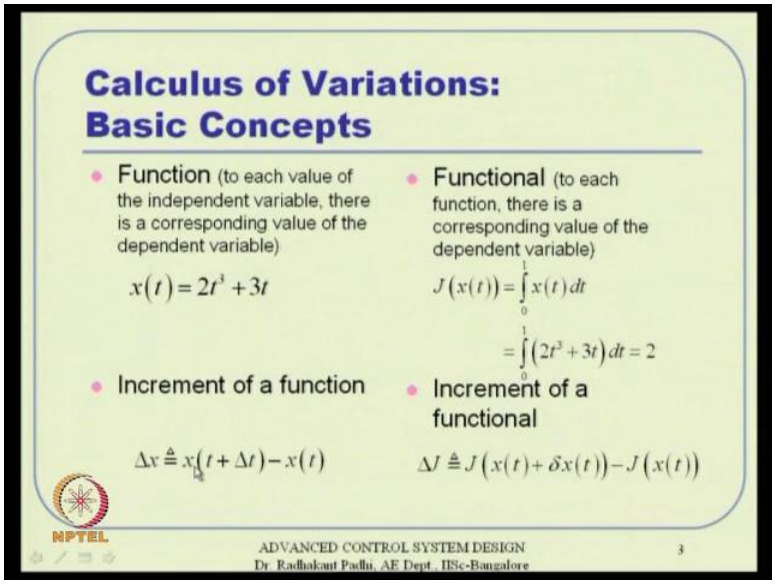

Now what are the basic concepts of calculus of variation, I on one side you see calculus regular calculus other side. You see some sort of calculus of variation concepts. Actually in calculus normally we talk about functions that mean function is a something like to each value of the independent variable.

There is a corresponding variable and the dependent variable look; we know what you mean by function anyway. So, this example what you see here x of t is 2 t cube plus 3t is essentially a function of time t where as in the functional side you have to talk about some sort of a I mean remember the ultimate answer is typically a scalar in the functional side, but this is essentially a functional time also but it is a function of time through another variable which itself is a functional time. That means you can visualize it is some sort of a function, actually where the ultimate result is scalar basically.

So, here is an example if you have or if you take the same x of t and then you integrate it out over 0 to t. If then you ultimately get some sort of a number two, actually it is actually I mean if the moment you change this function then obviously you'll get a different answer where thus for J but that in a j is not given directly as a function of time, about it is also given as a integral of x of t dt. I mean evaluated from 0 to 1, actually these are the difference function is a direct function of some independent variable.

Functional is of is a function of independent variable through some other function. Actually when you talk function we talk about increment of a function when we talk increment I mean when you talk functional we talk increment of functional and as long as you interrupted this J is a function of x then the concepts are fairly similar actually.

So, where you consider here, you consider increment of a function here. You consider increment of a functional that way.

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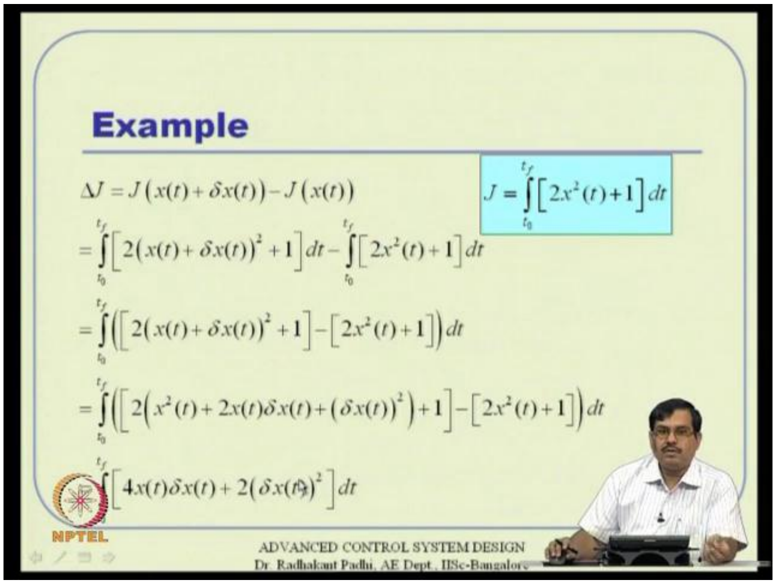

Let us, a see an example: if you have a delta J which is just this definition so depending on whatever j you have we can actually try to evaluate that. So, delta J is J of that minus J of and let take J something like this 2x square plus 1 integration t 0 to t f then is a simply put this instead of x. You put x plus delta x that means x plus delta x whole square here and then this remains 2x square. So you try to simplify these two, because these integrals are for the same. I mean the limits of the integration are same, so you can combine them together and then try to simplify out and then ultimately this one turns out. Actually we cannot neglect second order terms probably and then tell look this is the first variation and all there is we will talk about that in next couple of slides actually.

So, this essentially gives us the total increment of a functional what a by definition now. This increment of a functional can be divided into first variation second variation like that the way we do that in regular calculus actually.

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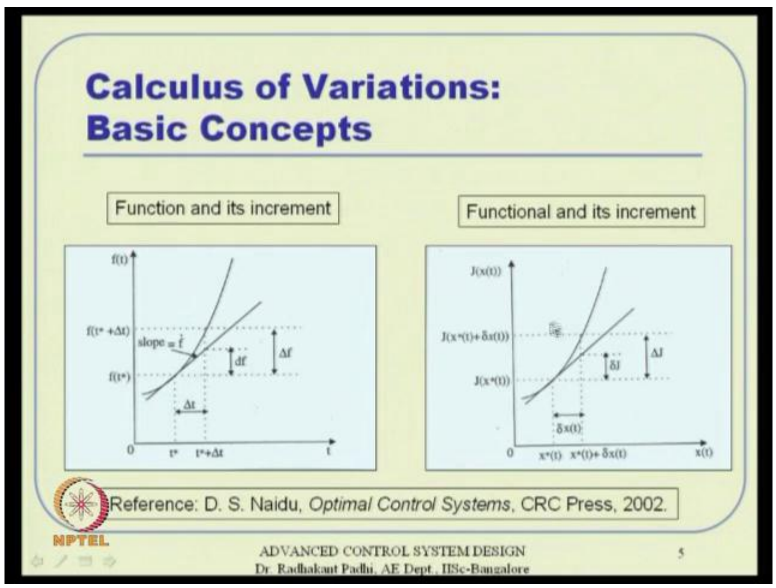

So, see that this way, this function and its increment is given like this. So, if you talk about a function f of t star plus delta t that is the value, but if you want to kind of approximate that value and think like that then you take a delta t this side, and then take a one first the slope there and then wherever it goes up to that point. You have consider that is a first a d f essentially, it is not f of t star plus delta t minus f of t star I mean that is the total difference actually.

So, similar concept you can bring it in the functional side as long as you interrupted this j is a function of x if you see the x axis here it is a x and the x axis. Here is t that is the difference, actually so x of t itself can take a different path if through this function and that one if you plot it here then the J will pop up and then you can talk about the similar concepts that happens in the various inside actually.

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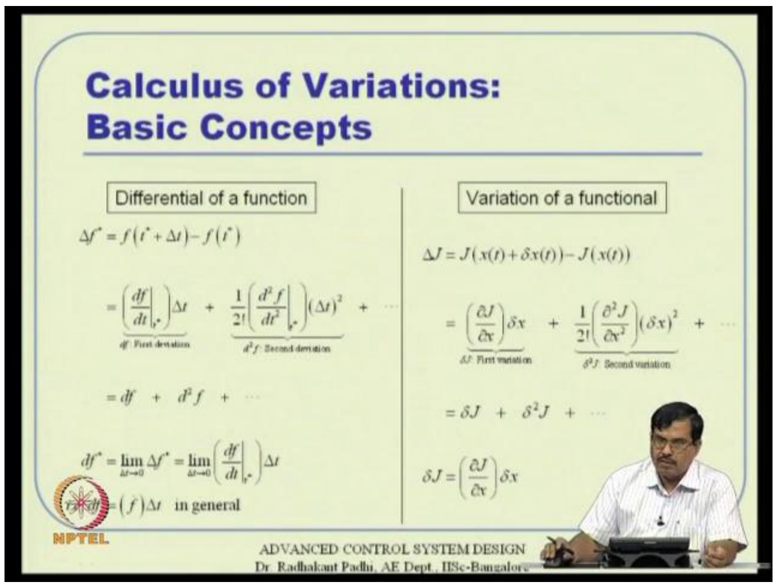

So, differential of a function and variation of a functional, these are two same or similar concepts very similar concepts. Actually we talk delta f here which is like f of t star plus t minus f of t star obviously and there is also similar way. Here you take a like a Taylor series expansion. Taylor you expand this by Taylor series and first order term will cancel out. I mean the f of t star and f of t star will cancel out here. We are left out with this first order term in delta t second order term in delta t like that actually.

So, all these terms together though we consider them as something like a first order deviation second order deviation things like that and similarly, here also we can talk about similarly exactly similar things with respect to the functional. Now, actually so here we define this df and d square f and think like that here we define something like del J del square J and think like that and this delta typically means variation actually and that is what it means actually. This first variation means del J by del x into delta x into this one and then a second variation means this kind of the thing which essentially come from the Taylor series actually.

So, the point is as long as we interrupted this J as a function of x the concepts are very similar but the x itself is not an independent variable that depends on time when that itself can be a function time actually.

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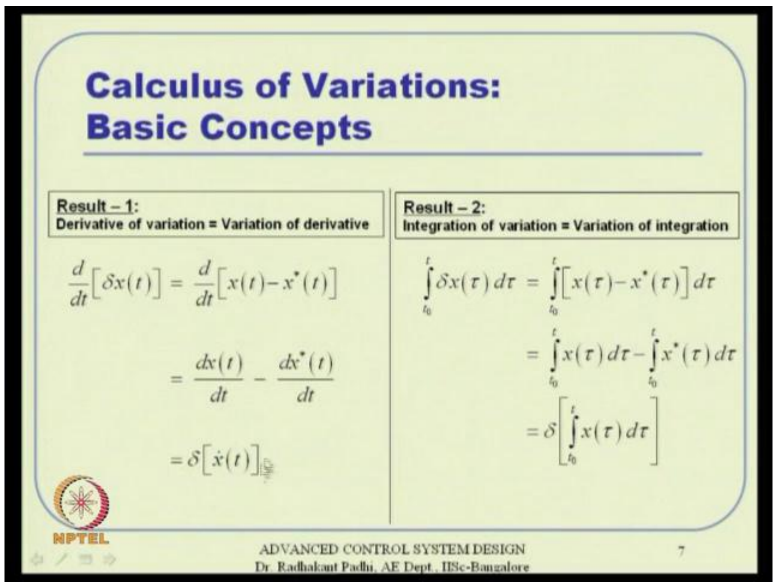

So, that is what, it is now there are very two standard results, which is this, may sound intuitive where they are very powerful that way, because our implications are large. Actually first the term of a result one tells that and derivative of a variation is nothing but variation of a derivative and then second thing tells about integration also, integration of a variation is nothing but variation of the integration.

Actually how does it come simply from definition and if you take derivative of a variation. This variation and you take derivative that and then you take by definition. I will put it that way x t minus x star t, and then I can separate it out using this laws of calculus and it turns out to be like that once I take, I interrupted these are the x dot of t minus x star dot of t then this is nothing but variation of the derivative actually.

So this result is obvious and very similarly, this or this side will also turn out to be like that another. Why this happen you primarily, this happens because this derivative and integration operators are typically linear operators.

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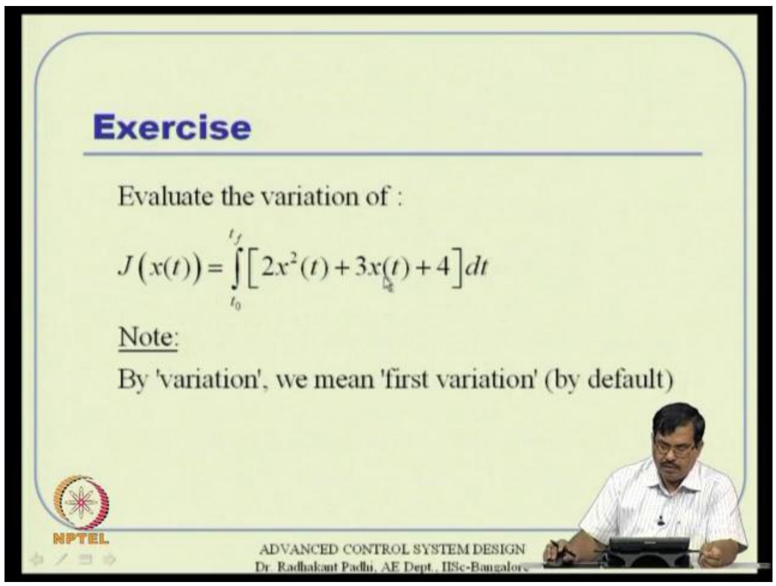

So, that is why it happens that way again we talk about some sort example to clarify our ideas, so also note that is if somebody tells us as only variation but by definition by notation we mean first variation that is by default actually. So, let us calculate the first variation coming from directly from the definition that we talked about here and like this whatever we did here, and then we talk about the variation using this result whatever is here so let does it tell to the same conclusion basically.

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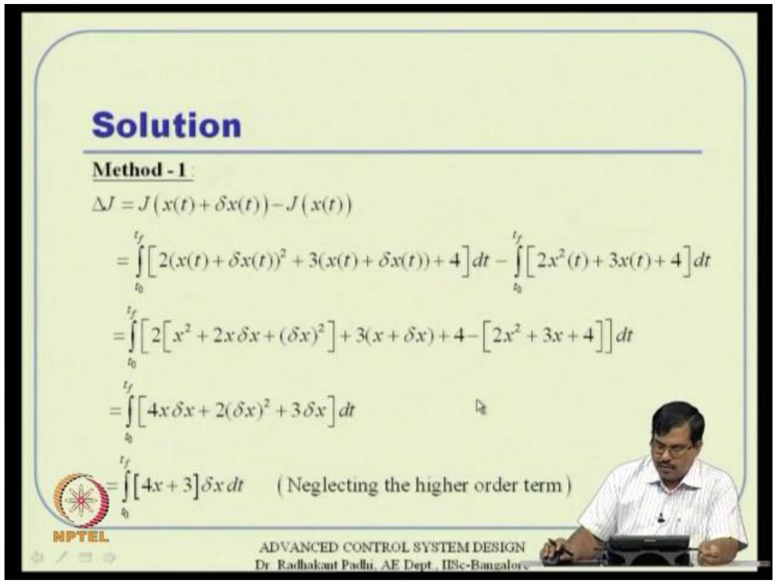

So, this is an example here, so method one we just follow the definition part of it. We just again substitute x plus delta x minus J x, I mean J of x 3 then carry out the similar analysis that we carried in the previous example proceed with the simplification of the algebra and ultimately we get something like this.

So, we neglect the higher order terms of course, by having this otherwise this is the result. Actually I mean you simply substitute all that and then there will be a quadratic term expand that then the integration are same limit. So you can combine them together canceled out there the common terms and then you are left out with a something like this actually.

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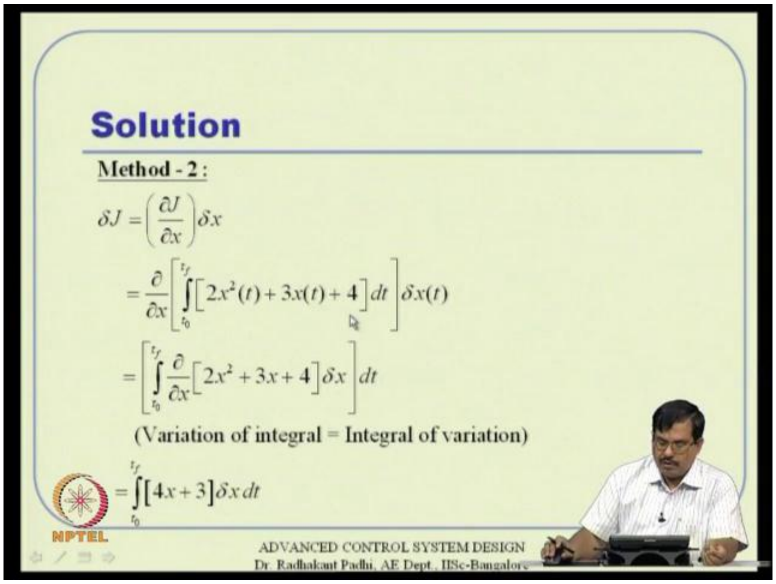

Now if you directly apply that the formula that we had this is a nothing but that so by definition this is that then it is like the using this fundamental theorem that we discuss or here we can just push this is derivative inside the integral and then tell this is, what it is and then obviously it turns out that variation of a integral is a integral of the variation of course, and we are like if you take derivative of all these. So, I mean this result is, because of that actually and then if you take derivative of that. It essentially 2x square partial derivative of x is 4 x and then 3 x partial derivative of x is 3 like that actually. So, what you got here the first variation is exactly same as that so you may probably like to do this use this result sometimes to get this algebra simple actually.

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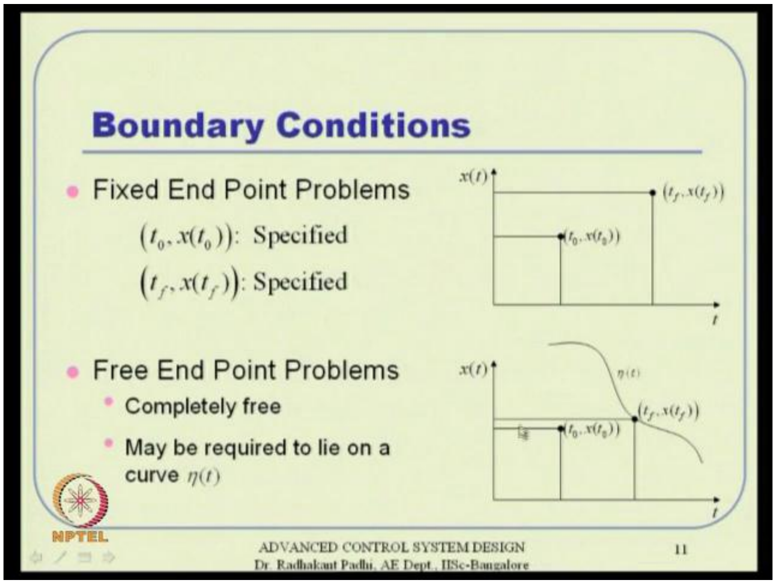

Now in calculus of variations, we encounter various boundary point problems and there with two different class of boundary problems a typically like this one is, they were call fix to fixed end point problems and the other one is free end point problems and in fixed end point problems the initial time as well as the initial states are a specified and similarly, the final time and final states are also specified.

That means you have a liberty of I mean you have to start from here having to end there but you have a liberty of going anywhere. You like you can go this way, you can that way whatever way you like actually but you are initial time and initial states are must start from here and the final time and final states even here. Actually that is kind of a fixed point problems and free end point problems are either completely free or they may be required to lie on a certain curve actually for example, here the t 0 x 0 is let us a fix initial time and initial state. However the final time and final states are suppose to lie on this curve eta of t as long as the lie anywhere we are and these problems are very relevant also in engineering applications.

For example: if you launch a satellite then it does not matter on that trajectory of a satellite, where use the in ultimately because once you join the trajectory of the satellite then from there onwards. You will continue to follow actually so that way these kinds of problems are very relevant actually here.

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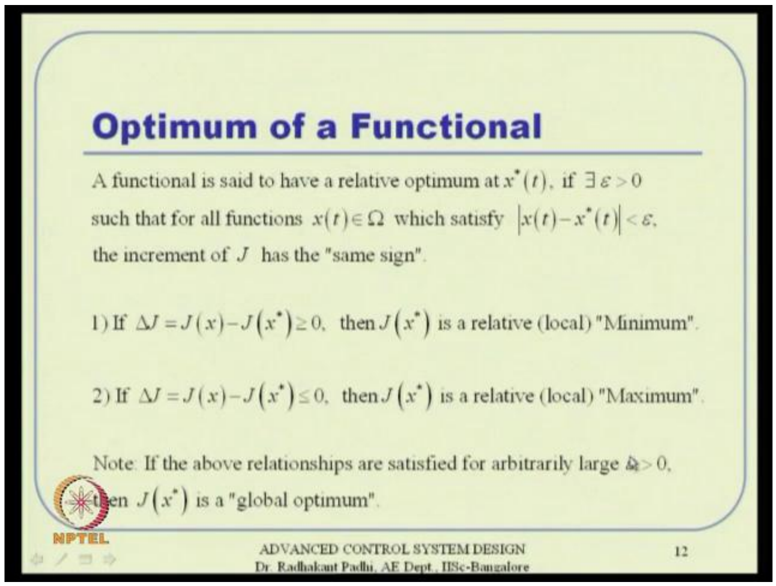

Now when talk about optimum of a functional and optimum of functional is very similar to what we discuss last time in optimization. Here we are talking about delta J which is something like J of x from minus J of x term, where x is nothing but around x star that means x is close to x star and then if you take, if you consider this term then it has to be always greater than equal to 0.

That means it is sign insensitive, basically with respect to delta x basically and similarly, it has to happen less than equal to 0, then it is a maximum point right. If my neighbor if I consider and neighboring path and then it will always lead me into a higher cost function then obviously the path that I got x star of t is obviously a minimum path actually.

Similarly, if I have the other condition that if I if my any if I take any other neighboring path then I will end up with a lower value compare to this x star of t then obviously x star of t is certainly a maximum path actually, and if these relationships are valued for arbitrarily large epsilon that means I do not have to confine myself only to the neighborhood of x star t then obviously J of x star is a global optimum that depending on what condition you talk whether it is a global minimum or it is a global maximum.

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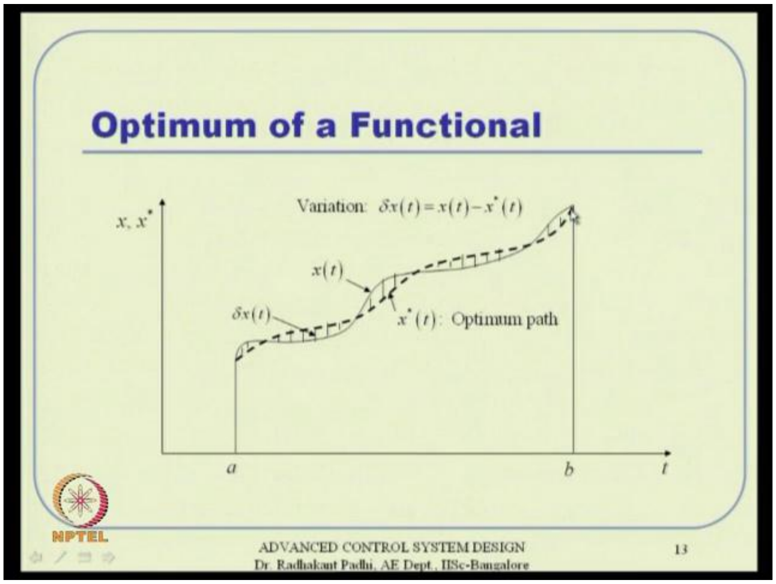

Now, similarly same thing can be pictorially shown that way, so what we are interested in essentially is finding out this x star of t. It is x star of t is our optimal path starting from. Let us say point a t 0 x 0 somewhere, and then you end somewhere here. Then let us say this is our optimal path that means, if I follow any other path there on that, that is x of t and certainly it is going to give me a non optimum value for the J of x, actually that means if I am talking about x star of t is a minimum thing then if I follow this dotted line x star of t. Then I will end up with the minimum cost value of the minimum value of the cost function, if I follow any of that path around that then obviously I will end up with higher value of the cost function, actually that is what did terminals actually.

So any optimal control problem typically we are interested in finding out this path, so that some costs function will be minimize or maximize, then it will also satisfy the necessary boundary conditions that we want to impose on the problem actually.

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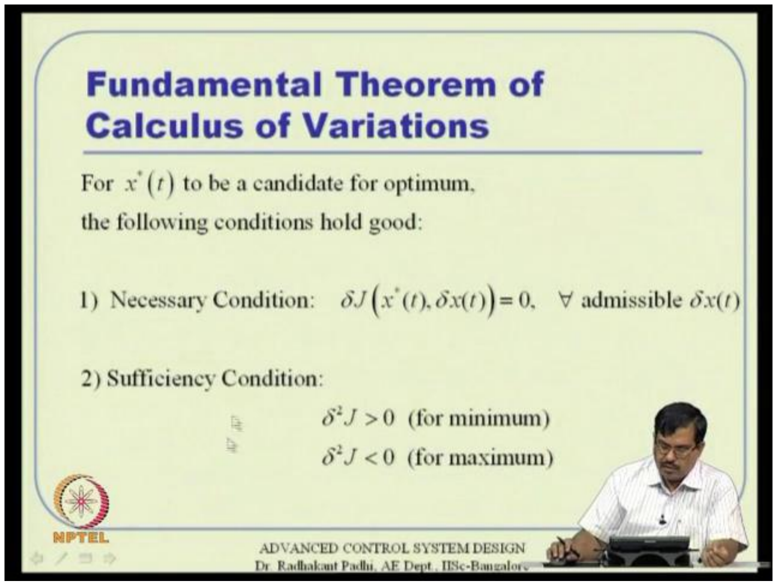

Then there is a fundamental theorem of calculus of variations, which tells us that like our in a static optimization. The first derivative was equal to 0, here it tells the first variation has to be 0, and it can be derives very very much analogues way. What we did before actually and then sufficiency condition is something like that, once this condition has satisfied whether it is a minimum or a maximum, we can verify using the second variation condition, which is like del square J and it has to be greater than 0 for minimum and this less than 0 for maximum, actually we have define this term before by the way like, if you go back a little this first variation second variation all that you defined here this is the second variation term.

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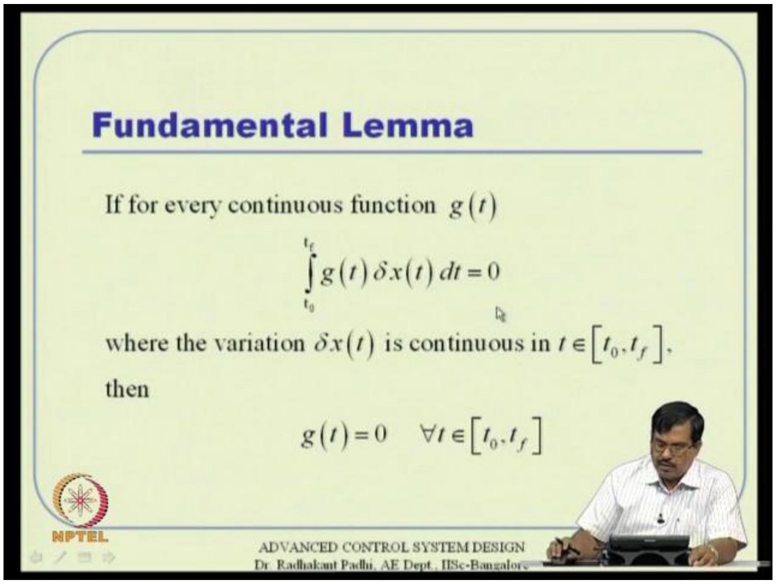

Now there is a beautiful fundamental lemma, which sounds very-very intuitive and obvious, but it is not. I mean it is implication an extremely great, actually what it tells something like this, if for every continuous function g of t this condition holds good no matter whatever is g of t when the condition that is given to us is for every continuous function g of t this condition has to be satisfied.

Where the variation delta f of I mean this delta x of t what you talking is certainly continuous in this interval t 0 to t f, that is only condition this has to be a continuous function and this integral equation, what you are seeing here has to be valid for all continuous functions g of t, and if that happens there are infinite situation of course, but this theorem beautifully tells that if that this such a case happens then the only the solution that is possible here is g of g of t has to be 0, actually there is no other thing that can happen actually. It is very intuitively you can show that but I will not so much actually.

We can show it by contradiction actually, that is the end actually. So, what we tell here is that if you just haves this kind of a equation for every continuous function g of t then the only possibility that you are having a g of t, has to be equal to 0 for all time in this interval, t

0 to t f that is a powerful theorem actually.

So using this fundamental, this necessary condition and this fundamental theorem what we are doing what we are seeing here, this necessary conditions of optimality can be derived actually.

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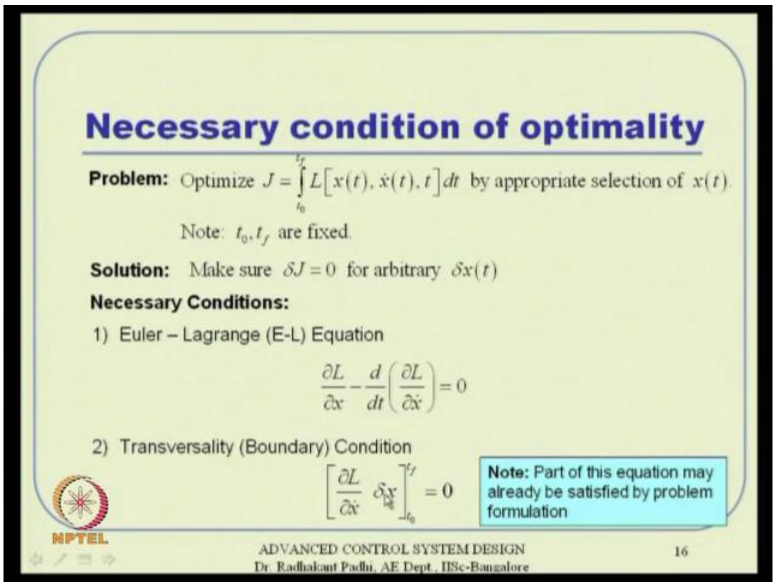

That means if you oppose a problem like this that means you need to optimize this cost function. Cost functional J, which is given like this where L is a kind of a function that you want to select for your optimization problem.

So, you want to optimize this J by appropriate, select selection of x of t and we are considering here that t 0 t a for fixed values. Then we want to obviously make sure that first variation of J J delta, J is 0 for arbitrarily selected delta, x of t that mean no matter whatever I select delta x of t, this has to be same and hence it will turn out that you can, you are always this first variation and then tell there are some copy sends multiplied by delta x t and thing like that and then when it is, when it should happen for arbitrary a delta of delta x of t then we exceed this condition and tell the only way it can happen is that coefficient has to be

0, and hence this is this conditions will pop up naturally, which will the first of that tells us that del L by del x minus d by dt of del L by del x dot is equal to 0 which is very famously known as Euler-Lagrange equation and then associated with that will, we will turn out that we have a boundary condition or trans versality condition, actually and you can also note that part of this trans versality condition can be already satisfied, was the problem formulation itself. If you are talking about a fixed initial condition problem let us say then delta delta x 0 is certainly 0 as I know variation of that that value x 0.

So, in that situation this condition does not throw any other additional kind of information. You already have that information basically so similarly, you can pause that both the end conditions end conditions are also fix that means x 0 is fixed x of t is fixed in addition to t 0 t f been fixed, then this condition is not necessary actually, because in both the situation this will tell delta x f is 0 and delta x 0 is 0 that mean in both the cases we will end up with only identities actually 0 equal to 0 that way.

So, the (( )) if I allow that means is let us say x f is free the certainly, this is not 0 and hence this condition del L by del x dot evaluated at t f has to be 0. So, that that conditions that equation in this condition or Tran's versality condition can be exploiting to that way actually. Proof is also not that way I mean as I as I told it can be derived using this first order variation. What is this fundamental necessary condition that first variation has to be 0 and fundamental lemma we can exceed and try to prove actually?

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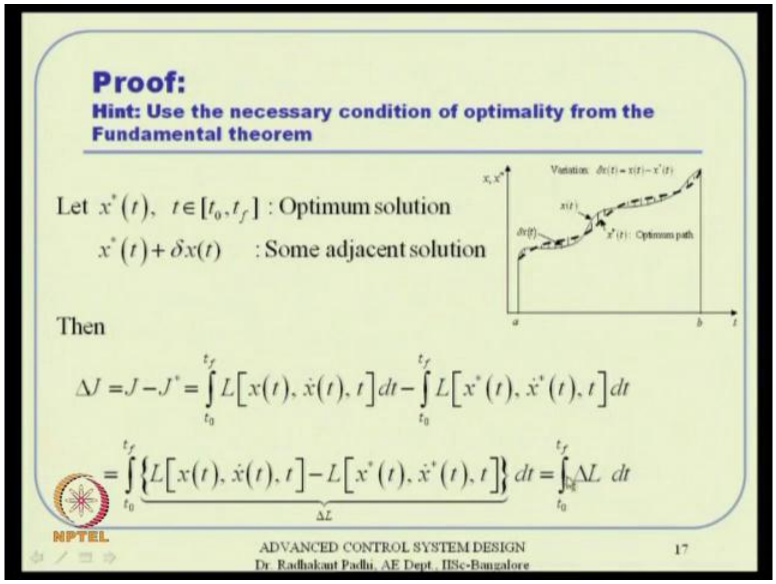

Very quickly we can see this if I take about x star of t is optimum path, which is I and x of t if is neighborhood neighboring path then I want to analyze this delta J which will pop up one will pop up, because of x of t another one will pop up, because of x star of t and I want to take the difference between the then between the two and then because the limits of the integral are again same. I can combine then and it turns out that this delta J is nothing but integral of delta L of t I mean integral of delta L del t.

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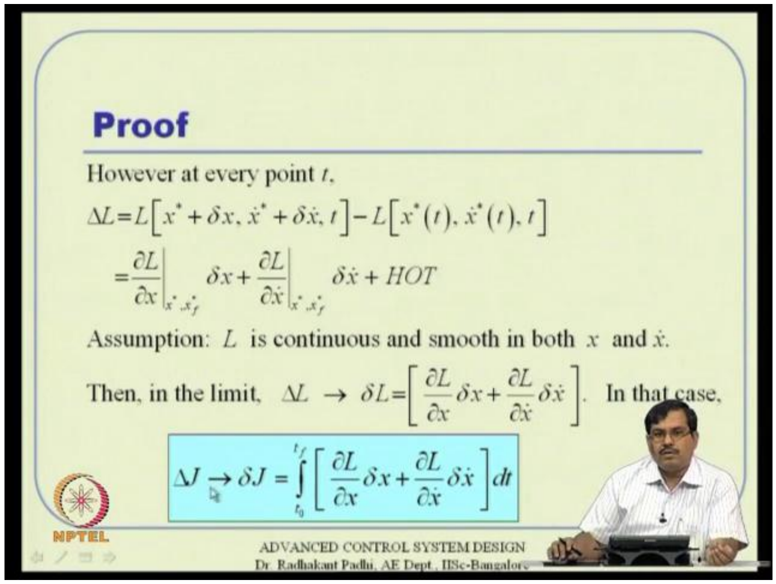

So, if you simply take delta L and integrate it out over t 0 to t f, we got the answer actually yes, can we do that of course, you can do and this delta L turns out to be something like this by definition L of all that as I remember this is x and this is x star. I mean this is x dot x star dot plus delta x dot is x dot this is x.

So, what you, what you see here is replace in terms of this, 1 plus this delta x terms actually so once you do that and you neglect higher order terms and thing like that it turns out to be something like this, this first two terms will be kept other terms will be higher order. So, we will try to neglect that once you neglect that this delta J in the limit in the limiting sense turns out to be first variation of J, which is nothing but that actually.

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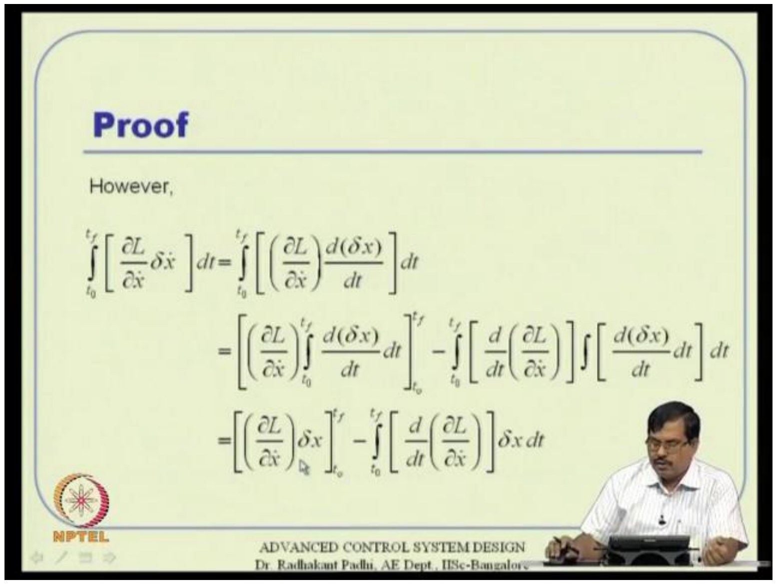

Now we can see, this is problematic term, because the variation of derivative, we do not want to talk about that. So, we take this integral thing and then by definition this is delta x dot is again like that and then we apply this integration by parts tell that this is the first function that is the second function. So, I will integrate that one and keep the first one there then take the derivative of the first one integrate the second one thing like that actually.

So, then this we can exceed this other condition that integral of derivative, derivative integral things like that so this one will go and we are left out with only delta x here and similarly, we will also we left out with this only delta x so no matter whatever is the coefficient, we are left out with variations in terms of delta x only that is the kind of transformation actually.

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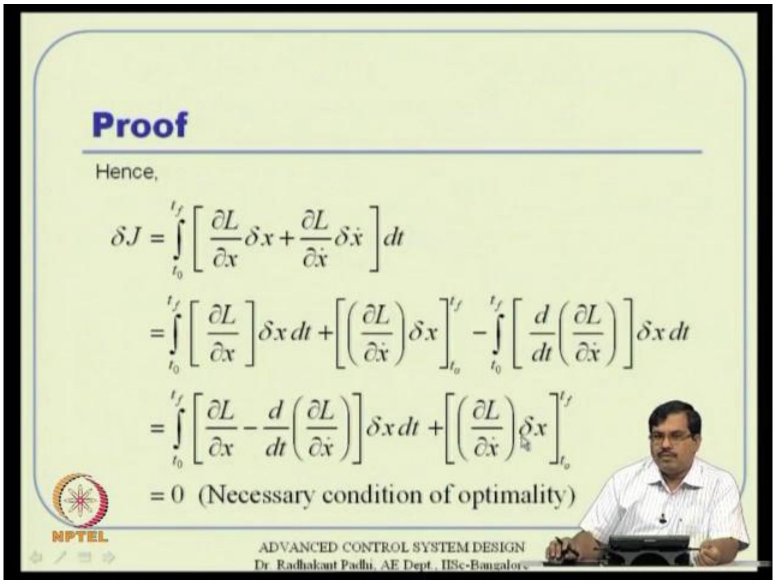

Now we can go to go back this as prove, I mean this delta J definition and then plug in there, whatever we got here this first term is as it is we keep, it is keep as it is second term we plug in whatever values we got from here and then we are left out with these terms actually. First term is as it is second term is first in one term like this and another term like that what we got here actually.

Now we combine this delta x delta t terms one side and on their boundary conditions sort of thing one side and this is to be equal to 0 for all such a variations for delta x as well as delta x of, actually for all variations of delta x and delta x which should all happen actually. So, then that by necessary I mean there is by fundamental lemma we can tell that coefficients have to be 0.

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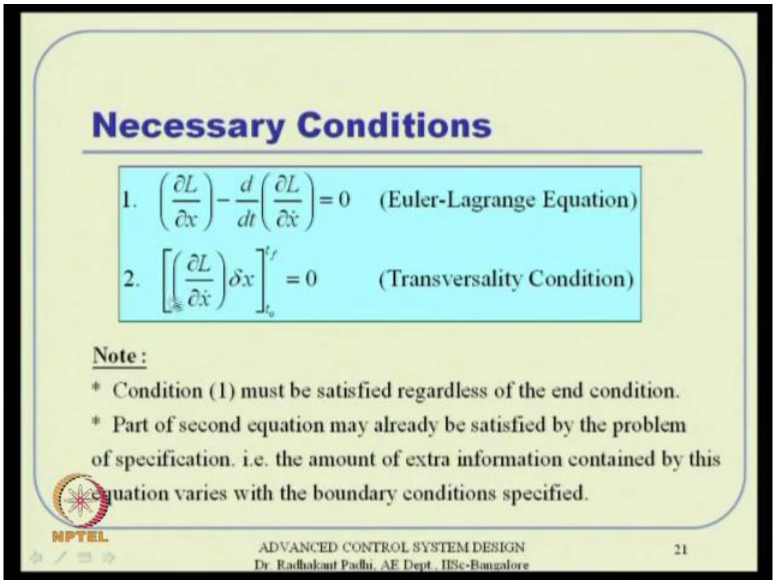

And hence we got the necessary conditions, one is the Euler E-L equation Euler-Lagrange equation the second one is the trans versality condition, which we just I mean we summarize that here actually, these two conditions that is what we can derive it quickly basically that way and also remember that the condition one must be satisfied regardless of the end condition that means it does not matter what, what boundary conditions you are putting this equation is a path equation needs to be satisfied from t 0 to t f or the time actually and this essentially gives us a trans versality condition actually and as I told before this can part of the equation can be already satisfied by the problem formulation. If something is free then the associative corresponds the corresponding coefficient has to be 0 actually.

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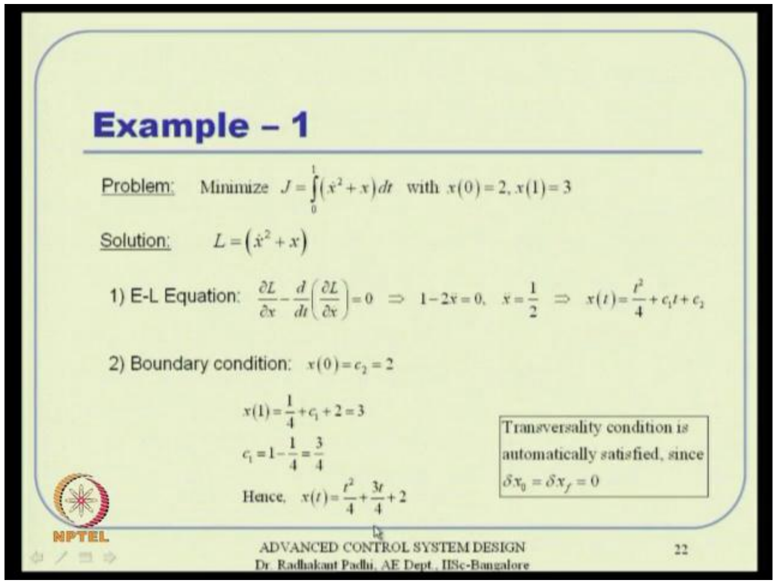

That is the way, it needs to be exploited actually very quickly, see an example; we want to see minimize this cost functional with respect to these boundary condition. Remember this t 0 is 0 t f is one and x 0 is also fixed two, and x f is also fixed at three, so it is a fixed end point problem actually.

So, the solution sense for this is L whatever is there inside the integral first, we applied the E-L equation which will give us dx del L by del x. That means you have only one from here and then del L by del x dot is two x dot and then take derivative of that that is two x double dot.

So, we left out with the equation that x double dot is half and hence if you take integral twice then you'll end up with x of t is something like this. So obviously c 1 c 2 needs to be evaluated based on the boundary condition. What you have so if you put the boundary condition at x of 0 then that is first two terms are 0. So, obviously the c two is 0 here x of 0 is two.

So, you put two here and these two terms are anyway not there t equal to 0. So, what you get is c two equal to two and now we go back to the other condition x of one is three you try to put that also. We are left out with only c one so c one you can quickly compute, so once you compute c one and c two obviously the solution is that way and in this case the solution turns out to be that way actually.

Now the beauty here is or the critical point to note here is the moment you change any of the boundary condition a little bit that means you keep everything else, same cost functional same initial condition.

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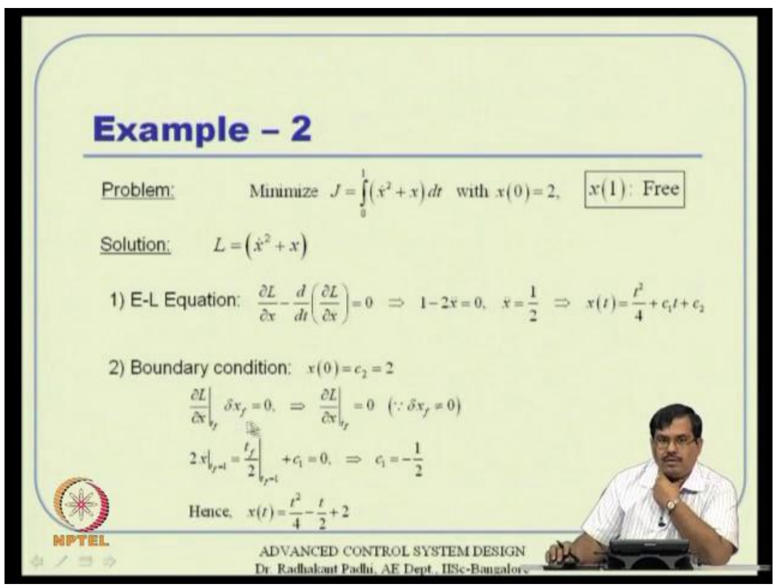

Let us say now we talk about x of one is really free that means previously it was fixed value three, now we do not care actually. It can be any value then what happens we go back to that same situation the E-L equation will remain same anyway. So, we will end up with the same solution here but the boundary condition sense, one boundary condition is same.

So, we will get the c two as same is two, so we are left out with that one but the other condition need not known to us x of one is no more three. What we had here so we cannot apply directly as we did that here, what you do then we exceed the trans versality condition.

We go back to that and tell we have that weapon here, we will try to apply that and then we will apply only an only a t x, because t 0 x delta, delta x 0 0 anyway that, that does not give us any further information.

So, we will apply that here delta x del L by del x dot evaluated at t f has to be this has to be 0 and hence and this has to be 0 for all variations of delta x f and obviously that means that coefficient has to be 0 and say and this now we have to evaluate this actually del L by del x dot which is nothing but del L by del x dot is to be x dot actually.

So, this two x dot evaluated at t f equal to one that gives us this thing t f by two evaluated at t f equal to one plus c one equal to 0. So, obviously t f one this is one by two actually so one by two plus c one equal to 0 so that means c one is minus half, so what solution you get is x of t is t square by 4 minus t by 2 plus two compare to that solution here this is t square by 2 plus 3 t by 4 plus 2. So, this is the there is a drastic change in this in the solution nature itself just by having a different boundary condition actually.

So, I mean the message to say, if I mean kind of give here is everything that you pause the problem is important, actually the moment you change this cost functional obviously the answer is going to be different even the boundary conditions. If you change obviously this very different problem, it is a need not be very close I mean need not be even close to the other boundary condition problems actually.

So, even if you release them boundary condition is do play an important role actually, that is the message there moving on we also, what we derived here in this condition and all that we essentially assumed that t 0 t f r actually fixed now the condition is, they can also be free. I mean I do not have to be restricted too much on this initial time and final time. I can make initial time fix and then t f is free or vice versa our various combinations of that.

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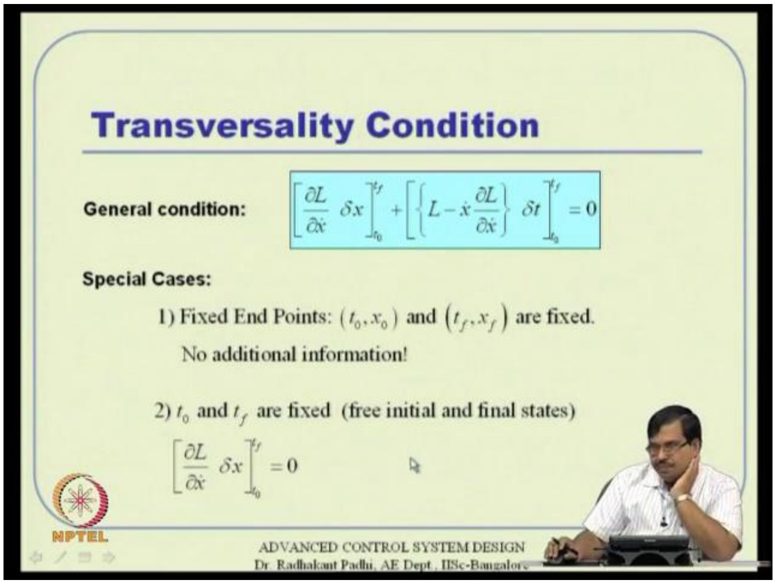

So, in general the transversality condition E-L equation will remain same anyway but the transversality condition senses this is true. If you bring in the flexibility in time initial and final time also, so we know do not I mean we not only of the first term or a first term is what we did, before plus this additional term which contains this various. I mean t also at t 0 and t f remember this variation of time is allowed only at initial and final points of time, where you start, where you end you have flexibility on the way. You do not have flexibility really time is independent, Independent variable it starts developing. You cannot vary that as it develops actually but you have certainly of a liberty of where to start the problem and where to and when to end to the problem actually.

So, obviously if you have this Trans versality condition, which is very general then the special cases will turn out to be like fixed end point conditions. So as we discuss before, if a everything is fixed t 0 x 0 and t f x f all pairs have to I mean both the pairs are fixed then obviously all these variations are 0 at both at t f as well as t 0 and hence it does not give us any additional information.

Now only if t 0 and t f are fixed as case two then this part does not give us any additional information. So, what we got is what you derive before, so this will hold good and because of that the coefficient the coefficients in the initial time and coefficient in the final time will be zeros that we will have to exploit actually.

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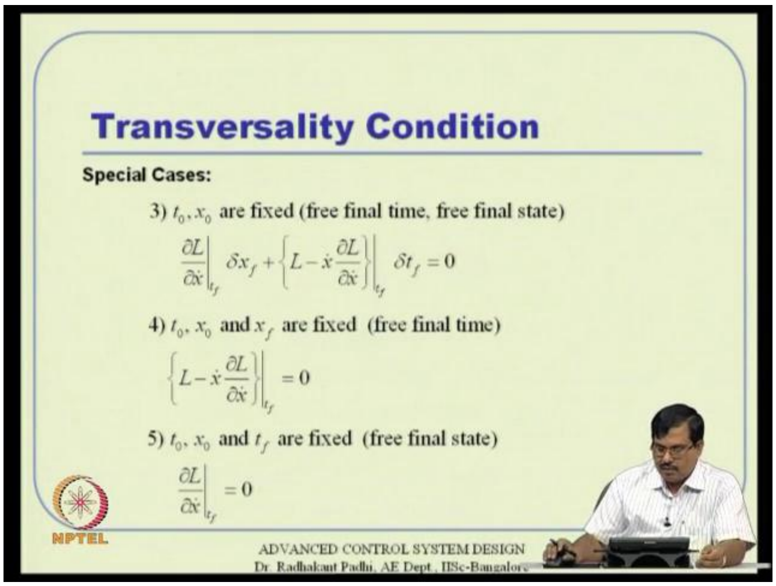

Now case three, we can bring in further coefficient generality and tell t 0 x 0 is fixed but t f x f are free that means we are talking about free final time when free final state also, so in that situation we get the initial time. It will not be giving us any information but the final time sense we have to keep this condition as it is.

So, this will give us the necessary condition for such for dealing such cases and this means a probably individually, these two has to be 0. They also that means we have to have them the same number of boundary conditions as number of differential equation that will pop up naturally actually.

Now you can squeeze this condition little further and tell t 0 x 0 are fixed anyway but in addition to this let me fix t x f also but t f is free. I mean and this is a beautiful class of problem actually, if it is a slightly- slightly difficult problem also in the way but you can visualize this. This missile guidance problem especially you have liberty of a, you start to your missile to an if some t 0 value that you know and where you are starting it that also you know and where you are falling also you know, because you obviously have to fall on the target actually.

So, wherever is your target that is your fixed end point actually, so that means x f is also fixed on the target location. Let us say initial time of learns is known your learns position is known and the final position of the missile is also known and t f is normally. I do not care actually as long as I fall on the target in very precious way so normally t f does not care play such a big role even though you think. If I have to kill then I have to kill in a faster time anyway that is true in a way but in general missile guidance problems are not a such critical for t f condition. Actually they are very critical about condition that means the final x f has to be on the target, that is more important then when you fall on the target actually, so that is the very class of problem very a relevant class of problems actually.

Now if you think about the other way that means you tell well I do not have to fix- fix x f but let me fix t f so this is probably a problem where you have to think about something like aircraft guidelines. Actually you start with a let us say city one airport I mean city one airport and you want to go to city two airport obviously then I mean the final things are not on your hand. That means depending on the runway clearance and things like that but but you have to reach that a city to at a certain final t f actually and as long as you are around the city two at the final t f you are probably and after that the terminal guidance another things are airport authority anyway.

So, these are all the various cases where we can relate our real life problems in a very good way. I mean the again that the message here is optimal control gives us the all these frame work a very need frame work to discuss all these. So, called difficult problems that we cannot deal using only stabilization control theory basically that is the message there.

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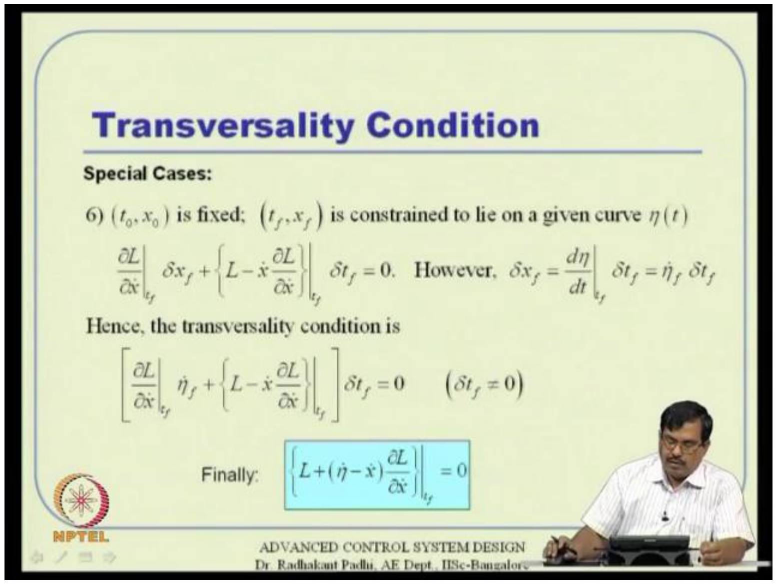

Then still further generalize this and tell if t f t 0 x 0 is fixed and t f x f is consider is not really fixed but it is constrained to lie on a given curve as I that this problem that I discussed before probably. Probably satellite guidance is launch vehicle guidance to through a satellite in it (( )) falls in that actually and hence also ballistic missile guidance, where are kind of like that as well so there also you are use some these concepts actually.

Anyway so coming back to this, this is the trans versality general trans versality condition turns out to be like that but the you are remember this x delta x f is no more completely free. It is constrained to lie on this that means delta x f is something like this. So, I can put this delta x f back in that way and try to combine these two and then where, I'm left out with only delta t f as long as this delta t f is there then I'm constrained. Anywhere I have to the delta x of f has to be given by that actually-actually.

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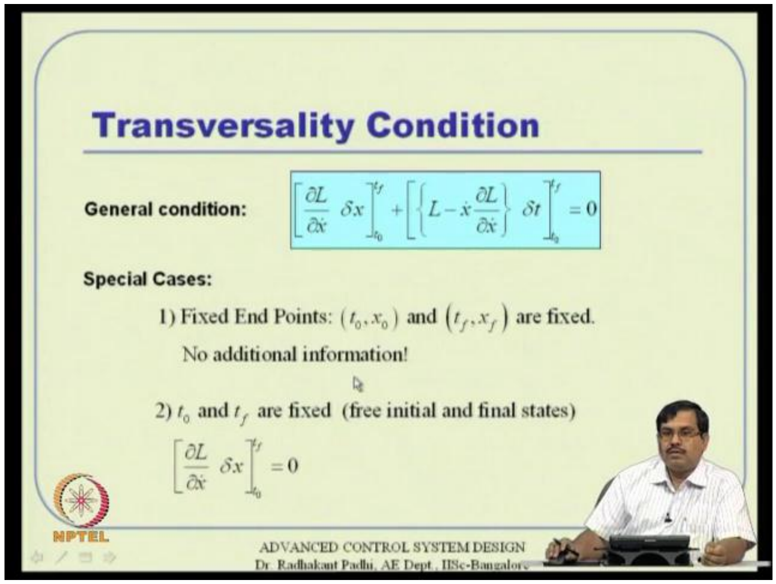

So when I, when I constrained look at this equation, all that it gives me is some sort of a equation that way as long as I satisfy this equation, which is the coefficient of that actually then I'm done so like that the depending on various cases like a various-various situations that you want to impose the problem from this general generic trans versality condition. You can derive several Trans versality condition allows several boundary conditions essentially.

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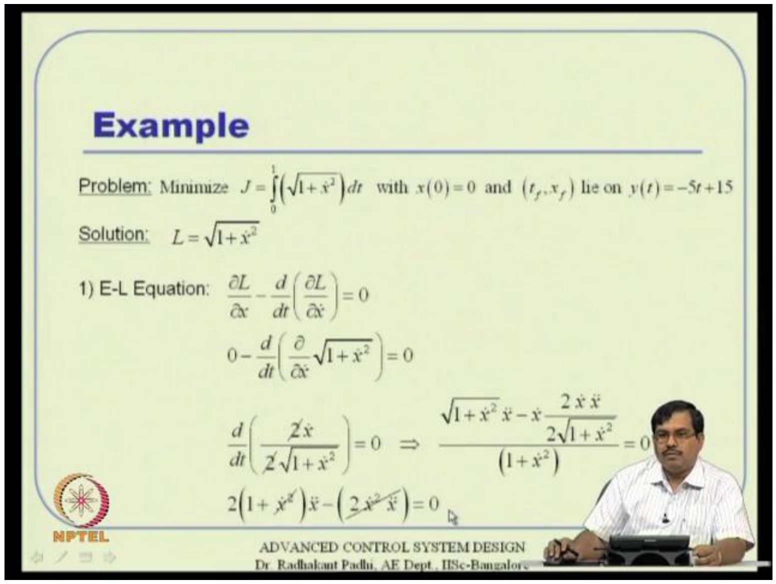

Again an example we will talk about minimizing this cost functional with respect to x 0 equal to 0 and t f x f lie on this particular curve. Now so E L equation sense is very clear Del L by Del x minus d by dt of del L by del x dot equal to 0. There is no x, there is no x term where it leads the partial derivative of that is 0 and partial derivative of x dot is there any way so that one I have to take first and then I will take d by d t of that one actually.

So, if I take partial derivative of this square root of one plus x dot square with respect to x dot then that is what it is and then this particular term I have to take derivative of that again actually. So, if I do that all these calculation thing like that it turns out to be something like this and as essentially we can cancel out the terms that are like plus minus terms and all we are left out with something like, I mean essentially this all that it gives us is this coefficient is non zero and hence x double dot is 0 that is all you will end I mean we are leading to actually.

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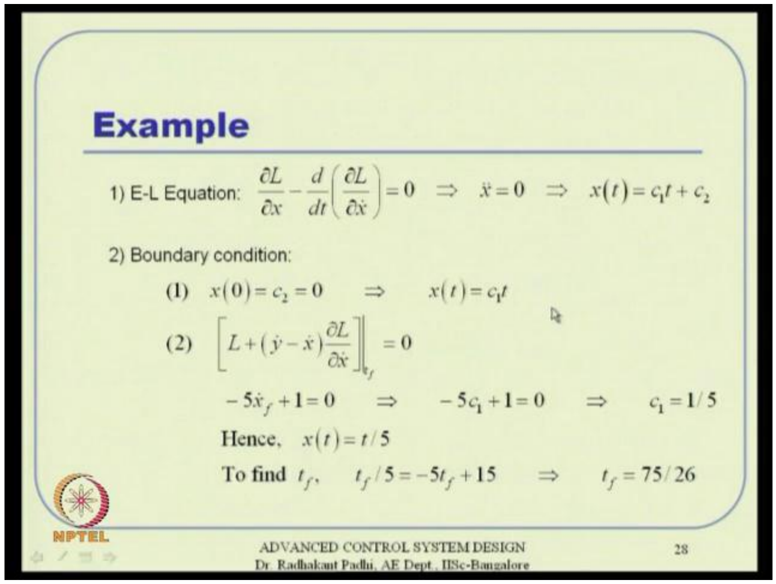

Now if x double dot is 0 then obviously the solution is once you integrate it twice. It is like c 1 t plus c 2 and then you can bring in the boundary condition and tell my first boundary condition is x of 0 is 0. So I will put it there that gives me c two equal to 0, so my x of t is really c one of t. That is as simple as a linear curve in time actually but this particular thing is constrained to lie on that curve whatever curve is that, that is another linear curve you know that.

So, obviously I have to satisfy that particular situation. I mean that condition so, I put it back here this trans versality condition and I am left out with this equation where I can bring in this x dot of f. Remember x dot is nothing but c 1 actually x of t is c one t so x dot is c one x simply. So, you put minus 5 c 1 plus1 plus 1 equal to 0 that means c one is equal to one five one one over five actually.

So, essentially what you get is c is x of t is essentially t by 5. That is all you got actually, that is the solution, that is the optimal solution actually, so and this of t by five will lies on this curve actually. So, pictorially if you want to see what goes on then probably you can see this time base directly then minus 5 t plus onefive is something like. It has to pass through onefive obviously at t equal to 0 and minus five t means something-something like this actually. This is the constrained curve and then you are solving a solution which is one fifth of t when t by 5 basically so that is essentially something likes this.

So, you are essentially I mean looking at that t f where it ends there actually that is your t f and the corresponding value and all the things will be there. So, t f turns out to be if you just equate them together then t x t f turns out to be like this value seven five by two six actually. So, that is, that is the t f that is the solution the-the t f t over five is your optimal solution and that satisfies all sort of conditions actually.

Now all these things that we discuss so far are without constraints, they are all free optimization problem. There was also I mean on the only condition constraint was boundary condition that is all but you can also a path constraints on the way and that is how it is relevant to our problems, because on invariably we talk about state equations. Actually state dynamics and whatever solution we need to find for control optimal control need to satisfy the system dynamic constraints anyway that is valid entire path actually.

So those are more relevant problems and let us talk about variational problems with and without constraints actually in a more generic sense and first thing we talk about the one more thing is all these things that we discussed is all with-with respect to scalar x. Now that is also not reality, because number of states can be more that one so let us generalize that to vectors first and the first we discuss with respect without constraints and then we will come back to with constraints actually.

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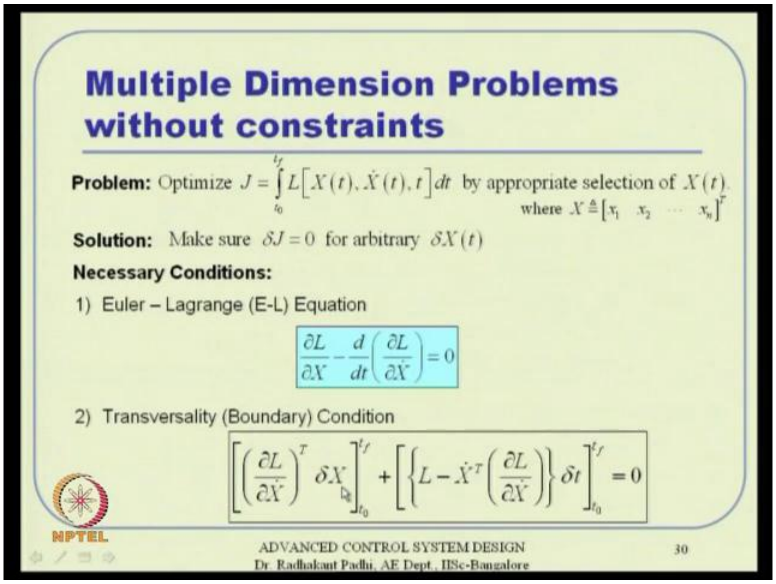

So, without constraints what is the multi dimensional problem that means you with the cost functional? I have to optimize J is something like this remember normally. J is scalar values even the X is a vector J is a scalar and here X we talk about like n dimensional state. Vector I mean dimensional vector in general because still we do not know, what is X when you talk about calculus of variation in general sense.

So, you have to make sure that delta J is 0 for arbitrary selection of this vector delta X of t and if you carry out the similar analysis one more time. It turns out that the same E L equations will be valid and similar Tran's versality condition will also e value. The only difference here is these equations are now valid in terms of vector or matrix equations sense and if you see this, this a trans versa needed, because that is are raw vector after transpose and that'll give to column vector that the entire-entire multiplication will turn out to be scalar like that actually and here also del L by del X. So, remember L is a scalar typically but Del L by del X will certainly of vector, because X is a vector. So, this is this is n equation in a variables and similarly, the Trans versality condition also inbuilt this-this variation in all n variables variation in time so that is generality but the equation form and everything remains very similar actually.

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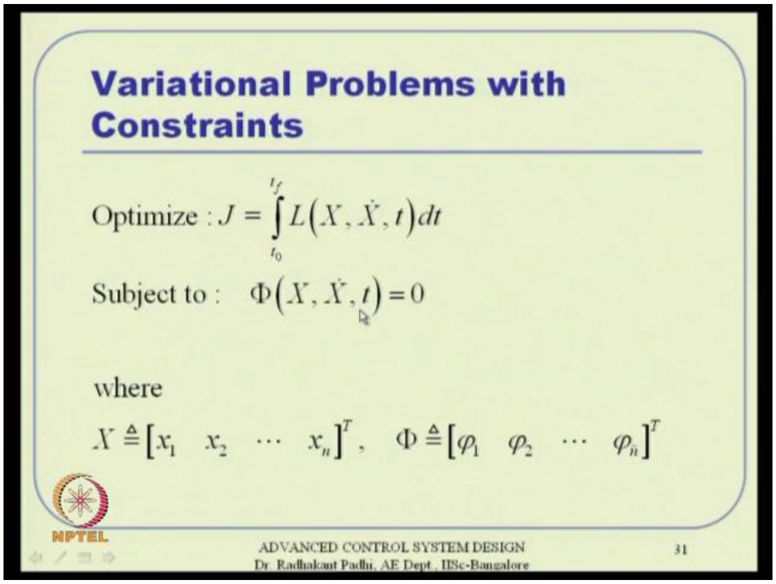

Now how about with constraint, so with constraint if you discuss this J is like this and subject to this constraint equation. Now remember this X dot equal to some other thing but in generally I mean in generally without like considering only that special class in general. We can talk about generic non-linear function which contains both X as well as X dot in addition to time that is equal to 0 essentially. What we discuss here is X dot is f of f of x c I mean later-later on this is that meaning actually.

So, optimize we want to optimize this cost functional with respect to I mean this subject to this path equation constraint. Actually how do you do that and if you remember static optimization problem and all we conveniently did that using logarithmic multiplier and here also we try to do that and this constraint equation need not to be of the same dimension instead that is also another issue. So x is n dimensional vector, where is the constraint is a UN tilde the dimensional function basically give that is what we are discussing here.

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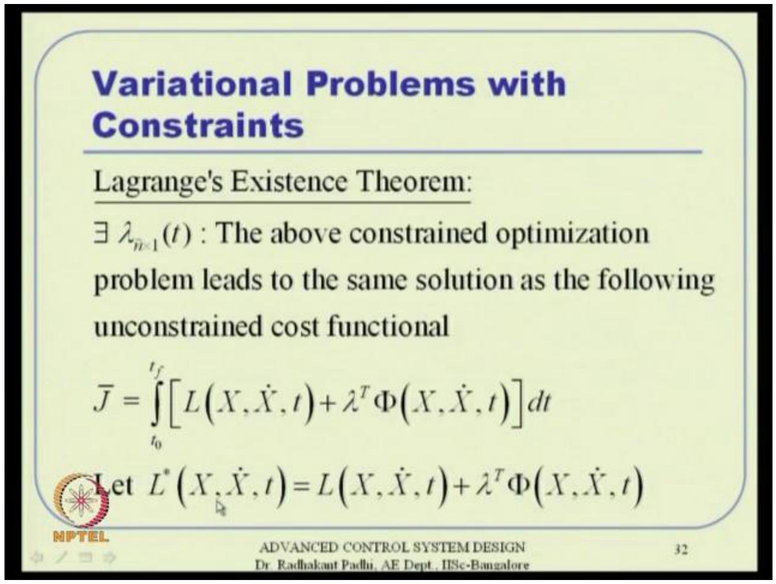

And going back to the Lagrange theorem again it is it delights on the existence theorem. We tells us that the exits a un tilde the dimensional vector lambda of t.

So, that the above constrained optimization problem least to the same solution as the following unconstrained cost functional, remember lambda is a function of time now and it is very throughout this path. So, we want to in bay, I mean augment this cost function a inside the integral basically. So this argument constraint a bar contains L, what is coming from here plus lambda transpose this term. Actually so that, that is what we have here now this (( )), you can consider this as a free optimal problem optimization problem, when consider lambda as a free variable as well actually.

So, in that sense this L star what you are talking inside the integral is nothing but that L plus lambda transpose five. Where remember L star is now a function of lambda as well so that is that you should keep in mind actually.

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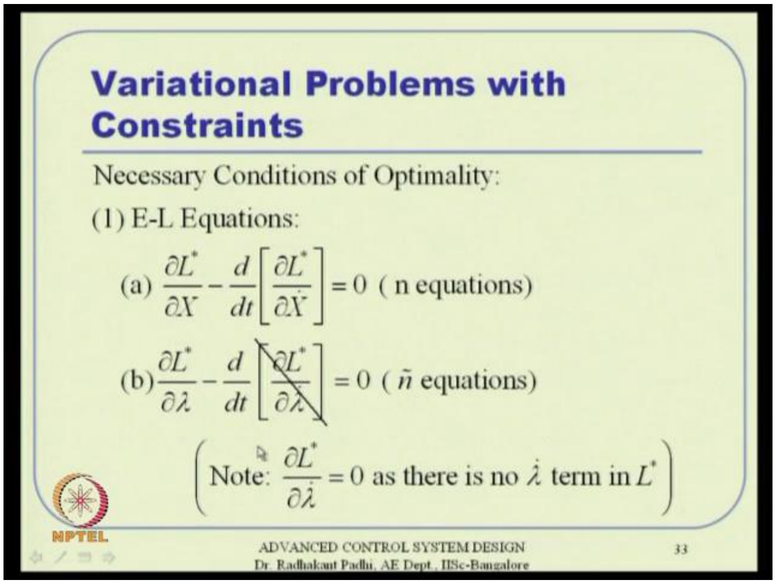

Now obviously the necessary conditions optimality, we have to apply it to twice once with respect to x and once with respect to lambda. It will give us, this will give us the n equations and essentially you will see that this will I mean this will give us the un tilde equation and this will certainly contain the same a same constraint equation, that we started with you see that (( )) actually.

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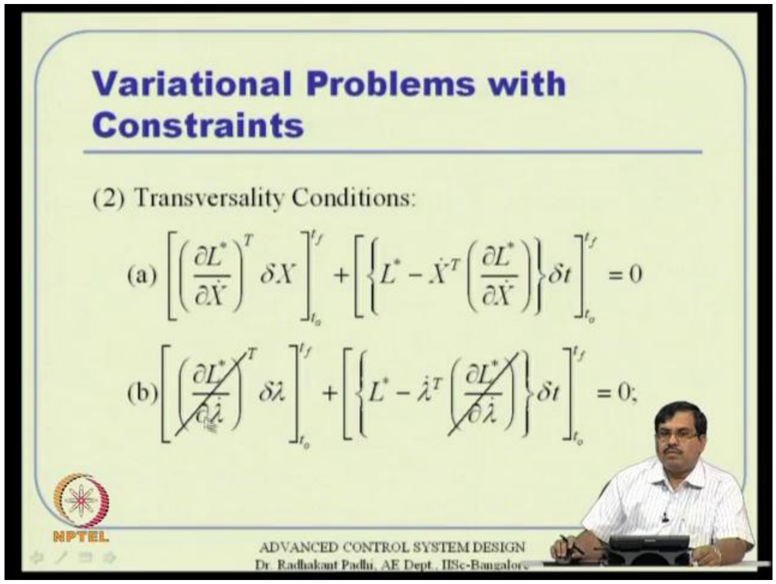

Similar Tran's versality conditions in first, we apply with respect to X and then next X to apply with respect to lambda actually.

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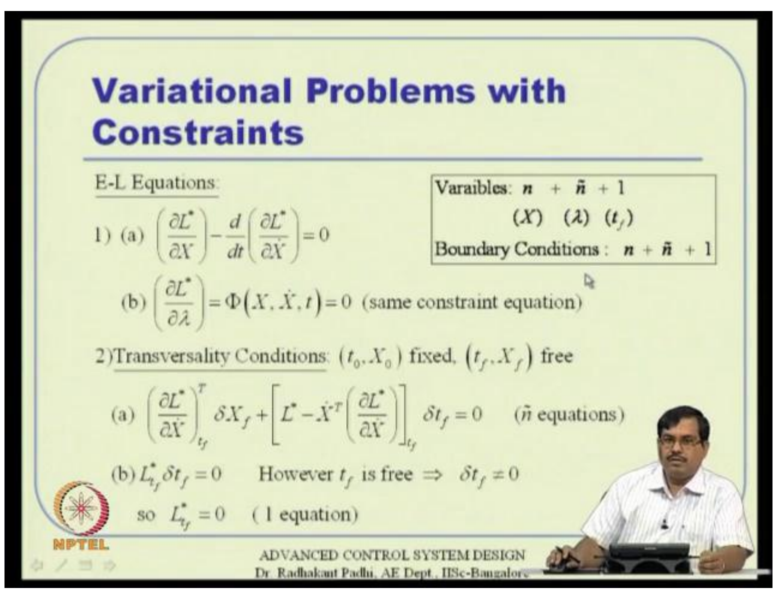

So, if you apply with respect to X I mean this E L equation then the first equation is that the second equation is nothing but that one so we have one set of E L equation and second set of equal E L equation is like that but remember del star by del lambda nothing but five of L star is like this so del L star by del lambda is nothing but five.

So, then you can see that this is this equation turns out to be simply five equal to 0, which is same as the constraint equation. So of the same constraint equation becomes part of the necessary condition. We do not have to kind of account, it separately that and think like that once you once you do the necessary condition optimality, once you plug in there essentially the same condition props up again actually.

Similarly trans versality conditions since you take a t o X o fixed and t f and X f free let say then you are like dealing something like this (( )) only applied a t f actually both the terms and if it is t f is also free then in addition to that we will get something like this. Actually so depending on what all conditions, we have we can talk about additional boundary conditions coming from there actually.

Now variable sense we have got X I mean X is n dimensional X lambda X un tilde dimensional t f is one dimensional. So, we have this (( )) number of boundary conditions since we have this x 0 fixed then this sort of conditions will give us un tilde equation. This will give us one more equation so, in a way the all the problems are always like that the number of variables and number of boundary conditions are same a only a matter, how do we do exploit that actually then as per as constrained equation generalities is concerns so these are all like a various constrained equation things like that.

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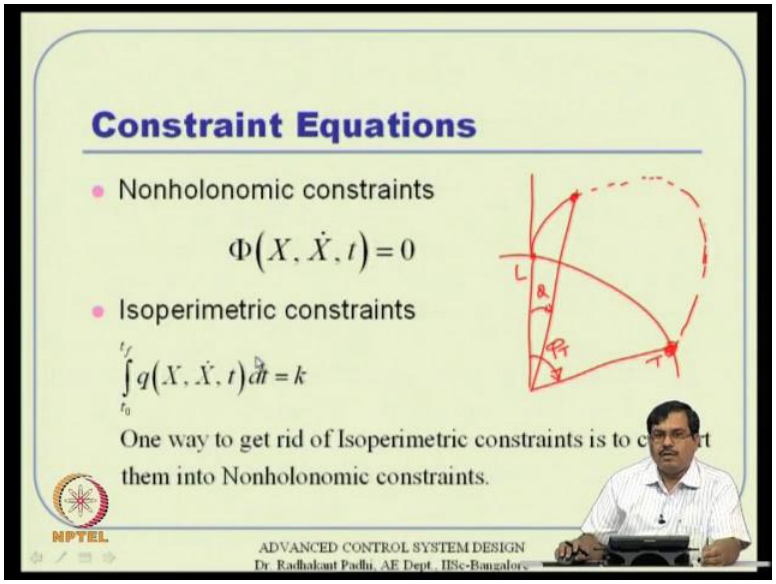

We can generalize that then they look at the constraint equation that we discussed before here what we discussed started with this is equal to 0. That is path constraint equation anyway and then we can tell if that is the case that is fine, we know how to do it for there is another set of constrained which can pop up, which can tell, I do not care about each of the values, I mean everywhere in the time as long as this integral since I got a constraint value actually.

So, let me give a simple example to kind of see, what is going on. Let us say you start with something like a let say, so we have a lenses point here. Let us see, we have a target location somewhere here now what you do I mean if they (( )), let us say while join these two lines with respect center of earth now with respect this angle (( )) with respect. This is also let me see, that is a reference line basically this and then tell my lows of vehicle. Suppose to go somewhere I mean my whatever the (( )) missile or whatever then it is going to fall like that as long as I do fall then I am I got my target actually (( )) that way but you can see that as long as I fall there. I had I am done that means I really do not have to constraint too much myself actually.

So, how do I do that now you consider this angle development starting from this reference line and this angle development let us say the final angle is something like five t? This is this is five not nothing do with the constraint five; I mean this is a angel five basically.

So, e as long as I a kind of I mean cover this angle ultimately then I am done actually. So, how do I formulate this I mean let us say I join any other point in the trajectory? I join a that way and tell this is my actual five at some location, I mean some point of time whatever is my launch vehicle launch vehicle did that is what the angle I have covered actually this is my five angle.

Now as long as this five equal to five I mean at t t f ultimately when the problem is over t equal to t f my five has to be same as this five t basically. So, that mean this constrained that I am posing here is in terms of the range angle, what is this angle is called range angle. Actually that total covered range angle has to be equal to five t as long as that is there then I am actually, so again these problems are also relevant in practice also. That is the message, there so they are called isoperimetric constraints and the question is how to handle that actually.

Now you can see that there is a difficulty here about there is a very clever of solution to that as well actually. Now what you do is we define this additional side variable. Let say because you remember these all are impressed functions of time X dot X and all that and it is integrity (( )) these are all integrated over I mean dt, they all integrated over time actually.

So, if I convert it something like some variable d by dt f some variable then I can apply d y dt f of that integrate then I integrate over dt, then I am done with that, because this integral derivative canceled out actually.

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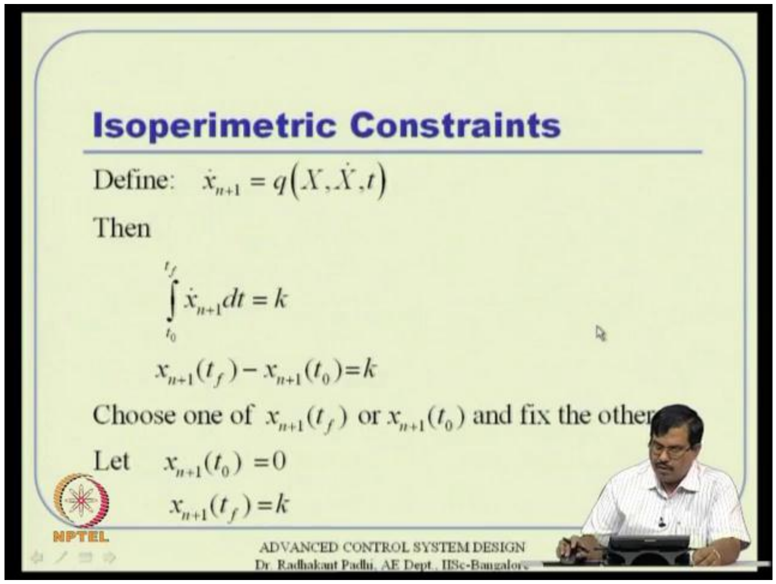

So, keeping that in mind I will define some sort of a new variable, which is d x n plus one divided by dt that means x n plus one dot is nothing but that then this is m equation that I adhering it turns out to be like this. I put it back that way and then tell now it is a derivative and then there is a integral.

So, instead of doing derivative integral now you can evaluate that and tell this is nothing about like that final value final value minus initial value of the integral variable not X dot but x n plus one simply so this is an addition (( )). This is this constraint, what we are talking here equivalent to having both the things together, I have this dynamic equation and I have this equation and ultimately actually. Now if I this is remember, this is the different between the variable values at t f and t 0 so if I choose one or one of that then I can fix the other one actually.

So, for example, if I choose this t 0 equal to 0 then the t f has to be going back to that example. This t 0 that means pi of 0 e 0 here, because that is the reference line anyway so I can choose that and then locate the final difference is nothing but the total range angle that I am looking at actually. So, that is these are various or if you generalize this problem a little bit you can talk about like this angle become three sixty degrees. It will take the total revaluation and come back to the same position that means it is also relevant for satellite gradients problems actually.

So various (( )) engineering problems can we handle that way and using both these non holonomic constraints and isoperimetric constraints. Now the second thing there is something isoperimetric so far. What we discussed is all equality constraint now the problem where the beauty of optimal control is like a not only up to get restricted by these thing but you can also tell I will bring in any quality constrained as well that means the state values and control values will be given retracted by certain bounds .

That we know a priori and if I know the bounds a priori that means let us say my controls are reflection should be lying between plus or minus thirty degrees or my let us say my altitude of the vehicle should not go beyond thirty kilometers. Let us say because after that the pressure is very low then all these things I can inverse that into the trajectory (( )) problem and then tell the solution-solution that I am getting from that problem is utterly going to be stabilizing. It is optimal think like that actually where then we are not going to discuss too much on that that will lead us to optimal controlled d for into optimal control actually and here is just want to get a flavor of optimal control and try to kind of explode as much as possible.

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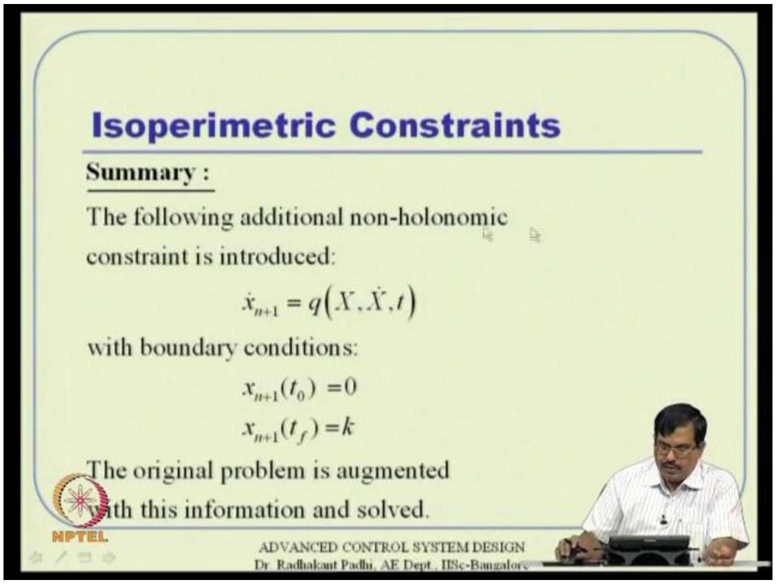

So, this is a isoperimetric constraint and then all these thing that I discussed in this class are taken from these two references anyway but essentially this entire class what we talk is a kind of a over view of all these calculus-calculus of variations and we can see next class that using some of these constraint kind of.

I mean concepts that we discussed here, we can actually very cleverly a very easily rather formulate optimal control problems for various class of systems and I will also try to give many engineering examples where this formulation of the problems like cost function boundary conditions. All are relevant in engineering practice especially in aerospace-aerospace in engineering and then we will one further for this linier class of problems something plays that actually. We have with that I will probably stop the references to follow is something like this.

The first one is the essentially one which essences talks about both static optimization and dynamic optimization you can you can also see this particular book which fairly recent book talks about the somewhat this calculus of variation concepts and lot of this optimal control. Theory as well and what we discuss next class and all I also taken from a very classic book which is like optimal applied optimal control by sen hoeven and all that I tell in the next class anyway I with this I think probably stop this class actually.

Thank you