Advanced Control System Design

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Lecture No. # 23

Static Optimization: An Overview

Hello everyone, so far we had seen many topics for linear control theory. And especially this stabilization, conception, controllability, observability followed by its usage for control design; and we also saw observed a design as well. So, we will proceed to a very kind of different concept now, where you tell control theory, I mean control design can also be done from optimization perspective; and that is where, we need to have some optimal control coming up and all that. But before that, we need to have some appreciation of what is called that static optimization or parameter optimization followed by I will take you through something some overview of calculus of various, and then based on those things, we will be able to synthesize optimal control.

And especially this particular course will primarily concentrate on linear control theory even though we will talk a little bit on linear control as well. So let us proceed for this particular lecture, where I will talk about static optimization or in some books it is also called parameter optimization, where things are not changing with respect to time, it is also stationary.

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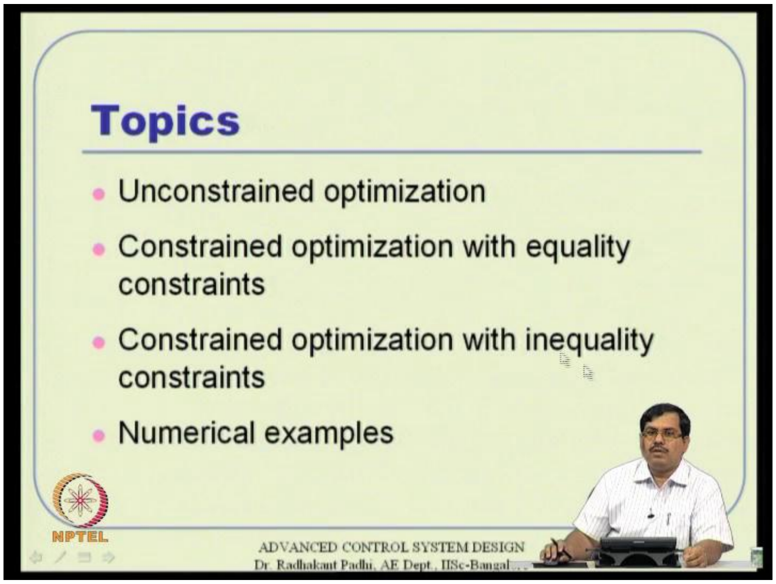

And with respect to a stationary object, we are going to minimize or maximize certain objective function. So the topics will be like this; first is unconstrained optimization so, there is no constraint acting on the optimization, then what are the standard results. Then we will talk about constraint optimization with some sort of equality constraint, and then we will see something overview on inequality constraint as well. And on the way we will see lot of numerical examples.

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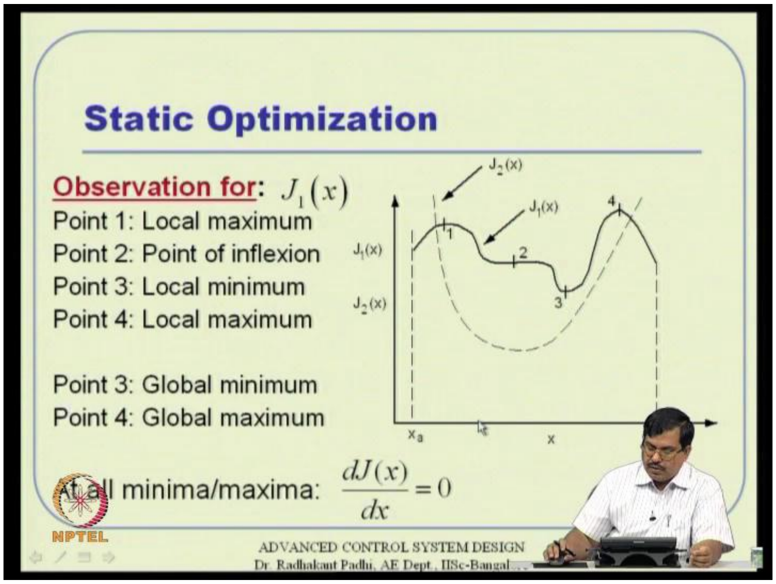

So, let us start with unconstraint optimization. What we are telling here is, let us concentrate on the; that is J 1 with a scalar. We will start with a very simple thing, where the objective function is a function of only one variable x. And we are analyzing this function J 1 and we want to answer this question, where all its local minimum and maximum exist and without going through too much of math, you can observe that these points 1, 2, 3, 4 are kind perceivable candidates, that we are bothered about. And even without analyzing too much of math, you can also clearly say that point 1 is a local maximum, point 4 is a local maximum, point 3 is a local minimum, point 2 well we cannot say too much of things right now. But it looks like a candidate for either minimum or maximum or what is mathematically called as point of inflation anyway so.

And the common thing that we were worried about is, we just observe everywhere, point 1, 2, 3, 4 is that everywhere, the first derivative is turns out to be 0. Everywhere, if I take a slope here at point 1, point 2, point 3, at point 4 local slopes, the slopes are all 0. So, it kind of gives us an idea that very perceivably, whether it is minimum or maximum problem, the necessary condition terms have to be like derivative equal to; first derivative equal to 0. And that we have to see little formally, this is just an observation at this point of time. Now, coming back to this curve J 2, suppose instead of J 1 I have J 2 and I want to see the difference between J 1 and J 2.

Suppose I have J 2 instead of J 1, then it turns out that I have only one minimum and because I have only one minimum between x a and x b. Whatever answer I get, even though I use only local conditions, that is going to be global as well, because I have only one solution anyway. Now, coming to; but that is in a same thing cannot be said about J 1, because suppose we have point 1 and point 4, both are local maximums, but point 4 will be kind of global maximum within this domain basically. But this particular lecture and this particular course we will not worry so much on global minimization or maximization. We are all interested in local conditions and we also concentrate mainly on necessary conditions, not sufficiency either.

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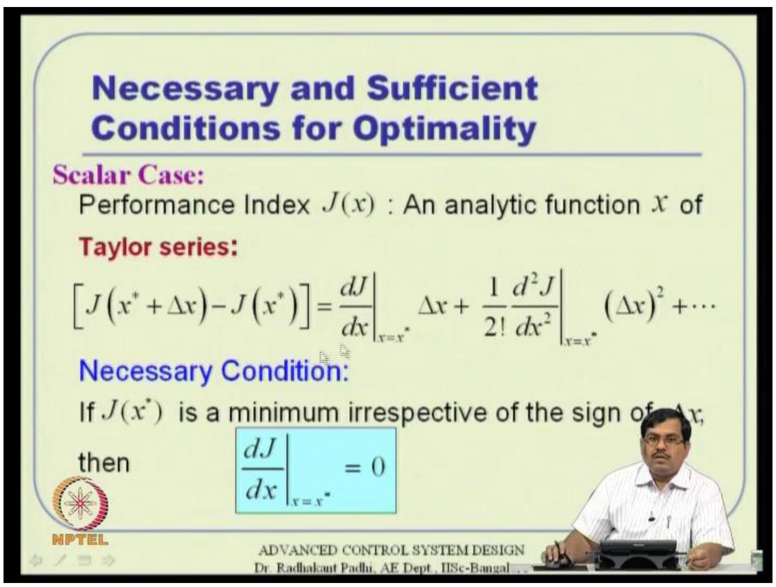

So let us see this analysis little formally. So, what we are interested in we want to have a performance index that is our objective for minimization or maximization depending on the problem you have to select one. And then, we are interested in analyzing this function at x star, that means, J of x star plus delta x, minus J of x star that is using Taylor series I can write it that way. And also remember that Taylor series is satisfies a nice convergence condition, where it turns out that the first term is determinant and the second most dominant compared to the rest of the terms.

And after the first term goes, the next term becomes most dominant compared to the rest of the terms like that. So if I take this minus that then it turns out to be that the series like that and what I am interested in by either minimum or maximum is that this function that I am looking at J of x star plus delta x minus J of x star. That is the difference between two values of the function in a neighborhood of x star. This quantity has to be sign independent of delta x. That means, either I go to negative side or I go to; let us say point1 then, either I got to negative left side of point 1 or right side of point 1, the function value has to decrease. That means, that the visual that I am looking is, it has to be either greater than 1; I mean greater than 0 or less than 0 irrespective of sin delta x.

But I cannot say that, as long as this term is non 0, because delta x is a sign sensitive term, it is a linear term. So, to make this left hand side sign insensitive, I must necessarily have the first derivative equal to 0. So, that will make sure J of x star is either maximum or minimum irrespective of sign of delta x. So, that means this necessary condition turn out to be that, this is equal to 0. Remember, this particular term is dominant compared to the rest of the series that is the reason why, we want to make that this first term equal to 0.

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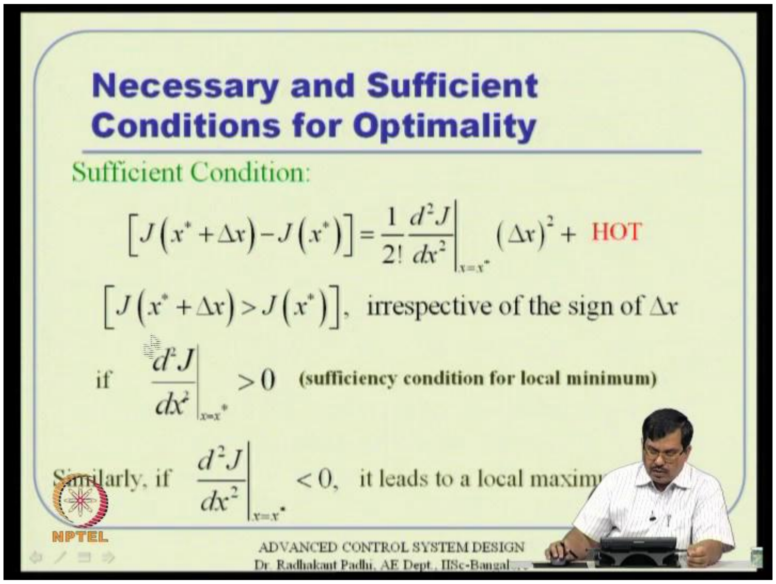

So that turns out to be necessary condition, what about sufficiency condition? So, after this has become 0, the series starts from there, this left hand side. So, let us see that left hand side we are left out with that, so, this first term is a quadratic term. And if it is quadratic obviously, it is sign sensitive already. And then, if I just have this condition that means, this secondary vector turns out to be greater than 0, then this left hand side is certainly greater than 0. That will makes that, this quantity is greater than that quantity, J of x star plus delta x is always greater than J of x star. No matter whatever the sign of delta x so, that means, this gives me a sufficiency condition that secondary way, it has to be greater than 0 for minimum and secondary way, it has to be less than 0 for local maximum.

Now, the question is, if it happens that you got the first derivative equal to 0 and the second derivative happens to be either of that then you have done anyway.

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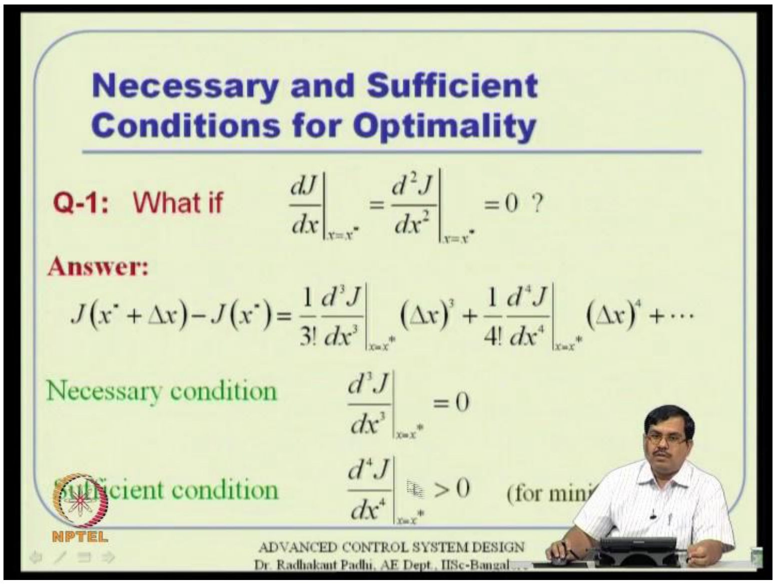

What about when the second derivative also equal to 0, then this term is also not there. Now, in that situation you have to analyze the third term and the third term means, this particular term here again this becomes sign sensitive, because delta x cube. So, because it is sign sensitive now, so, we have another necessary condition to tell that the co efficient must be 0. If it is nonzero, then certainly it is point of inflation and we are kind of done. But if the hope is still alive, then that means, the x star is either still maximum or minimum. Then, another necessary condition has to be the third derivative has to be equal to 0 and again the following sufficiency condition will rely on the fourth derivative this time and this analysis continues that way.

So, I mean, the idea here is suppose you want to get a minimum or maximum then, first thing we apply necessary condition, get the solution for that. That the extra values will pop up from there and for various extra values, you will check this condition, the secondary derivative condition. If it is either strictly greater than 0 or strictly less than 0 we have done, otherwise if it is equal to 0, then you continue with the third term. And then maximal; and you just observe that, this is also equal to 0 and if it is not 0 then it is a point of inflation and if it is 0, you go to the fourth term and strictly greater than 0 or strictly less than 0.

If it is strictly greater than 0, it is minimized, x star is the minimum point and if you strictly greater than 0, the x star is a maximum point. So, this conditions will rely on these odd powers of this derivative and all that odd derivatives and even derivatives essentially.

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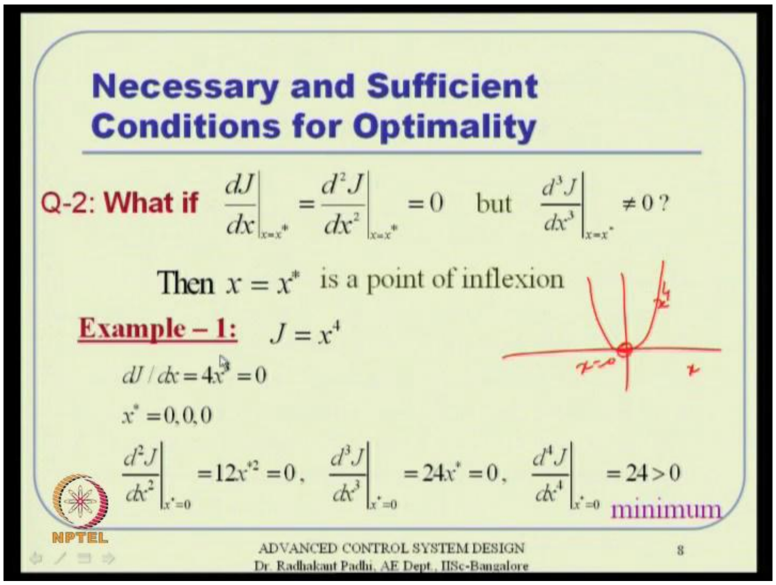

So, we will see some example very quickly. Suppose, J is like x fourth and the first derivative tells us that, this is 4 x cube, which is equal to 0 and hence extra solutions turns out to be all 0s. So, you take the second derivative is also 0. So, we cannot say anything about it right now. Let us proceed with the third derivative, which is even 0 at extra, which is more important. So, we will go to fourth derivative and fourth derivative turns out to be strictly positive and hence it is x star value which is 0, 0, 0 is certainly a minimum point. It is very clear from the picture also simply draw this picture like x versus x fourth, then the curve will turn out to be like this, this is x fourth curve. So, obviously this is say the minimum point at x equal to 0. So, that is what graphically it means.

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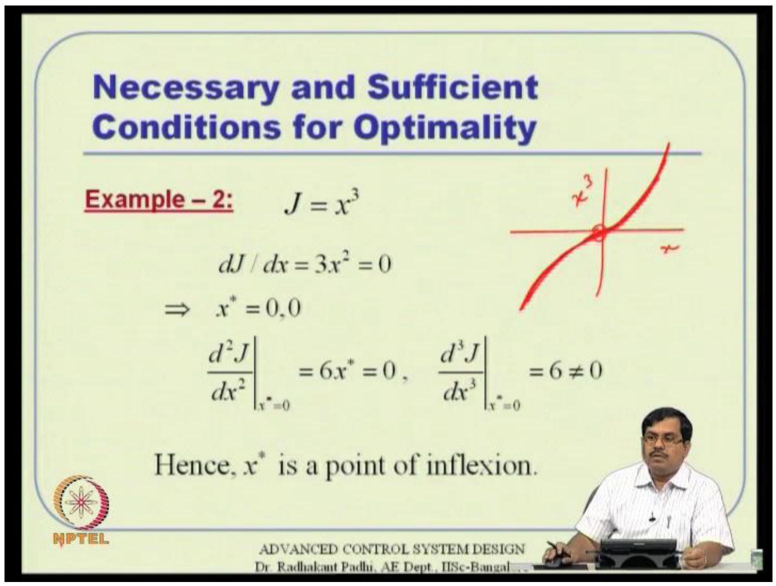

This is what analytically we can get. Now, let then the objective function little bit and tell that J equal to x cube, then what? You proceed with the same analysis, again we have two solutions here not three, all are same by the way. But then, secondary term turns out to be 0, however the third derivative is not 0 and hence it is a point of inflation. Again if you want to draw the curve, then this is something like that will happen. This is x versus x cube, then obviously this point has a point of inflation. The slope is 0 here at that 0 point, but after that this goes; I mean of the right hand side it increases, on the left hand side it decreases. So, that way, this is the point of inflation itself.

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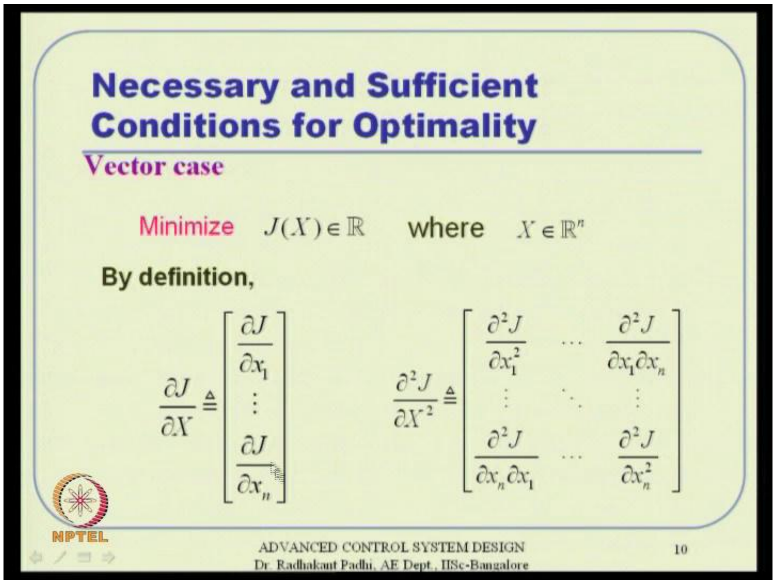

These are all about scalar case. Now, what about the vector problem, those are more practically relevant, because subjective function even though it is a scalar, the variable this that it takes can be more than one and that is more realistic. Thus you are normally your objective of minimization and maximization will typically act on several variables. So, if you take various combinations of those variables, then you will get some sort of minimization. So, how do you handle this case? The analysis is fairly similar to what we have done, but here by definition, we have to have this jacobian of J and ((.)).

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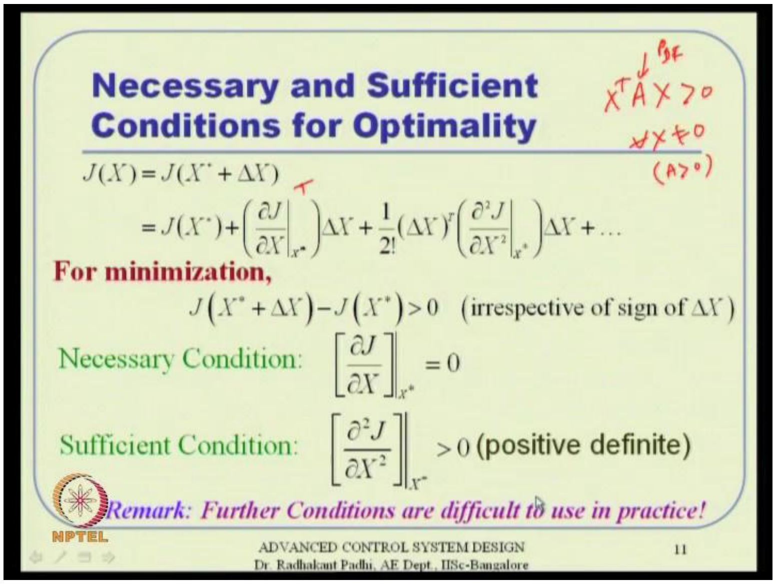

And then the Taylor series idea will stay as it is. We apply the Taylor series of J of X with the X star and then we analyze this Taylor series turns out to be that way. We notice that, this is this turns out to be like may be transpose out there because of this vector matrix mutation compatibility has to (( )). So, then second term like that and third term onwards is little bit more complex, (( )). So, if you take this one, minus that term, again this first term onwards it will pop up. And also remember, now that the function value, the difference that we are looking at so this term minus that term has to be sign in sensitive irrespective of whatever delta X components sign.

Delta X will have several components. No matter, whatever the sign of each of these components whether positive or negative. We must take this sign insensitive thing. So obviously, that means, geometrically speaking no matter, which direction we want to travel, your function is to either increase in all direction or it has to decrease in all direction. So, necessary condition from this analysis is obviously, we want to this first term 0, that means, the gradual vector that we are talking that the jacobian let we are talking J has to be 0. This delta by delta X is gradual vector now, evaluated at X star raised to be 0 and the sufficiency condition will be dictated by that, which tells us that this entire quantity leads to be positive.

That means by definition something like suppose, we take; tell that X transpose A X, if you remember that there is other things little bit. And then this is strictly taken 0 for all x not equal to 0, then A matrix is (( )), this A matrix is a positive definite matrix. (No Audio From: 15:00 to 50:09) So, symbolically we write that A is greater than 0, so, that means, coming back to this idea, that this quantity is strictly positive all the time, then this matrix has to be positive definite. So, sufficiency for necessary condition of either maximum or minimum this first derivative has to be 0, this is a gradient vector now.

And the sufficiency condition tells for minimization, this particular matrix evaluated at X star, that means, it is a pure number now, it has to be positive definite. So, the procedure remains almost similar to what we have done, first we will equate these equations and remember this will turn out n equations. And this is like a n dimensional vector now so this is the vector equation. So, we will have n equations and n variables, because from X is a multi dimensional vector now. So, you have n equations and n variables to solve for, once you solve it, you will get a star.

And at x star, you can evaluate this and then if it definite well this is certainly a local minimum point. And further conditions are difficult so, we will probably try to skip it in this review lecture.

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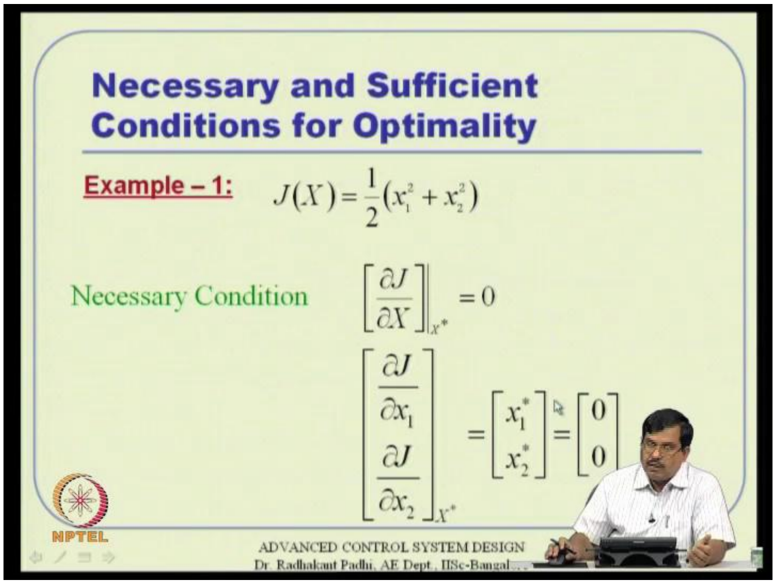

Now, come back to an example J effects is let us say half of x 1 square plus x 2 square. Then the necessary conditions told first gradient first you take gradient vector make it equal to 0. So, del J by del X 1 is turns out to be just like x 1 del J of del X 2 turns out to be x 2 evaluated at x star value so, that means x 2 star now. This both has to be equal to 0 0 this is direct solution now these are no more equations or if it equation then also you can solve. Because you have number of equations are equal to number of variables. In this case we got direct solution as 0 0 originally turns out to be a candidate solution.

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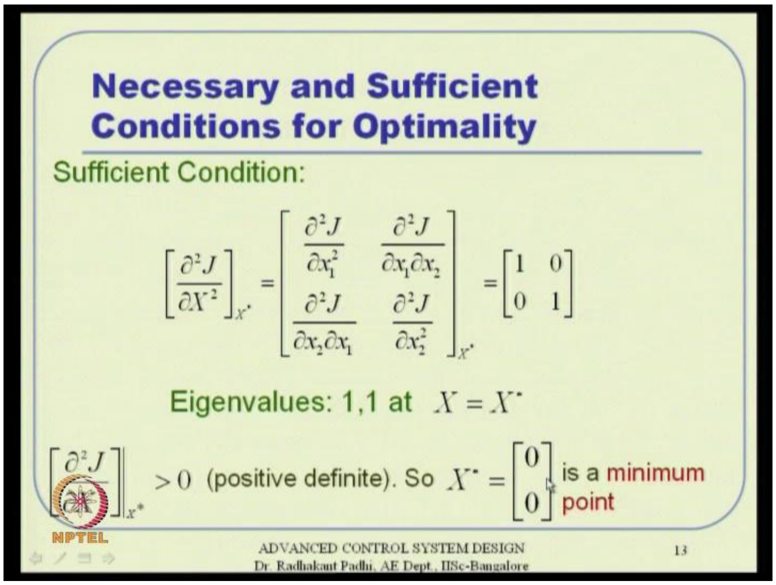

Now, let us see whether this candidate solutions satisfies these sufficiency conditions as well. So, sufficiency condition tells us that this matrix evaluate at star and this case it turns out to be an identity matrix it is independent of X star. So, identity matrix Eigen values are obviously 1 1 it is a diagonal matrix 1 1 both are positive and hence this matrix is positive definite. So, that means, this second this jacobian matrix evaluated at X star is positive definite and hence the X star candidate solution that we got from here has to be a minimum point solution.

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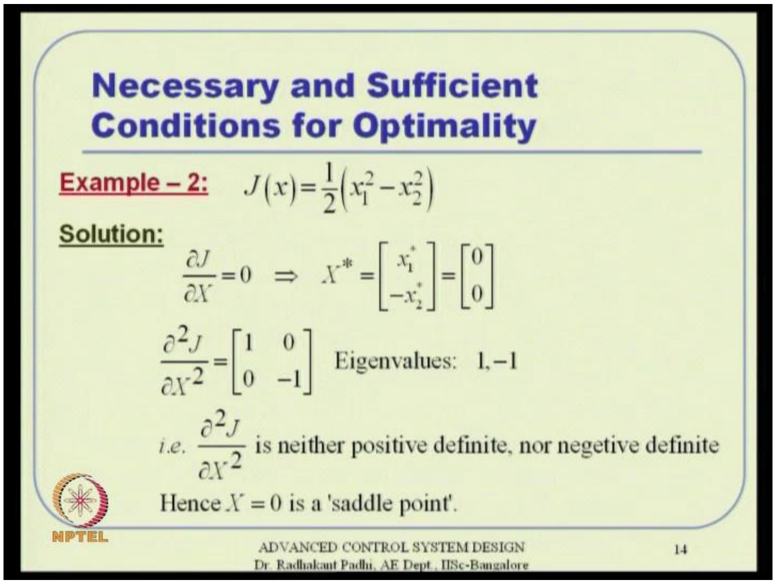

So, that is how we proceed with that now, suppose you change this problem again it will be and it take a different problem now. Let us say instead of x 1 square plus x 2 square we just change the problem to x 1 square minus x 2 square. Then what happens? Will gain start with the same idea we will just take the first derivative and that will give us the x star candidate solution again it turns out to be 0 0. However when you take second derivative the Eigen values turns out to be one positive and one negative and hence it is neither positive definite nor negative definite. And hence it turns out to be a subtle point, in other words one side the function gets a minimum value and the other side it turns out to the function gets a maximum value itself.

And these are typical text book examples will be like horseback something so one thing. So, one side it will be going positive I mean one side it is increasing and the other side it is decreasing. So, picture if really speaking so, picture that way then it is let us say s 1 is this way and s 2 is this way, x 1 this way and you have J that way. Then this function take is somewhat like this so, this is a 0 solution no matter whichever direction you want to go you will get a solution which is minimum at 0 0. Now, the same thing cannot be said for these this will have something like a one side I mean I do not know whether I will be able to draw a picture correctly, but let me try. So, this will be like one side it is minimum, but the other side if you see it is something like it will happen that way.

One side it is going to increase, but the other side it is going to decrease. So, that way I mean this may not be (( )) picture to visualize you remember but you can see the horse break whatever people put on one side it is like this the other side it is like that. So, those are the type of things what we call as the subtle point it is a multi dimensional thing, it is not the no more called as the point of inflation it is called subtle point anyway. Those are all free optimization both in scalar as well as multi dimensional case now the free optimization is also not very relevant in practice. In practice we have to optimize certain cos function subject to certain constraint equation.

And constraint equations if we take this then how do you handle those so, first thing we will talk about equality constraints then we will move on to what happens when there is inequality constraints.

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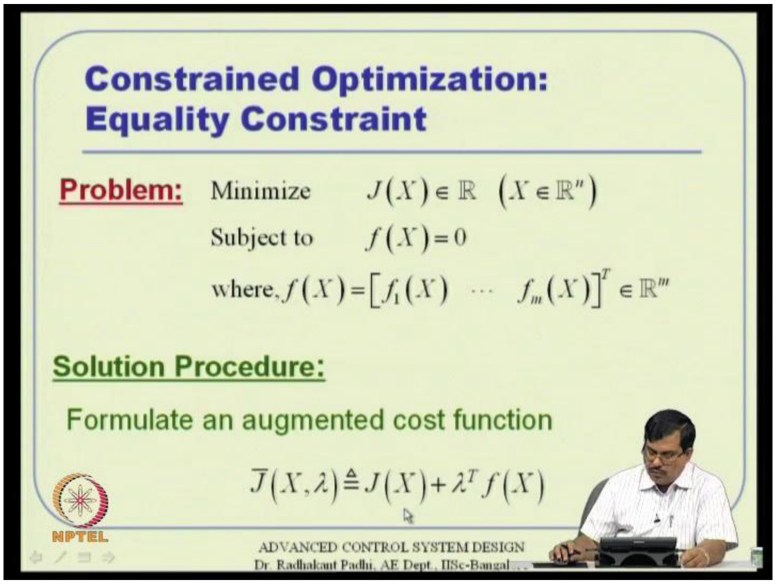

And these are all in multi dimensional case now. So, our objective is something like this if we want to minimize without those losses that we will talk minimization only. Maximization will be part of the deal it is just that we have to take care of the conditions appropriately. So, we want to minimize a objective function and when you talk about optimization for typically the cost function turns out to be always scalar. Unless you talk about objective optimization that is a different subject all together. But normally the objective function of the cost function is a scalar quantity, but the variable that it takes can be multi dimensional.

So, you want to minimize this cost function subject to this equality constraint and this equality constraints need not be one constraints it can be several constraints we have f 1 of is to 0, f 2 of X is to 0 like that f n of X is 0. That means, the number of variables are n, but the number of equations are m they need not be same either basically. You see that this is a m dimensional constraint, it is a n dimensional free variable the objective function is a scalar quantity. Now, how do you solve it, whatever solution we intend that it has to certainly satisfy this constraint first it does not satisfy the constraint we are not interested that is not the right solution.

So, how do you that it comes from a great theorem that (( )) is a great mathematician that this problem can be equivalently solved by putting by formulating an augmented cost function in terms of lamda. Where that is defined as like that I mean theoretically speaking lambda transpose f of x so f of x is equal to 0. So, that means it is kind of 0, but you consider that (( )) and just concentrate on this augmented cost function. Where lambda is a free variable let us say that means lambda takes an appropriate value in the process of optimizing J bar itself. So, we just consider this J bar as a free optimization problem, where we have to deal with this extra variables lambda remember this lambda is same dimension as f.

Because ultimately this is a scalar that means you have m such lamdas, lambda 1 lambda 2 to lambda m basically.

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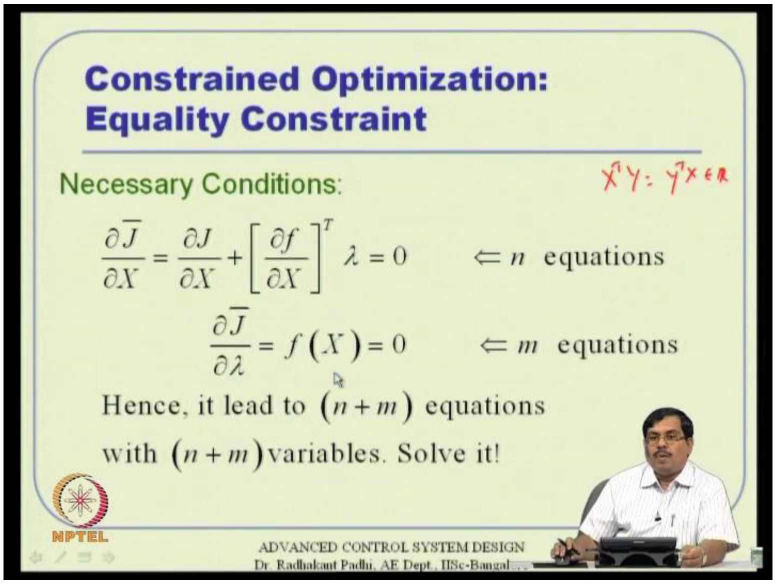

So, we have J bar which is J of x plus this lambda transpose f of X, where X and lambda both are free variables that is the idea here. So, how do you do that now necessary conditions turns out to be one with respect to and the other one with respect to lamda.

So, if you take derivative with respect to X then this is J by del X plus lambda transfers del f by del X we can also like that (( )) as long as the multiplication is a scalar quantity you can have X transpose Y, Y transpose X also and then this is a scalar quantity. So, similarly, you can alter the sequence of derivative here if you want you can keep it also basically (( )). So, we just take an alternative way and tell this del J bar by del X is like this del J bar by del X plus this one del f by del X transpose. So, this will give us n equation this is a scalar and this is vector and so, del J bar by del X is a scalar is a vector quantity and this is a vector equation, give us n such equation.

And how about by del J bar by del lambda that also is a free variable we have to take derivative with respect to that also remember that. This does not contain lambda this is gone and this is only simply lamda, lambda appears linearly here basically. So, the answer to that is simply for f of X. So, that this second condition turns out to be f of X equal to 0 and that will give us an equation and this f of X equal to 0 means this m constant equation. So, it is a part of the necessary condition that we want to satisfy a for this free optimization problem that necessary condition turns out to be the same constraint equation.

That means, we are not violating the constraints. So, we want to solve these two equations these two set of equations together where we have n equations coming from these and n equations coming from that. And how many variables and obviously we have X which n dimensional variables and lambda which is m dimensional variable. That means, we have m plus n equations with n plus m variable so, obviously we can solve this and once you solve this you get the solution for X and lamda. Where we are not typically interested in the lambda variable solution we are interested in the only X variable solution.

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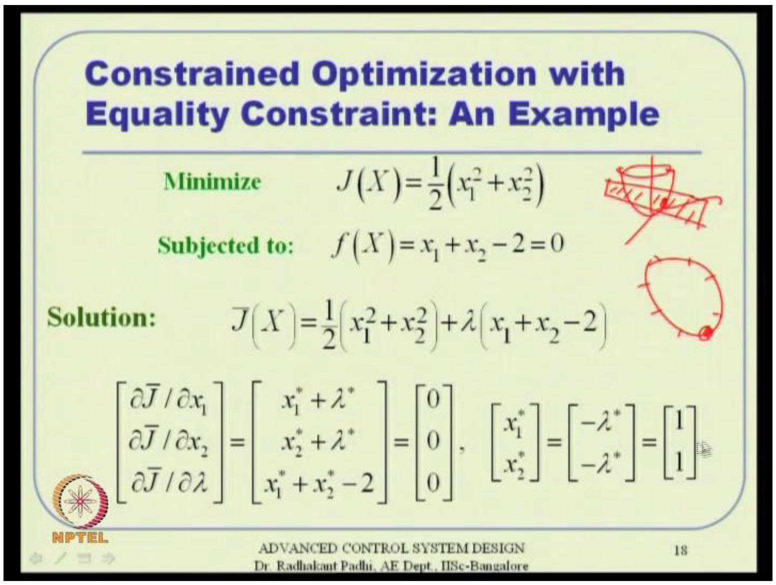

But lambda helps us in getting the solution that is the tricky now let us take an example having we will go back to the same example, where we earlier got 0 0 as solution. Now, will we want to examine the same cost function minimize that subjected to this constraint equation now. Remember this is a plane equation x 1 plus x 2 equal to 2 basically. So, this cost function that we have earlier that the subjective function the solution is no more at 0 0 value then because we are interested in analyzing some sort of equality constraint with respect to a plane now. That means wherever this plane cuts this objective function on that boundary surface will get some sort of a relief. Then on that relief we are interested in wherever this minimum maximum happens.

So, we are not bothered anywhere else we are only bothered about that particular (( )). So obviously the 0 0 is no more a solution kind and just now let us see if we get that answer or not. So, by procedure we formulate an augmented cost function like this so, that means this is what it is this J plus lambda times this entire equation equal to 0. Remember if it is given like equal to two then you take it as minus then make it to 0 the augment. So we have only one constraint equation so only one lambda basically. So, the solution terms have to be the necessary condition we have to apply the same thing with respect to x 1 with respect to x 2 and with respect to lambda as well.

So, if you that the with respect to x 1 is like this with respect to x 2 is like this and with respect to lambda is the same constraint equation again basically. So, we have got three variables x 1 star x 2 star and lambda star and obviously if (( )) the solution then x 1 star is minus lambda star x 2 star is minus lambda star. By the way all the star notations of the some books follow this they tell it is the optimum solution. So, when you see the star optimal solution basically so, if this x 1 star and x 2 star is minus lambda star minus lambda star both are same. And then you substitute like it here we will get that lambda star minus 2 lambda star minus 2 equal to 0. That means, lambda star is minus 1 and hence minus lambda star is plus 1 so, you get x 1 star x 2 star you get 1 1.

So obviously somewhere here in the function somewhere here that means the minimum will be somewhere here. Now, if you generalize that a little bit more this is a specific case for quadratic cost function with linear constraint. By the way this quadratic function a linear constraint are heavily studied in both static optimization and dynamic optimization optimal control primarily because we have this close function simultaneously, ultimately we are getting is a nice close function and if you can formulate the quardiatic of cost function and linear constraints (( )).

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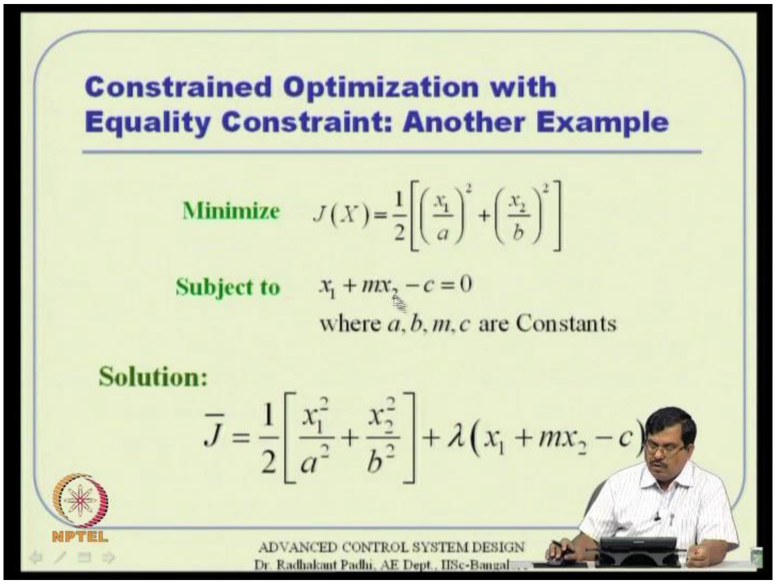

So to generalize that a little bit you can take little more general quadratic cost function into two variables and then you can take a generic kind of equation here. And then you can proceed with the same algorithm and then finally, get a solution which is very generic that is by blocking various values of a b and m n c you will get a general solution ready already basically.

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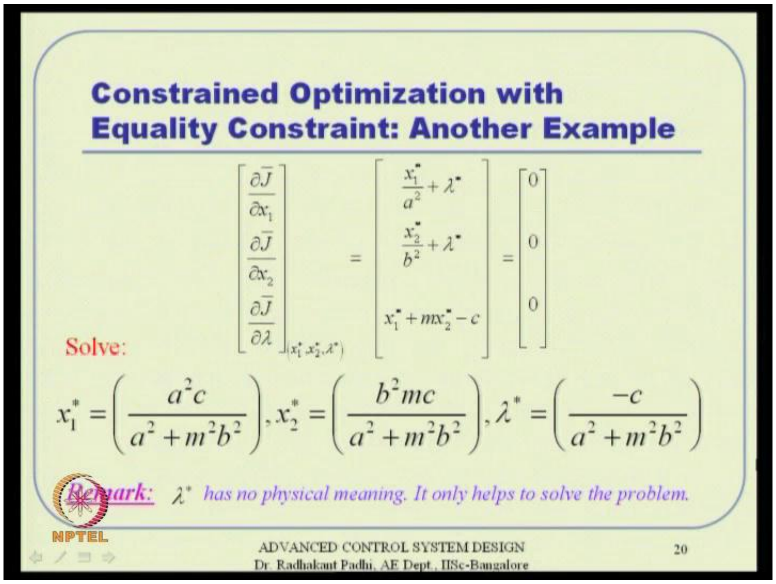

Also a small remark that your lambda star that your ultimately getting has no physical meaning, it only helps to solve the problem, that is true for optimal control also basically.

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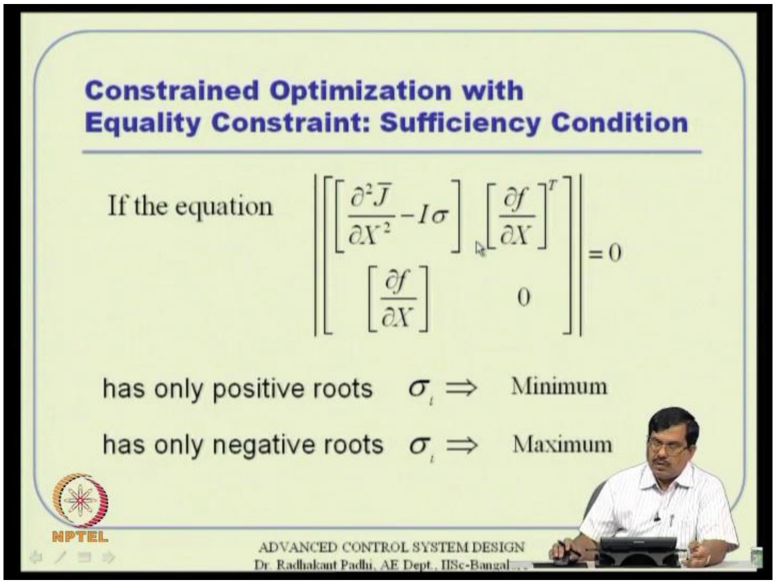

Now, what about sufficiency condition for constraint optimization then the sufficiency condition turns out to be that way it is no more only this way this matrix. You have to formulate this equation, you take this minus I sigma and augment this and you have to take this transpose here or here depending on whether you have single constraint or multiple constraint. And that will very apparent from the dimension of what you are getting ultimately this has to be a square matrix. So, once you put it there and make a determinant equal to 0 you get a equation for sigma and it is if it is only positive roots then it is minimum problem, if it is all negative roots it is maximum problem.

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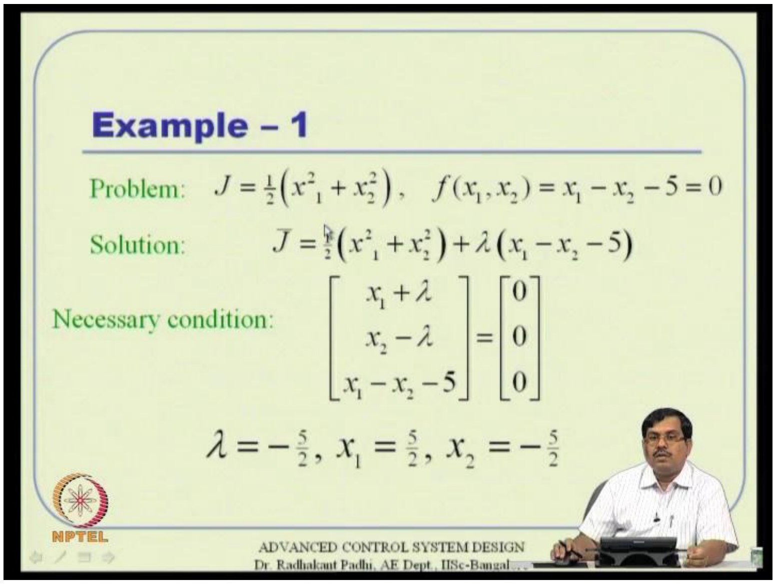

This is a maximization basically so, that is the sufficiency condition for equality constraints problems and you can solve that (( )). So, for example, if you take the same quadratic function with a little different constraint equation. Then you can proceed with the same idea then you get necessary conditions when you apply it gives you that kind of a solution.

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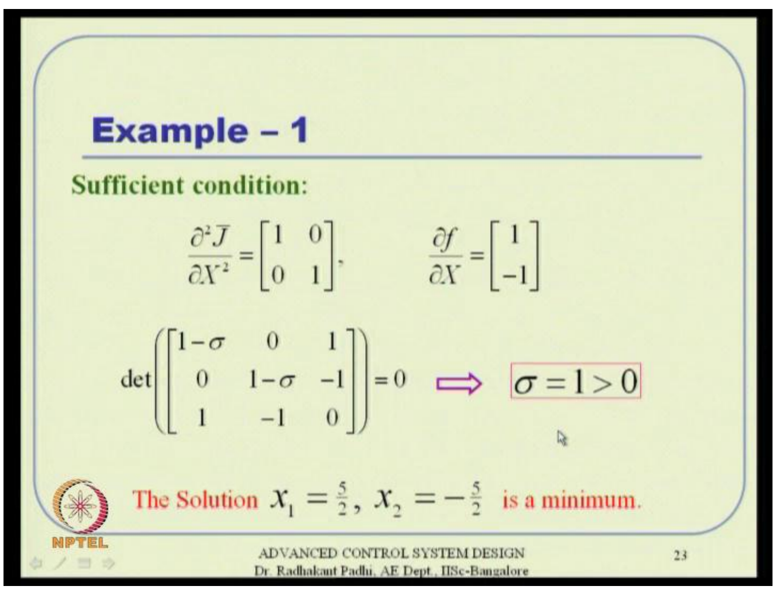

Now, what about sufficiency of this secondary turns out to be identity here so I will take secondary here just look at this secondary somehow I am not yet done I have to do this other thing. So, I take del f by del x turns out to be that way and put it here and this is a transpose of that. So, I put it here and then ultimately I form the determinant then the sigma solution turns out to be one which is certainly greater than zero.

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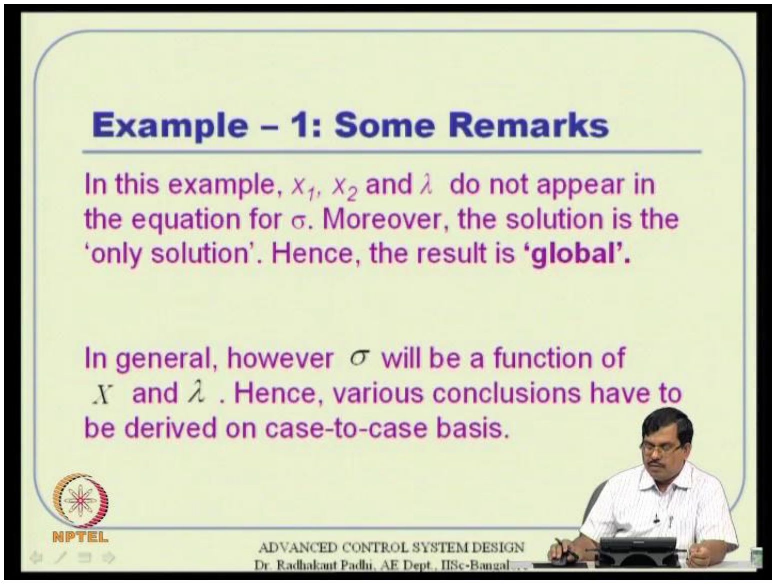

Strictly positive so that means solution is a minimum solution so, some remarks what you have is in this example X 1 X 2 and lambda do not appear in the equation for sigma. What you get is the equation for sigma is all constant numbers and only sigma there. That means the solution there you are getting here is independent of X 1 X 2 and that lambda necessary condition throws. And obviously enhance the solution is the only solution we do not have this multiplicity and all that enhance the result is global. However in general if you see this algebra this will contain X and lambda somewhere and then if you have that those kind of situation you will have multiple things and you have to answer for each of these case separately.

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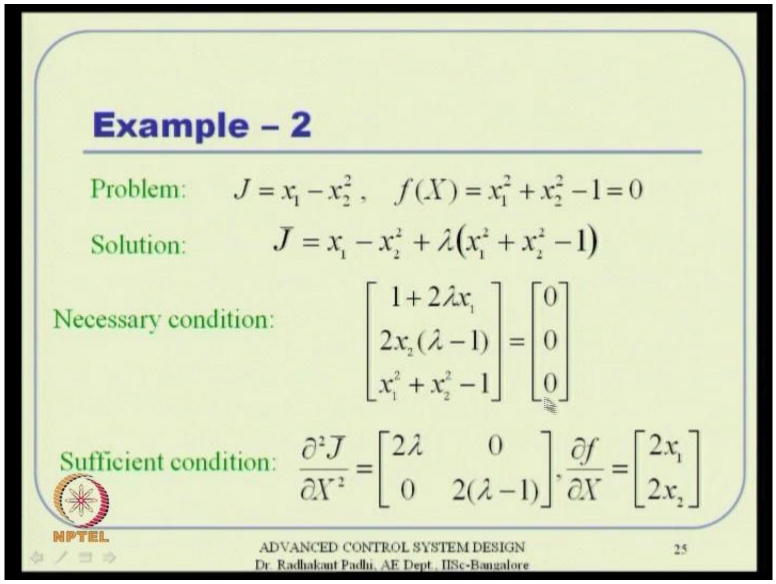

So, various conclusions have to be derived on case to case basis that is the remark this is a general example just like this. This is a subjective function and the quadratic constraint now so this constraint tells you that x 1 square plus x 2 square tells you equal to one is nothing but a sphere in two dimension and if you take this is a three dimensional problem x 1 square plus x 2 square equal to 1. Gives us some sort of a well if you if you plot in two dimension this is constraint problem in two dimension it is a circle basically x square equal to 1 is a circular equation. So, we have to minimize or I mean this constants subject to this circle equation and many times the solution may not exist also that is another problem.

All throughout this lecture and probably throughout this course (( )) and try to find the solution basically. Now, if your constant equation does not intersect with the cost function then obviously this solution does not exist. We will not talk too much on those existing situation along that way, anyway so we will proceed with the solution of approach. So, obviously this J is J bar is J plus lambda times one again this is single constraint equation. So, this part is J this part is lambda times this remember that this equal to 0 it is given in that (( )).

And the necessary condition turns out to be these three again and the sufficiency condition now is a function of lambda at least. I mean the secondary del of J bar del square of J bar del square turns out to be like that and del f by del x turns out to be things like that. Now, this is J bar so J bar will throw you lambda x and lambda and all that so this is del square J bar which is only then del f by del x contains x 1 and x 2 also. And we will put it back together that the sigma equation that we are talking will contain x is also some none the less.

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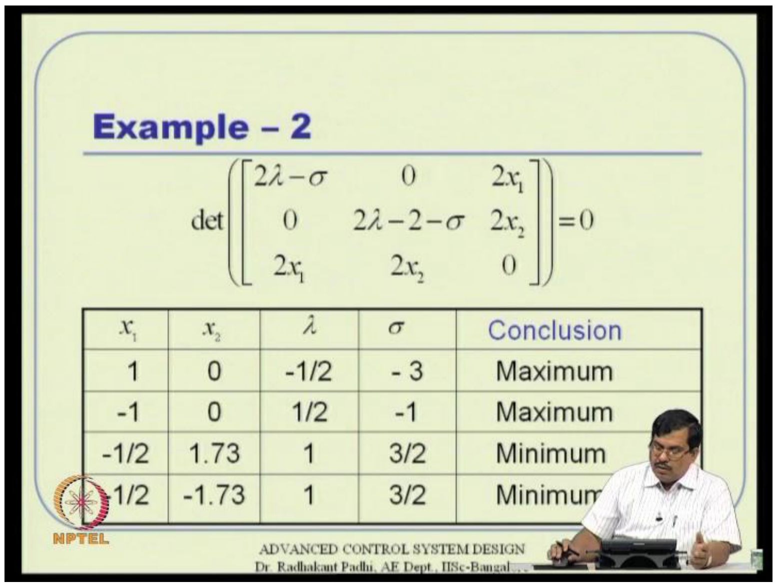

But the solution is a function of x 1 x 2 and lambda is here so you get various cases. Now, because this a necessary conditions also nonlinear this like a quadratic equation this is a multiplication term and this is also a multiplication term. So, various possibilities it will throw and this possibility turns out to be like that these four possibilities you will get. And then you have to get sigma values that is solution of these at those points if you evaluate that then the value turns out to be like that. So, these two values are strictly negative and these two values are strictly positive. So, what you are telling here is this combination what you are getting and 1 0 and that is a certainly a local maximum point minus 1 0 is also a maximum local point and these two are local maximum points.

And all these conditions necessary and sufficient are all local so, if you really want to find global maximum and global minimum out of this. Then you have to see, you have to evaluate the cost function on these two values whatever solution you are getting here you formulate one more column for the evaluation of cost function. Evaluate the cost function and take the minimum of these sorry the maximum of these two will give you these two will give you the global minimum. And the minimum of these two will give you the global minimum we are not too much interested in the global minimum. The typically gradient solutions are not suitable for finding out the global minimum anyway, but the concepts are like that.

So, if really you want global ideas, global answers then the cost function at various points then compare these two maximum points or multiple maximum points find out which is the most maximum value there then that becomes global maximum and similarly for global minimum. Now, what about constraint optimization with any quality these are all with equality constraints. And inequality constraints are anyway and these are also even real life problems are partly equal to, but the constraint equations will be partly equality partly inequality.

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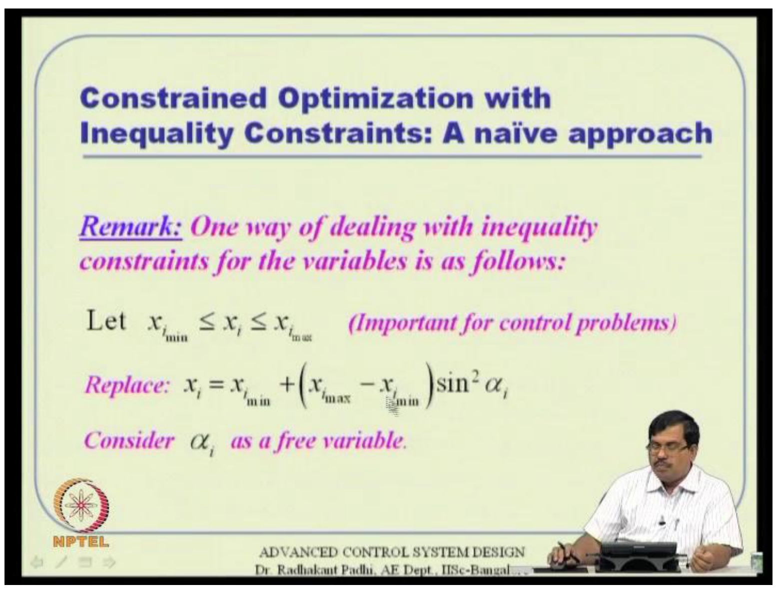

So, let us see how do you handle the inequality constraints one primitive idea is like this suppose your cost function is a function of several exercise and each of these exercise are bounded with between these minimum and maximum values. Typically if you these control design problem then each of these control variables will be bounded between certain minimum, maximum values. If you are interested only in satisfying the inequality constraints of the control variable so, what is the idea is let us I will replace this variable by another variable alpha I which I tie up with this that way. This is the equation that will formulate for x I and wherever x I is appearing in the objective function in the constraint equation everywhere I will replace this x I with this equation.

And then consider alpha I (( )) so I will solve this from alpha I and then I got a solution for x I first idea here if I formulate the equations. If you remember sin square I can take values between 0 and 1 and if it is 0 then obviously x I will get a minimum value, if it is 1 it will get a maximum value. Anywhere in between it will be kind of an intermediate value. So, the whole idea is to replace x I with this equation and then consider alpha I as a free variable. But unfortunately this approach does not work very well then so, many times this sin square alpha I the solution that your getting for alpha I satisfy sin square alpha I is bounded by 0 and 1.

It kind of falls according then if it is there then you will not get a solution for alpha I and hence you are not getting a real solution that is not nice and on top of that if you do not have the pre variable individually constraint that means there are equations and inequality constraints then this approach will not hold good either. Individually the components will be bounded between maximum and minimum value then only you can apply this.

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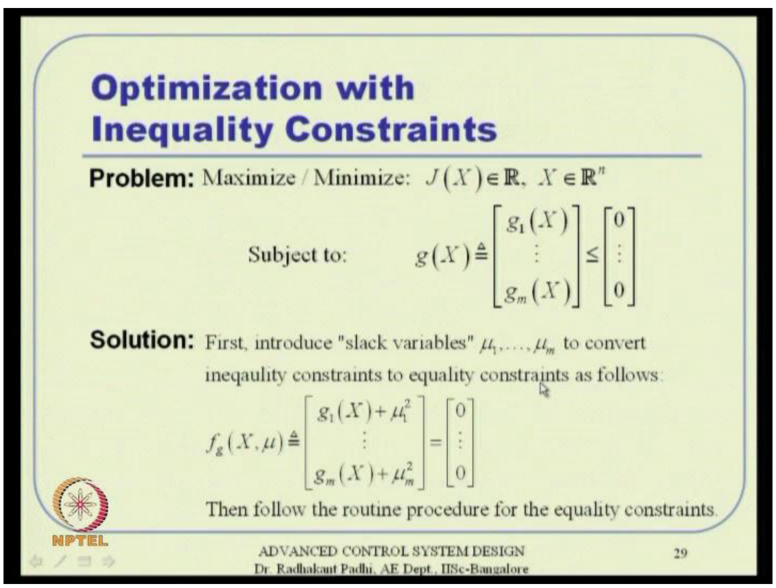

So, what are the other idea then so, let us talk about little formal way that we are interested in minimizing or maximizing this scalar cost function again the variables are dimensional. With this n dimensional constraint equation which are less than equal to 0 0 0 now and then equivalently we can talk about better (( )). So, we are interested in this inequality constraint reasons and tell how to handle this kind of thing. So, the idea here is we will introduce what is called as slack variables, these are not equal to 0 they are strictly less than 0. So, if I consider certain positive quantity is mu 1 square mu 2 square up to mu n square and simply add them to these equations then I can write it as equal to thing equal to 0 0 0.

That means this inequality constraint will be able to convert equivalent equality constraints by introducing the slack variables. Then once you are there you can follow the routine procedure for equality constraint, but the thing is we have to have some solution for mu 1 mu 2 up to mu m. And even with this less than equal to 0 we do not know how far away they are from 0 that is another thing. And we want to enhance the we do not want the conditions the appearing in terms of mu (( )) so, ultimately the conditions that we want to get should be independent of these mu 1 mu 2 up to mu n. How can we do that let us proceed with this we now have a equality constraint we have cost function to minimize so, we know how to do anyway.

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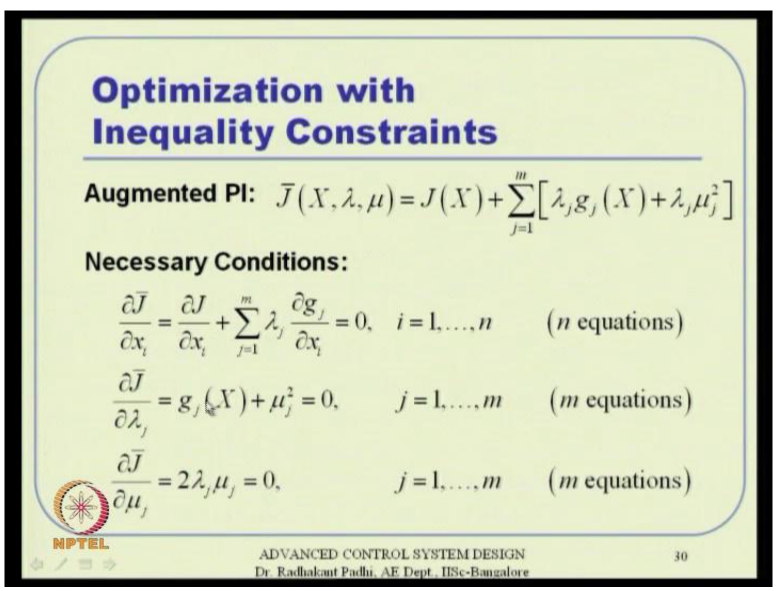

So, we formulate this J bar which is a function of X lambda and mu also mu is also a free variable so, we put it there together and then we use this lamda. This is the summation sign is written is alternative thing (( )) you can essentially write lambda transpose this vector equation. If you want to, but this is more reliably written in terms of summation sign. So, what do you do so necessary condition turns out to be like these set of equations has to be satisfied. You have n equations coming up from x variable x 1 x 2 then x n equations coming from lambda variable and one more set of m equations coming from mu variable.

Remember mu and lambda has to be same so, n number of slack variables and same number is constant equations that way the dimension of mu and lambda will be the same. Then analyze this term by term this gradient turns out to be like that and this one turns out to be the same constraint equation again the equality constraint. And the third one turns out to be two lambda as a mu z equal to 0 because this is appearing from here. If I tale del J bar by del mu then the term is here and that will turn out to be two time lambda J equal to 0 here. So, I have to use these equations to get some sort of a condition that should be preferably independent of mu values you do not know anyway.

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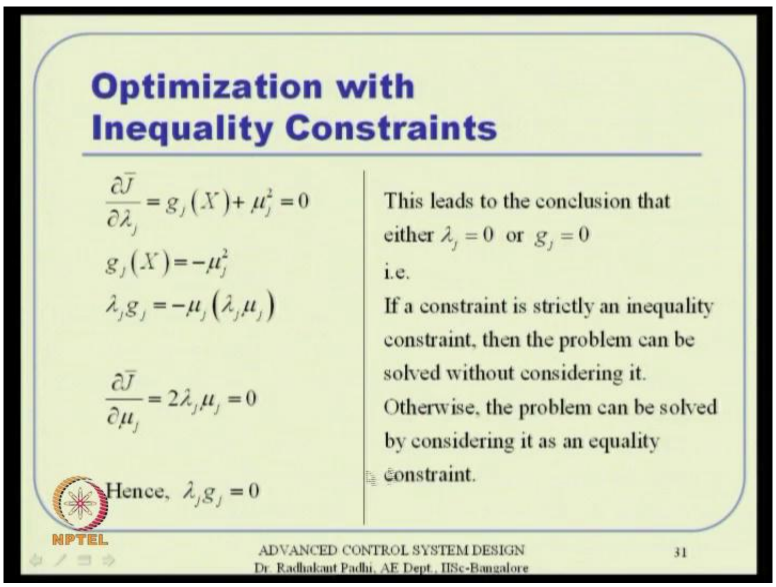

So, let us analyze the equation little bit so, del J bar by del lambda J this equation if you see if you take and this one I will take the right hand side. And g J of X is minus of mu J square and suppose I multiply g J with respect to lambda J this time and also with respect to lambda J that side so then I have this now third condition to apply. The third condition that lambda J mu J is equal to 0 so, if I apply that then this term turns out to be 0 anyway. So, if this turns out to be 0 then what it tells me that lambda J should also be equal to 0 so, this leads to the conclusion that either lambda J is 0 or g J is 0 because lambda g J is 0.

Right so, that means either lambda J is 0 or g J is 0 what does it tell you that means if the constraint is strictly an inequality constraint. Then the problem can be solved without considering it otherwise the problem can be solved by considering the equality constraint. Means either the constraint is active that is it is like a boundary it is active means it is in the equality side or if it is inactive we do not need to consider that if it is active we need to consider that it is the equality constraint basically. That is what it tells here so, what is the solution approach now we go ahead and do this so, far we have used these two equations in the analysis what about this one we have not used is we have to certainly use this.

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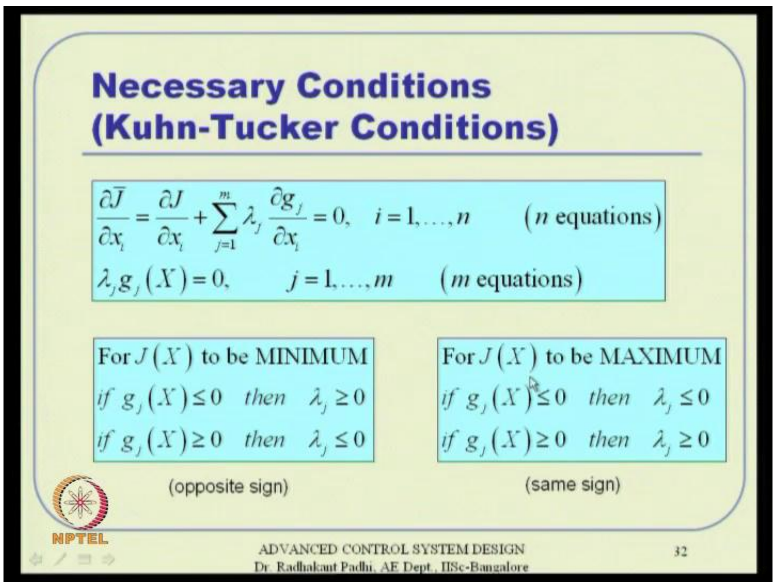

So, the procedure tells us that you formulate this set of equations and n equations. And now you consider this set of equations lambda g J versus this is m equations and this one contains only lambda remember mu is no more required mu is no more there here. Even though we started like that we are just concentrating on that part of the thing alright so and then you can carry on with this analysis. Now, I will try to kind of skip a little and then you tell J of x needs to be minimum. If these two conditions either of the two happens that means if g J is less than equal to 0 the lambda J has to be greater than equal to 0 or vice versa that means g J and lambda J must have opposite signs. And for maximum they should have same sign basically.

So, you have to get a multiple set of solutions from here and then for each of that you have to see whether all this I mean all these things I mean all these happens. That means if g J is less than equal to 0 then lambda J has to be 0 for all J specifically remember J is like one to m then similarly, for maximum these are all what is called contour conditions. So, these contour conditions are very in need because ultimately it gives you a precise condition to check.

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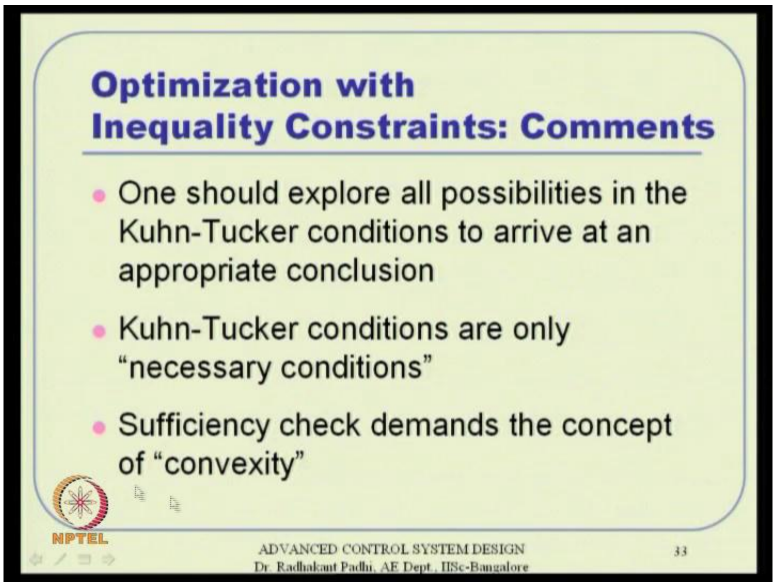

So, some comments that one should explore all possibilities in the contour conditions, to arrive at appropriate conclusions. And also remember that contour conditions that we are doing it is all derived based on these necessary conditions so, certainly the contour conditions are only necessary. Now, sufficiency levels which also requires the concept of convexity of a function.

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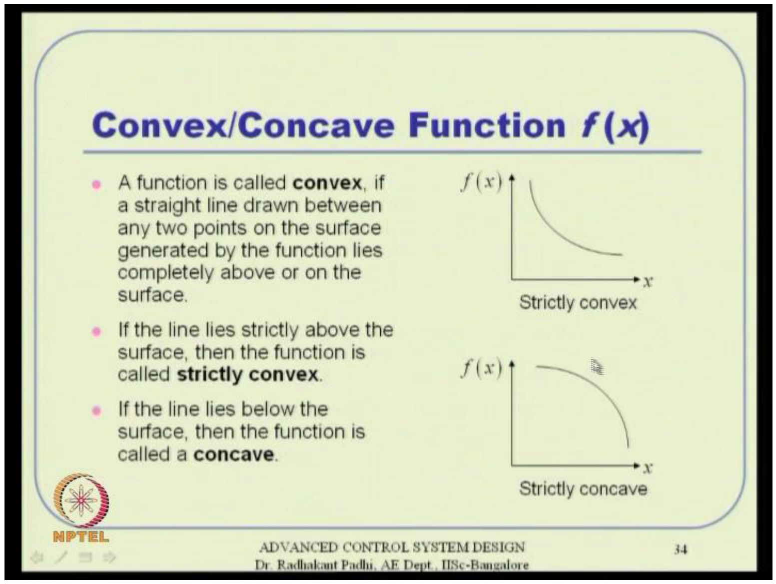

So, what is it the function is either concave or convex provided certain conditions are good and if it is something like this it is a strictly convex function if it is something like this it is a concave function. The whole idea is the function is convex, if you draw a straight line between any two points then the straight line should lie strictly above the curve. If I take any two points here I just join them together now the line lies above this curve. And in this case the line will lie below the curve so, the line is strictly above the surface all the time then it is strictly convex and then if strictly below then it is concave.

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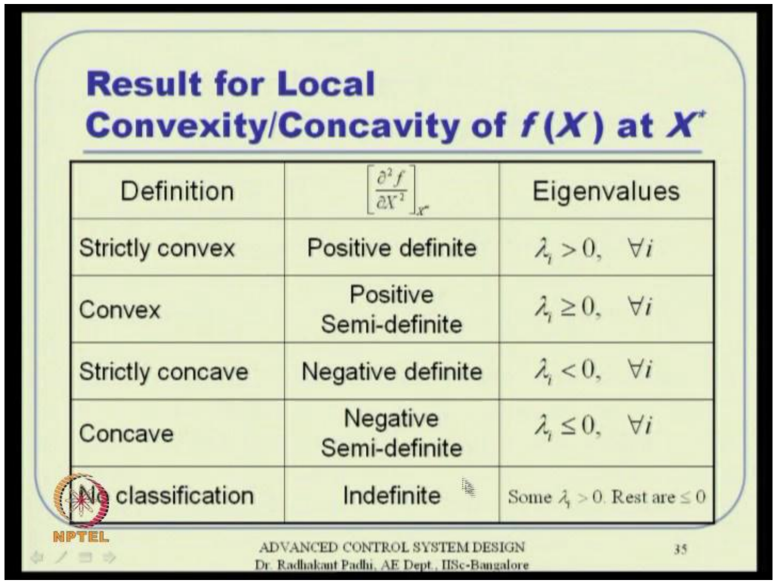

So, how does this convexity, concavity useful in optimization and the result turns out to be like that result will see here. But before that how do we check it this is simply a concept. So, for evaluating I mean for getting an answer that the function is strictly convex or strictly concave at a particular value of X star, we have the concept of positive definite and negative definite again comes. So, if it tells we can always evaluate this asymmetric and evaluate at X star value and then if this asymmetric turns out to be definite then the function is strictly convex. So, this is very easy now because we have a function f of X in the multi dimensional in general we have X star value and around that value we will be able to evaluate this matrix.

And once you evaluate this matrix we can take the Eigen values of that and the if it is strictly positive definite it is strictly convex function. The f of X is strictly convex at X star, around X star values that is what you are talking. So, then if it positive it is semi definite it is only convex, it is not strictly convex similarly, this is definite that means all Eigen values are strictly 0, then it is strictly concave and so on. And their some are positive and some are less than equal to 0 like that, then it is certainly in definite we cannot have any classification.

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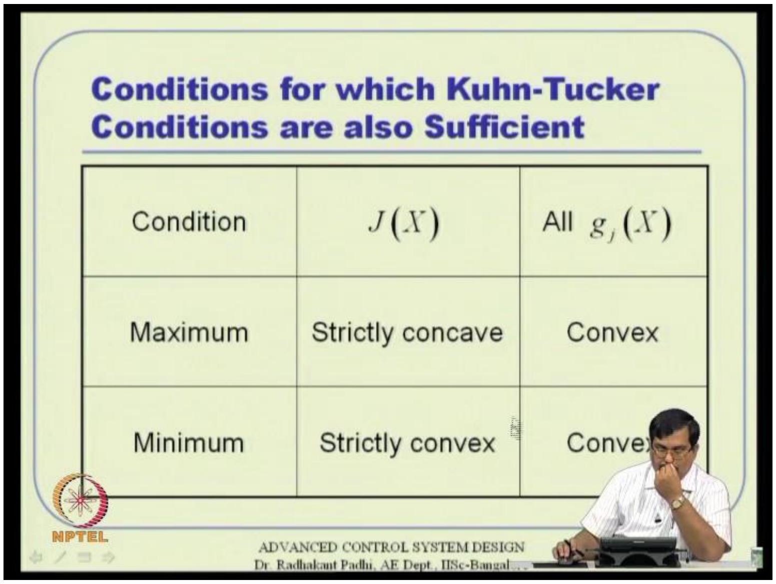

Now, the answer I mean the results tell us that J of X and then g J of X they are the two things that take us in the contour conditions we know that and results ultimately tells us that if J of X is strictly concave and g J of X is convex then it is the maximum point. And if it is still convex only here and g J of x is a convex this time then it is a minimum point. So, you have to see these needs to ne the constant equation needs to be convex and the cost function that you are talking is to be either concave or convex.

If you have a concave function it leads to maximization and if you have convex function it leads to minimization. So, these are like I mean conditions that are necessary for sufficiency check. Now, it turns out that under these conditions the contour conditions are also sufficient that means we really do not have to check too many conditions after that. If you apply contour conditions derive at certain answers based on these observations and then you go and see and verify some of these conditions are all good (( )).

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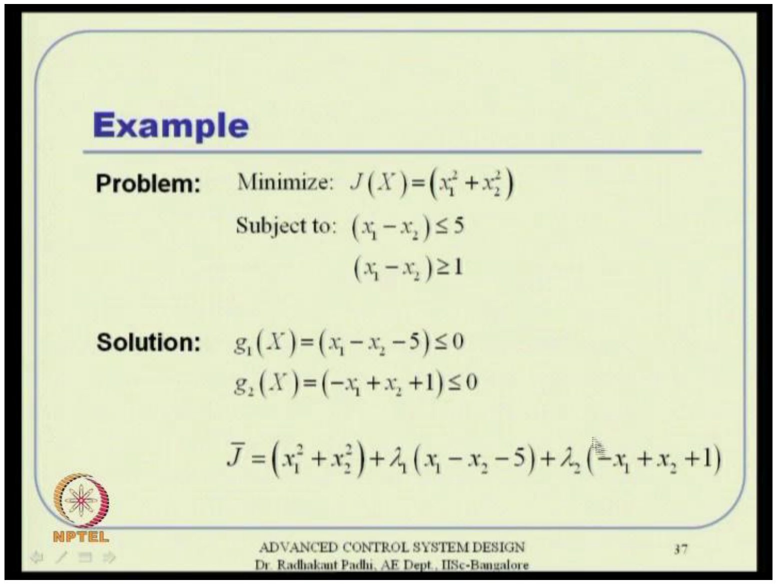

So, let us see an example before we wind up this minimize this objective function again the same quadratic cost function and then subject to these two inequality constraint. These two has to be one is strictly less than five or less than equal to five x 1 minus x 2 and x 1 minus x 2 should also be greater than 0 now greater than equal to one basically. So, now let us see that means you are looking for x 1 minus x 2 between one and five basically. So, this is greater than one or the same thing you are asking for solutions between one and five and you have to first split this thing into two such equation.

And take the negative sign of that and take this to left hand side, then put it in a less than equal to format. And that is what we have started our analysis with respect to that. So, all this inequality constraints I mean whatever inequality constraint is there all these things we need to put it in less than equal to formatting so, we put that and then we proceed with algebra here. And in this algebra we really do not have to talk about mu the slack variables here that only the analysis and development of the results basically. So, here we consider this as like a (( )) constraint that means equal to 0 then talk about J bar, which is like this lambda 1 time first constraint equal to 0 and then lambda 2 times second constraint equal to 0.

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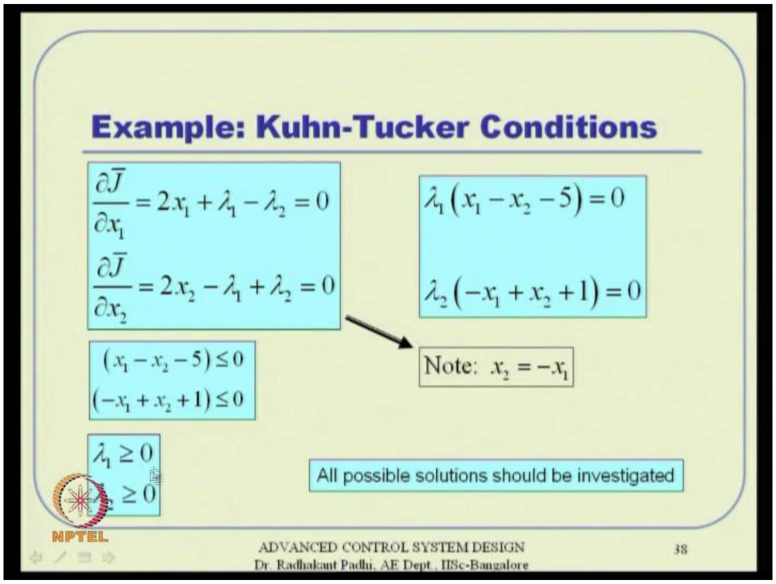

And then we talk about necessary conditions which tells us del J bar by del x 1 del J bar by del x 2 is here and then it is it will satisfy because remember x 1 is like lambda 2 minus lambda 1 by 2 and x 2 is negative of that and that means x 2 is equal I mean minus of lambda 1 so this are the only I mean all that we need to here I mean if you go back to that this set of condition tells (( )). So, this is with respect to x into this and this condition tells us that x 2 is minus of x 1, and then lambda 1 times g 1, g 1 is that, that is equal to 0 then lambda 2 times g 2 equal to 0. So, this set of four equations we will arrive at two coming from this percentage equal to 0 and two directly from lambda g I equal to 0 that way.

So, we have four variables here x 1, x 2, lambda 1, lambda 2 and we have four equations so, we can go ahead and solve that. And the only way to obviously satisfy this conditions remember these are necessary conditions what we saw, what we told here. So, it has to be both have to opposite sign or both have to be same sign it cannot be partially this way that way.

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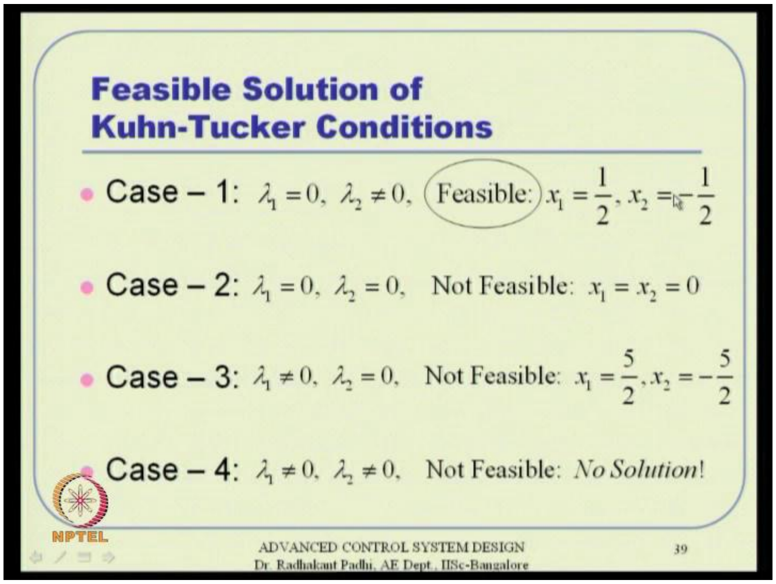

So, that the condition that we want to analyze here and all these conditions put together and then see this case by case analysis sort of thing lambda 1 equal to 0 lambda 2 naught equal to 0. And then what happens so and it turns out that this is the only feasible solution which will satisfy all the conditions. Otherwise all these case two case three, case four, these are all not feasible because it will not satisfy one of those conditions. So, and then they have to be satisfied with respect to the pair this is less then equal to 0. In corresponding lambda j has to be greater than equal to 0 (( )) and the this result do not hold good for all other case solution. That means for these case two, case three, case four something or other will fall out. So, certainly is not a feasible solution but case 1 comes out to feasible and hence this is the only solution basically. So, ultimately what solution you are getting is x 1 is half and x 2 is minus half basically.

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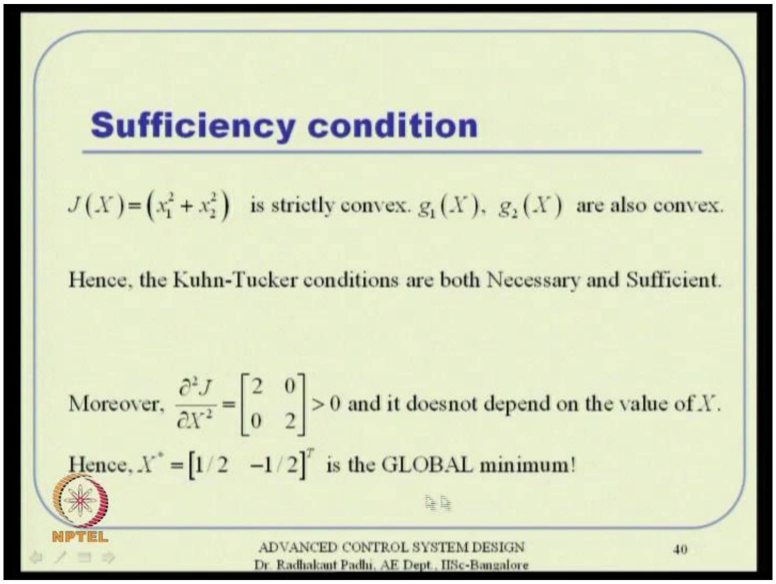

Sufficiency condition; we will analyze whether these are convex or not. So, we will talk J of X is x square plus x 2 square x 1 square plus x 2 square, it is obviously strictly convex and g 1 of X and g 2 of X are convex g 1 of X and g 2 of X if you see this way these are all linear functions are certainly. So, that Kuhn-Tucker's conditions are what we are getting are both necessary and sufficient actually. And also remember del square J by del X square which turns out to be these (( )) that means this is a del 2 J del X 2 is positive definite matrix. And it does not depend on the value of X, and hence whatever solution we got here is a global solution has global minimum basically. This is another way of kind of analyzing whether the solution is global or not.

So, this particular thing is probable all this is I will is sufficient exposer, we talked about various cases we start with a very scalar things which is we can go back to that. We started with a some kind of optimization that in a scalar sense, then we went with a equality constraint, then we I mean generalize that to multiple dimension, then went to equality constraint lot of numerical examples we saw which is an over view of ideas that how do handle that optimization problems. But remember optimization problems are typically not that easily solvable in terms of close function solutions whatever solutions we getting here nicely we are only valid for small problems.

So, for a big dimensional realistic problem we certainly need numerical methods to solve this, and these are all not certainly part of this course basically. But if you take a optimization book, you will see lot of numerical examples, numerical procedures, quadratic solutions. And there are many constraints that will that many concerns, whether the solution the iterative procedure will ever converge. If it converges will it ever converge to the real solution or it will get trapped in the local minimum all these things will be an issue as far as this I mean these practical difficulties are concerned. We are not so much bothered about that, but our motivation is towards optimal control. And we will not be so much interested in static optimization; it only gives you a some sort of a flavor to appreciate what goes on the dynamic optimization.

So next class, I will carry on with the ideas of calculus variation, in other words the dynamic optimization problems, and we will see how these ideas are useful for derivation of optimal control as well. So, this particular lecture, I have taken these two references; one is this estimation and control of system and kind of appendix, where all these conditions are discussed nicely. If you are interested in little lot more details about optimization theory, and (( )) algorithms all that, then there are many books; and out of that probably this one book is also a good book, you can probably see this. With that I will conclude this lecture thank you.