Advanced Control System Design
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Lecture No. # 20
Controllability and Observability of Linear Time Invariant System

Hello everyone, we will continue with our lecture series. Last class we discussed about stability of linear time invariant systems primarily and we will also see that controllability and observabilities are also important properties of dynamical systems in general and in this particular class we will talk about linear time invariant systems in particular because the neat results are available for linear time invariant system. These are also topics of research these days how to extend this to non-linear systems in general.

Some partial results are available but, since somebody wants to do research there still topics of research actually. But, anyway this controllability and observability of linear time invariant system has been well studied from for a long time primarily. These are kind of neatly proposed by Kalman and (( )) actually. So, we will see that the concepts and all but, before doing that we will also see various ways of evaluating this e to the power At which keeps on coming in the solution and as well as many other application actually.

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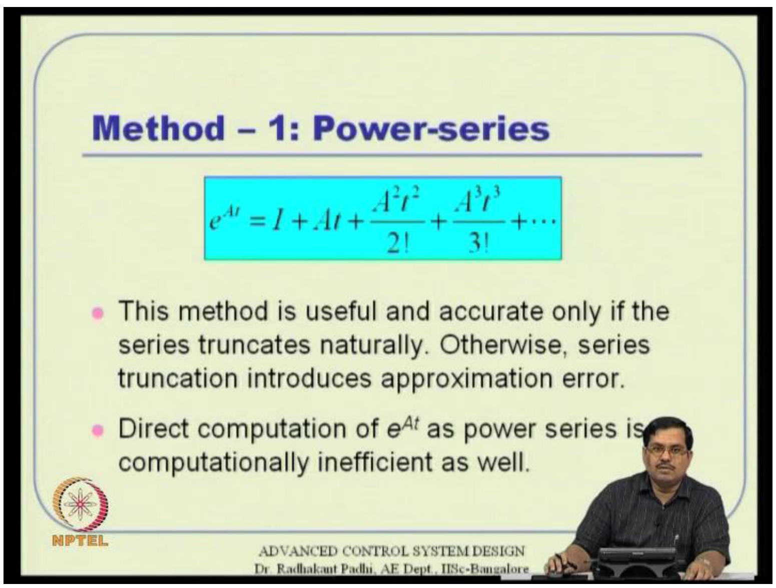

So, let us see the first thing is by power series, method one and this method is useful and accurate only if the series truncates naturally. That means if the A matrix is important some power it is 0 then, subsequent powers are also 0 and then this e to the power At is accurate. No problem otherwise you can truncate the series but, the series truncation will introduce approximation error actually.

And if you take too many terms means for example, if you take 20, 30 terms like that just to be accurate then computation of e to the power At, as power series is inefficient as well because too many computation you have to do actually.

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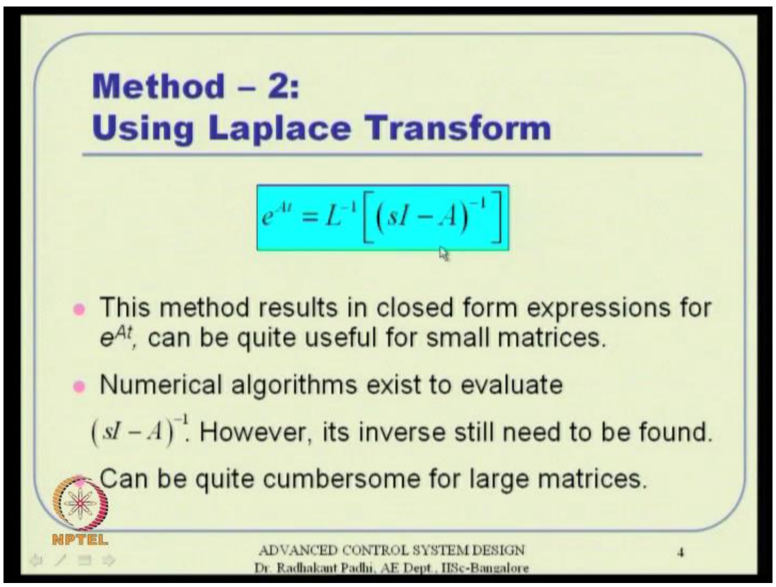

So what is that? the So people studied various alternate ways of computing this method two is using Laplace transform. We have seen this actually before as well. So, these method results in closed form expressions for e to the power At and it can be quite useful for small matrices. You can actually do this all this symbolic computation by hand and if possible by using some symbolic software's actually.

And we have also seen in one of the previous classes that you can evaluate this sI minus A inverse very efficiently in a recursive manner by using some numerical algorithm actually. However, you can see that Laplace inverse that means L up to that you can symbolically compute sI minus A inverse but, taking Laplace inverse still needs to be done either by hand or by symbolic software symbolic software's for large matrices can fail and doing that manually is very cumbersome and can be very inaccurate also if it if it contains large polynomials in each entries then, doing by manually is also not that good actually. So we will see what what is the alternate way.

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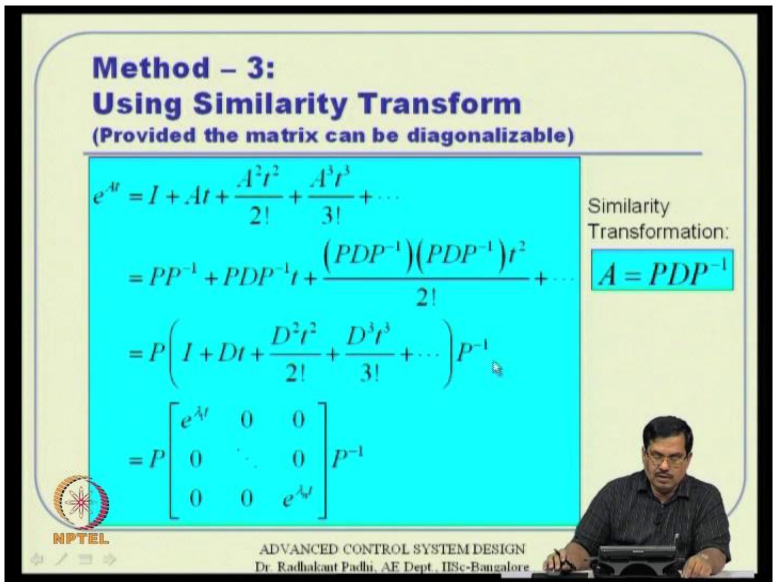

So people have thought about it. The method three is if you're A matrix is diagonalizable then a equal to PDP inverse where D is diagonal and we have seen this analysis before that e to the power At, this power series essentially you can you can introduce this PP inverse for I and A equal to PDP inverse all sort of things and then you can you can take out factor out P in the left hand side and P inverse in the right hand side because here all these terms these guys will cancel PP inverse and P all this will will be identity and all that. So you will be left out with e to the power kind of Dt.

And e to the power Dt is nothing but, e to the power lambda 1 t e to the power lambda 2 t all in diagonal were pre-multiplied by P and post-multiplied by P inverse actually. So, this is another way of computing. You can directly write this matrix and just compute, I mean just multiply it by P in the left and P inverse in the right and P is nothing but, the eigenvector matrix that pops up actually. If it is non diagonalizable you can still do that. We have seen that in the solution matrix solution approach it will result in the Jordan block diagram form and things like that you have to be slightly careful it may not be that efficient to do when the and the a matrix is not diagonalizable.

But it is still doable by introducing this Jt J square P and things like that instead of Dt actually Dt D square and things like that. So that is another way.

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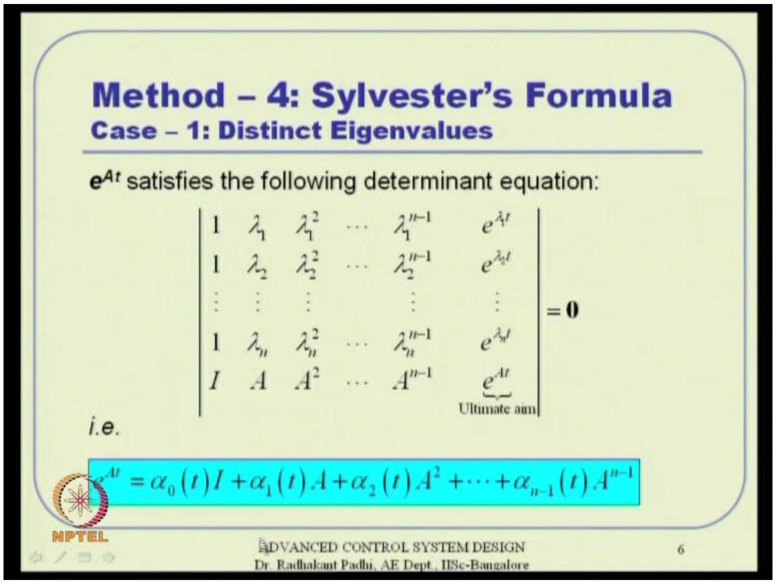

So, what is the third? There is a fourth way also which is a very powerful thing which which is something called Sylvester's formula. The Sylvester's formula tells us that it is always possible to write e to the power At as a finite order polynomial containing I A I A square up to A to the power and minus 1. So, if A is 1 by n then e to the power At is always possible to write that way.

Now, how do you evaluate these polynomials, these coefficient polynomials? So, those are not very straight forward. One one thing that you can notice is e to the power At will satisfy this matrix equation, this determinant equation in a symbolic manner. Remember these are all like we are talking about case one where we have distinct Eigen values and we will also see case two where you have repeated Eigen values. First is case one distinct Eigen values. So, if you have distinct Eigen values then this determinant you can formulate make it equal to 0 and then the remember this determinant is nothing but, a matrix equation ultimately because last entries are not scalars last entries are matrices by themselves.

So you cannot this is not really determinant in pure determinant sense this is more of a symbolic manners actually, symbolic sense. What you can do is you can I can evaluate this determinant like scalar determinant using the last row. If you do that it is easy this X 1 this determinant using the last row, we have a one term somewhere it is e to the power At n by n and that is what you want actually that is a matrix remember that. So, if you do that then then e to the power At you can solve for it. You expand this using the last row then solve for e to the power At.

So in that process you will get all these these coefficient polynomials that you are getting here and if you sit down with longhand, I mean algebra then it turns out that this these coefficient polynomials will satisfy this kind of equations actually.

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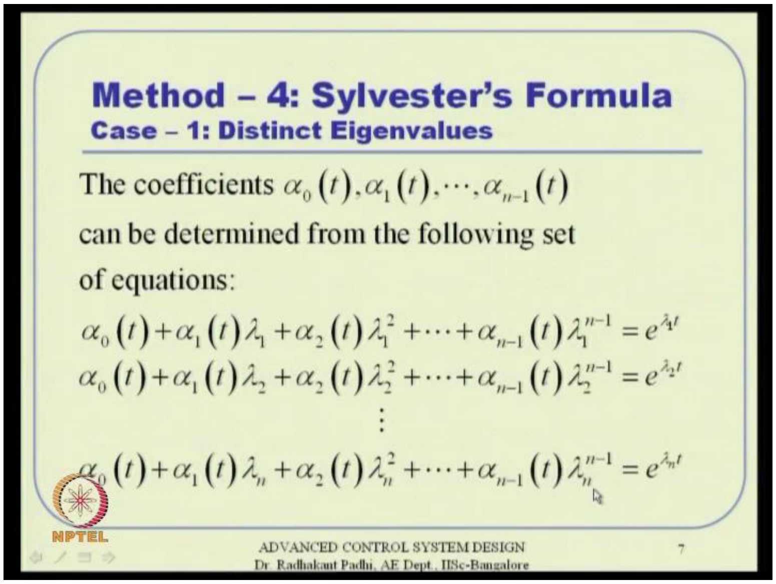

If you have distinct Eigen values then you can you can formulate these equations. Remember these are all scalar equations individually. You have about n equations and n unknown polynomials then this needs to be compute, this need to be solved in a symbolic manner remember that. It is not going to be a solved in a deterministic numerical way you have to solve it in a symbolic sense actually.

So it is possible to do that. So, that the beauty is and e to the power At can always be expressed as finite order polynomial of a matrix actually which is very neat. Then what about case two?

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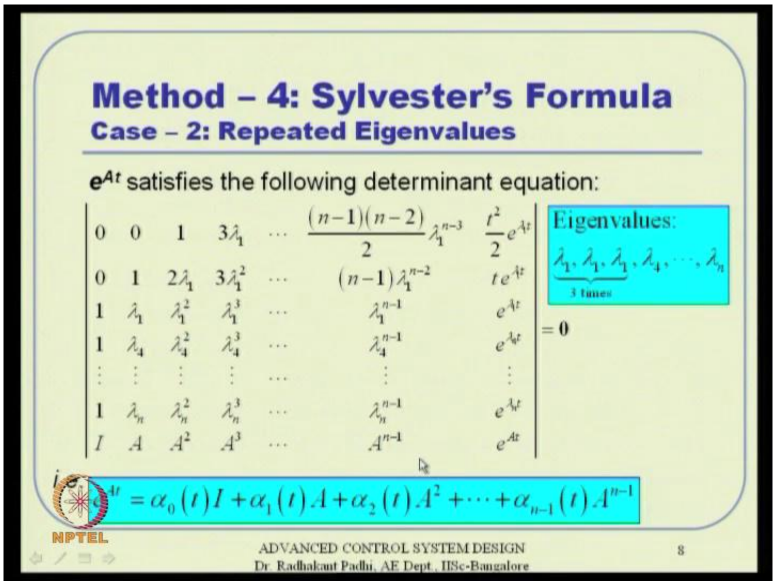

If you have case two; the same thing holds good. This is still possible to expand it in a finite order polynomial. However, the matrix equation the the determinant equation that you are talking about will be slightly complicated. So, if you take an Eigen values series where lambda is repeated thrice which a lambda 1 is repeated thrice and instead of lambda 2 and lambda 3 you have lambda 1 and lambda 1 and then lambda 4 onwards it is same.

So, these are all non-repeated and all that then this equation take this form actually. So, these are all you get a book it is there so you can see many of this. I mean include examples I will also talk one example from there actually. So, this is this is what you have to evaluate and ultimately you will get this and again if you expand it using this last row and then solve for it for At this polynomial coefficients will pop up and these coefficients will satisfy this kind of an equation this time.

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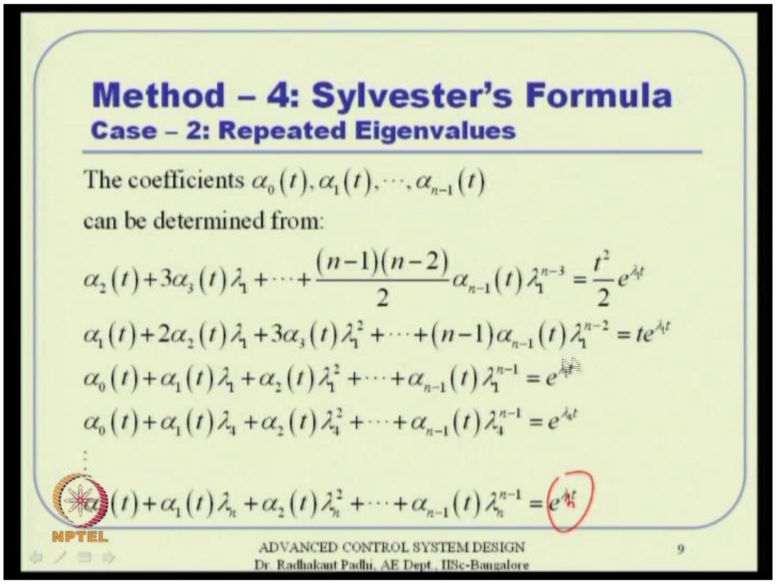

So, the first two equations are different because you have repeated Eigen values for two and three because of that the first two rows will be different and third onwards it is same actually. What you had before and so this is lambda 1 lambda 1 square about this is lambda 4 and lambda four lambda 4 square and things like that actually. So, this is how how this going to happen there actually. By the way there is a small print mistake may be this is supposed to be lambda n t the last one this is suppose to not e to the power lambda 1 t but, e to the power lambda n t basically.

Anyway, so if there is any further print mistake from I can I mean you can always go through get us book and see the accurate thing just to give a feeling of how do you make use of this this is an example.

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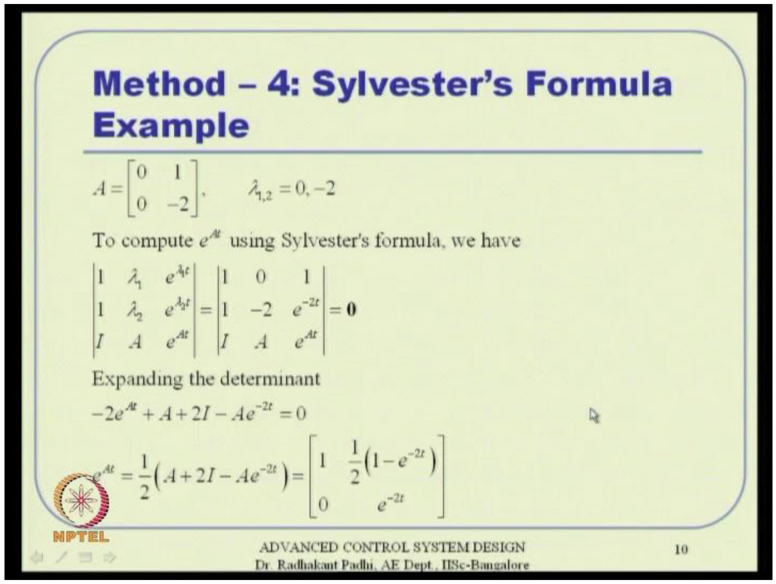

Where you have A matrix 2 by 2 and Eigen values turns out to be very neat because this is a triangular matrix. You can see that the Eigen values are nothing but, diagonal elements. So, Eigen values are 0 and minus 2 and then to compute the, these are all non-repeated Eigen values so you can directly use the first one.

And that turns out to be with this equation actually this is I mean this determinant equations are need to satisfy and if you plug in this lambda 1 lambda 2 values this is 0 minus 2 and e to the power 0 is 1 e to the power minus 2 t here and then this equation will evaluate using this last row. So, you can expand it using this so this is like 0 and then there is a minus 2 that that becomes n times I basically. So, that minus 2 becomes plus 2. So, this is two I which is popping up here and things like that actually.

So, you evaluate I mean evaluate using last row or you can you can evaluate any using any row also as long as long as you mean what you are doing actually. So, you can once you expand it is it is a matrix equation and then this matrix equation you can solve for e to the power At. This turns out to be like that. So, then you can plug in your matrices A and I always you know and e to the power minus 2 t also. We know that is that is the scalar term by the way and then you will get this one and you can also verify your results from for this particular matrix using other methods. Whatever other method we have seen this method like either power series way. This is this is actually a important matrix because it is a triangular with just one element on the top. You can you can see it yourself. It will truncate probably at a square level itself. Well I can verify that may be form I q onwards it may vanish or something.

Then you can also evaluate using this and verify yourself whether that is true or not actually. So, the the the reason is I mean why I wanted to discuss all this is people have studied this e to the power At anonymously before and then they have there are very efficient algorithms available to compute e to the power At in a rapid manner and these are also useful to this finite finite series polynomial is also useful in controllability analysis. That we will see just now actually immediately after this. So, that is the motivation why I wanted to include that here.

So, let us move on to the topic of today. This is controllability and observability. So, as you as I told before this controllability and stability are two different properties. Sometimes student get confused about that but, uncontrollable system can still be stable and then vise versa unstable system can still be controllable. They are different properties altogether.

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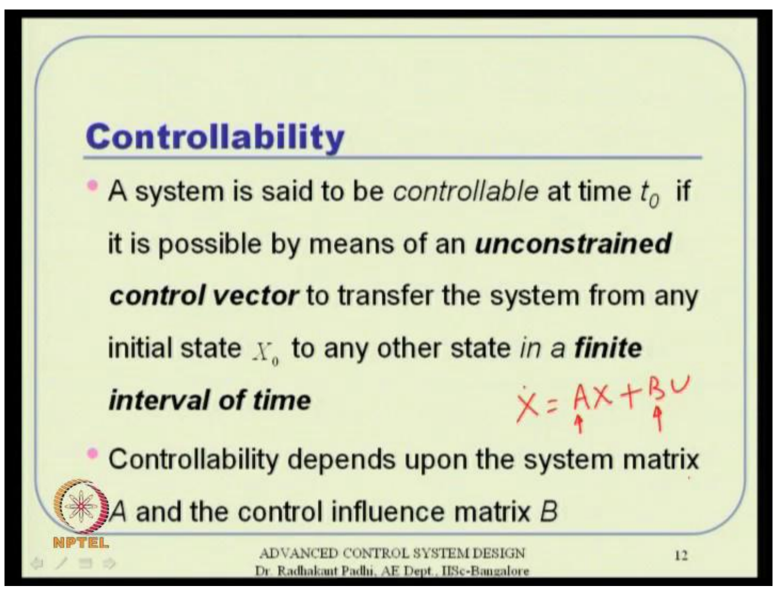

So what is the concept? Concept is this is the formal definition the system is said to be controllable if it is possible by means of an unconstrained control vector. that is That is more important actually control vector cannot be bounded by certain values and all that actually. Then you are then if the control is not really control it becomes a parameter and all that actually if the bound is it.

Anyway, so by means of a unconstrained control vector if you can transfer the system from any initial state X 0 to any other state in a finite interval of time that is also important actually. You should not take infinite time actually then a finite interval of time. This should be it should be possible to transfer any initial condition to any other condition actually. So, that is that is the concept of controllability if it is possible to do the system is called controllable. If it is not possible to do then it is uncontrollable and obviously if it is not possible to do that means the system is uncontrollable. Then no control design method is supposed to work.

That means it is not simply not possible to design a controller for an uncontrollable system. That is the beauty of modern control theory. Also in other words if you know that the system is uncontrollable, then it is futile to kind of attempt to design a control system. It is just not possible to do that now irrespective of whatever method you think about actually. So, that is that is why it is more important and obviously controllability depends on the system matrix A as well as the control influence matrix B for linear time invariant systems.

See if you if you this the system that you are talking about is h AX equal to A I mean X dot equal to AX plus BU so as long as stability was concerned, we are not bothered about B we are only bothered about A. But, as long I mean if you are bothered about controllability then, you we must talk A and B together because the effect of U comes through B and that is why this also called control influence matrix the influence is felt through B actually. In other words if B is 0 then no matter what control is there it is not going to affect your system anyway.

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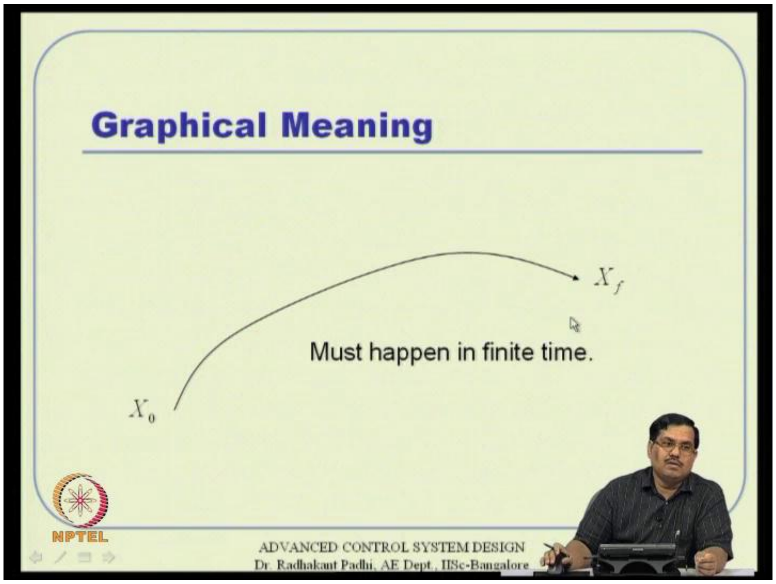

So, the controllability property is there in the pair A B it is not just not there in the matrix A only basically pictorially speaking. This is what it supposed suppose to be. So, we have some some initial condition X 0 in some finite dimensional space it can be r 2 r 3 r 5 whatever it is the depending on the dimension of your system and then from this initial condition you should be able to go to any final condition. Anywhere it is just a representative final condition. If you should be able to go wherever you want and this must happen in finite time using a non-constant control vector. That is the that is what it is actually.

So, if it is possible then I that if it is possible that means the control, the system is controllable. Then you can think about designing a control system from various approaches and then you can why like which approach is better basically whether whether your stabilizing controller is better or optimal control is better or robust control is better whatever it is. This will all depend on further specification and all that actually but, if the system is not controllable do not waste your time. It is not possible to design a control system actually.

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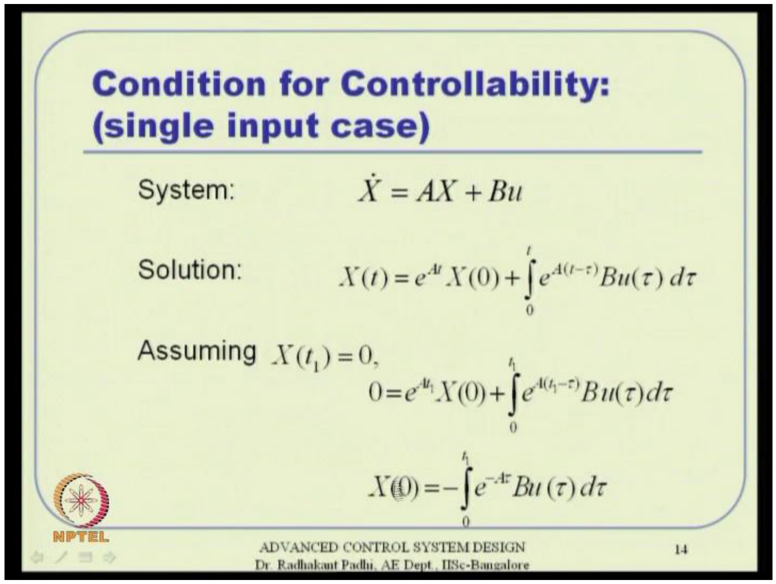

So, let us see that see that I mean, kind of I mean we are asking a bigger question here if the system is controllable or not. So, what is the condition? Is there is there any condition that is available to us? So, that we can quickly verify whether the system is controllable or not and for LTI system it is there. I mean the standard result is available actually. So, we will see how to how do we get that standard result first actually.

So, what you are talking here is a single input case first and multi input case. I probably may not be able to cover here but, you can the analysis is fairly similar actually. So what you are talking we are talking about X dot equal to AX plus Bu. So, that means you have to use this I mean I purposefully I written small that means it is a single input actually and the solution we know from for this system from when the initial condition X 0 and t 0 being 0 is like this.

X of t is e to the power At this is the homogeneous part and this is the forcing part you have derived this before and assuming X of t what you are doing here we are talking is we start with some t equal to 0 and at t equal to t 1, we are asking the question that instead of starting from X 0 to anywhere without loss of generality I will tell that the final point is 0. I can start from anywhere but, at t equal t 1 I have to come to 0 it is an equivalent problem. Anyway, I mean I can I can start from anywhere. Suppose in a two dimensional sense you can we can think about that instead of going from anywhere to anywhere the final point probably I shift my coordinate system to the final point. So, this becomes my X f and I can I can come from anywhere actually to this to this point. I should be able to come there.

So, this becomes my representative X 0 or this becomes I can start from anywhere. Ultimately I should be able to come there and the time it takes from here to there I start at t equal to 0 and the time it takes let me call that as t one actually. But, this remember X f is nothing but, 0 0 actually. I mean that is what we are interested in actually so X of t one is essentially 0 in so I can put that f t equal to t 1 0 equal to this this expression. So, I put just t equal to 1 any everywhere e to the power At e to the power At 1 0 to t 0 to t 1 e to the power t minus tau is a to the power t 1 minus tau like that actually.

So, I want to kind of solve for X 0 from here and its possible to solve because e to the power At 1 will come out of that integral. This is not a function of tau so I can take that out I can I can expand this that will be e to the power At 1 into e to the power minus A tau so e to the power At 1 I can take out and that is never 0 e to the power At 1 is never a singular matrix. You know that so you can cancel out and tell after that X of 0 is nothing but, that actually so what are we doing next.

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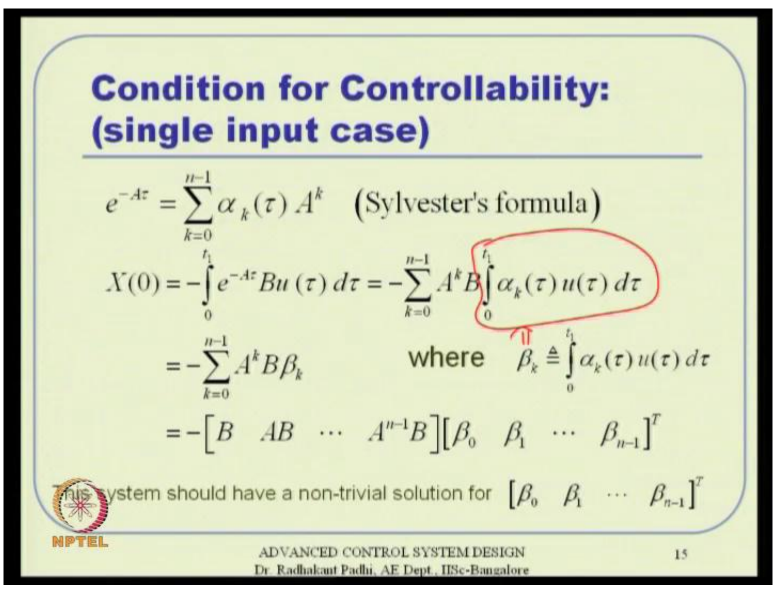

Now, we will invoke this Sylvester's formula no matter no matter whether you have I mean a being with distinct Eigen values or otherwise it is possible to write the in general e to the power minus A tau is that. So, k goes from 0 to n minus 1 alpha k of tau remember e to the power minus A tau that is what you are evaluating. So, this is nothing but, alpha alpha 0 of tau times identity plus alpha one of tau times a like that actually. That series goes up to alpha n minus 1 actually you have just seen that actually.

So, if you plug in that so we had this solution here. So, you want to plug in e to the power minus A tau with using Sylvester's formula actually. So, we will plug in here and then we also remember that A and B are constant matrices. So, A to the power k is also A constant matrix for any k actually. So, once I substitute then this B matrix will multiply that to every power of A and A to the power A to the power k times B is a constant matrix. So, I can take out from there take one from there integral part of it and I can be left out with that actually.

Remember this is a function of alpha k that means as k varies then the alpha k is a different different polynomial actually and similarly, A to the power k also. Basically, so what I am doing here is I am, I'll define this this term whatever term is there as beta k. That is that is what I am doing here actually. This term whatever I am getting here. This term I am defining it as beta k actually. So, that is the, so what I am getting here? I am in the series part of it the solution of X 0 I. In a compact form I can write it that way remember all this integral is with respect to tau. So, there is no tau dependency for say it is all integrated over from 0 to t 1. So, this beta k is a coefficient which is defined that way.

So, in an expanded form I can write it that way. This is all that I am writing from here to here is just I am writing in an expanding form where this matrix pops up and this is a column vector this is just for compact notation I've written with transpose. This essentially a column vector B 0 B beta 0 beta 1 and all that is a column vector. And what you are looking for? So we are able we should be able to go from anywhere to 0 actually. So, X 0 should be anything anything that state space actually.

That means this system of equations should give me a nontrivial solution obviously for these fellows and if I get a nontrivial solution for this obviously. Remember this a single input case that means B is just A vector. So, that means it is n by n full rank n by n matrix and if it has to have this this nontrivial solution then, this matrix must be full rank actually.

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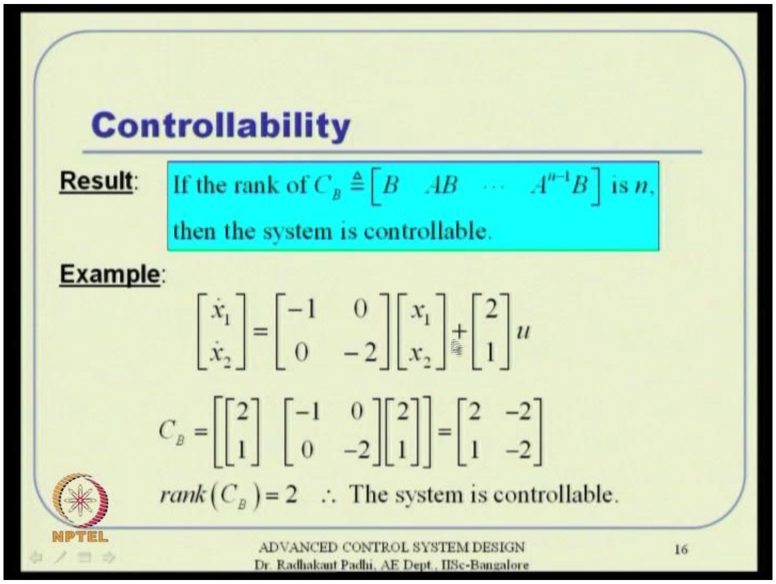

So, if so that is why you get this controllability matrix once you define that the rank of this controllability matrix should be n actually that is the reason for that.

And in general if you have multiple input system then this dimension will not be n by n we will have bigger dimension and essentially if you want to have a nontrivial solution like that then this matrix must be A of rank n. The matrix it will be like bigger matrix n n by some higher order dimensional matrix but, at least the rank has to be an n so that you can get a nontrivial solution for this. So, the ultimate result is if the rank of this controllability matrix which is defined like this; is n then the system is controllable actually.

So let us take an example. We have this small example like n where this very neatly given in diagonal form this is easy to work with by hand. These are all something called type problems. Also many times just to give us conceptual ideas what is a what is that? No practical significance for say actually. Anyway, so this is minus 1 minus 2 so it is easy this is our B matrix that is our A matrix. So, we have controllability matrix as B and then A times B so B is as it is 2 1 and A times B is you can multiply this I mean this two this is minus 2 plus 0. So, that is minus 2 and then it is 0 minus 2 so that is minus 2.

So, if you see this determinant determinant is minus 4 plus 2. So, obviously the the important thing is it is non 0 determinant is non 0. That means the rank of this controllability matrix is obviously two and hence the system is controllable. So, given any order polynomial you can test it very quickly as long as they are linear time invariant system, it is easy to kind of get very quickly basically.

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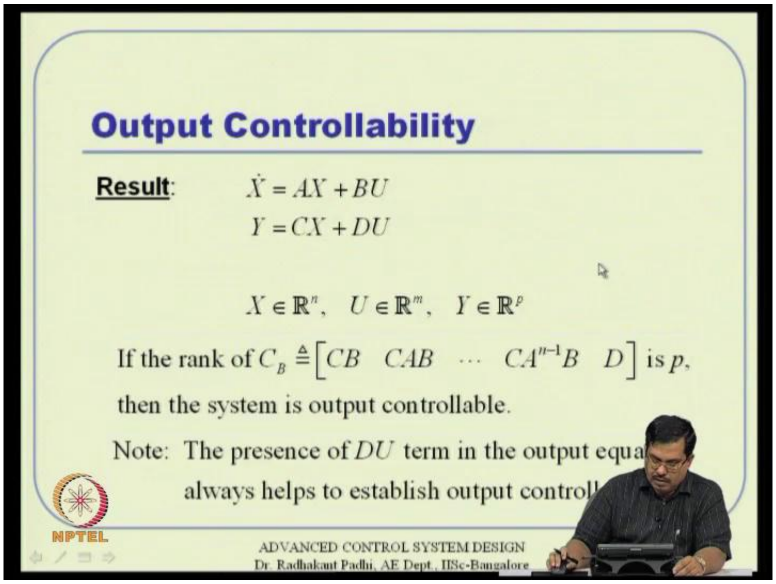

Now, the there is a related concept which is called output controllability. That means suppose your system somehow does not does not satisfy this full controllability complete controllability basically.

That is the question is can I transfer some output vector from anywhere to anywhere. Actually forget about the transferring the interested vector. So, that is called the concept of output controllability where the output equation will play a role. Now, earlier we never bothered about output equation. So, if your output the result is like this its like you controllability matrix will be CB CAB and things like that and then there is A D matrix there and the rank of this matrix has to be P where P is nothing but, dimension of y I mean that is what you are interested in actually. So, then the system is at least call output controllable that means in from in output sense you can transfer anywhere to anywhere actually.

So, that is the concept there actually and also remember that if you if your D matrix is non 0 that means you do have a control influence matrix directly. Then obviously y is controllable I mean this is algebraic equation after all actually. So, you can do anything and you can but, normally this D is not there with us. So, you have to work with only C part of it y equal to C X. Then it is possible to kind of derive this equation also and some of I mean who are interested I suggest that you can derive this as a corollary of the other one what you have done. What you have studied here it may pop up as a corollary of that one. Specially when D is not there probably it is easy to kind of do it and you can extend that later probably.

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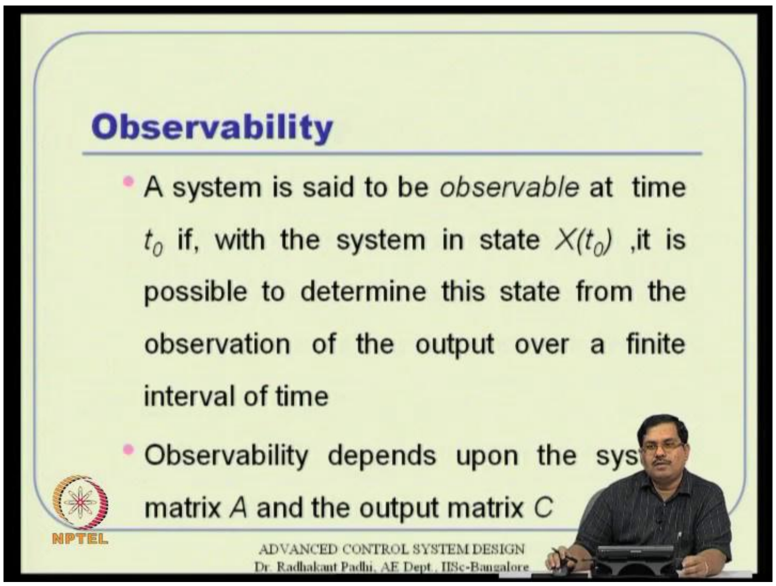

So, that is that is little thing about controllability. Now, what is about observability. Now, observability is a property again different property than than both stability and controllability. That means various possible combinations can happen. Remember that actually even if you take a stable system it can be controllable and observable also it can be just controllable but, not observable it can be other way round and it can be nothing actually either way so all these are different properties anyway.

So, the system is said to be observable at time t 0 this is the definition with the system in state X t 0. If with the system in state X t 0 it is possible to determine the state from the observation of the output over a finite interval of time. So, what are you doing here actually? So, you do not have the state information really here now what you are telling is I can observe my output because sensors are there with me so but, remember output is actually a smaller dimension right P y belongs to P and this P is typically, this y. What you are talking about in controllability sense is different from that y. What you are talking about in observability sense that I want to make here.

There are two different y's. This y is something called performance output. That your system has to be driven from somewhere to somewhere and the other other output that you are talking here and observability is sensor output which are measuring it actually they maybe same. They may not be same. You can measure something, you can control something else and we have I mean for example, you can keep on measuring the attitude of the aircraft and you can still control the height of the aircraft. How it will need not sense actually as long as the variable is controllable it is possible to do that actually.

So, that that normally many students are confused I thought I will just make it clear. There are two types of outputs in general; one is performance output that your controller is interested to drive and other one is sensor output that your controller information primarily comes from there. Where these are these are primarily coming from different sensor that you may have in the system actually. So, anyway so with that keep with keeping that in mind even then your system need not be sensor rich. That means sensors can be limited actually you may not get all the state information actually.

So, typically the y dimension is lesser than the X dimension. So, in that sense what you do I mean if I observe it one time that the information is just not available and even if I, even if the sensor is available, if just if I observe one time then it can also have noisy systems and all that noise can pop in actually which we are not talking here but, then observability as a concept related to observer design and that is related to filter design. So, those is those are all related concepts actually.

So what you are what you are telling here? If I if I measure my observation now take the output from my sensors it may not give me the state state information at one time. If I just do the measurement one time but, how about doing it in a repeated sense you keep on getting measurements. So, if my output is dynamically related to all of my states somewhere then at some point of time it should reflect whatever whatever measurements I am getting as a sequence. The information content is coming from state variable on the actually somewhere so is it happening.

If it happens then the system is observable. That means I can still do some math some I design an observer and things like that to recover my state information even though I am not directly measuring it actually. So, that is that is the whole concept I keep on getting information at discrete time interval. Let us say, so I keep on collecting all that and I if I take a finite duration interval measurements then I should from using those information together I should be able to reconstruct my initial state actually where you started so that is that is the concept of observability actually.

And observability depends on the system matrix a and output matrix e. Now, normally this D is not there actually anyway even if it is there observability primarily is relation between y and X. So, that is that is where and it is related to this this X dot equation because you are talking about the the effect in a dynamic sense actually. So, the the observability property is hidden in A and C together. That is the pair actually that that gives us. So, what is this result and the result is again straight forward. I mean I will not derive it one more time probably.

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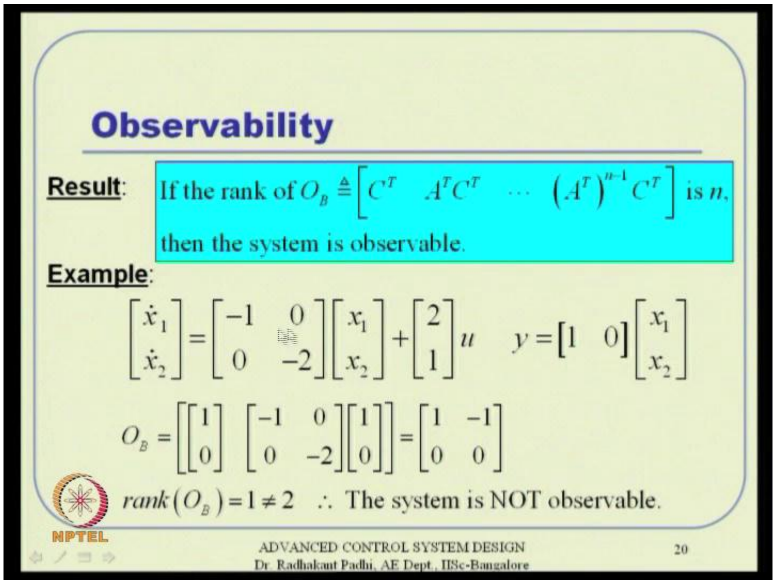

So this is result tells us that construct an observability matrix this time and the rank of this observability matrix has to be n where n is nothing but, dimension of the A matrix actually A is n by n. If it happens then the system is observable actually again. Again the example this time you have to talk about a output equation as well and your observability matrix is C transpose C is that so C transpose is 1 0 column vector and then A transpose E transpose so A transpose is a because it is a symmetric matrix anyway.

And then, C transpose is again that so if you multi first vector is 1 0 you have to keep it and second vector is minus 1 times one so its minus 1 and then the the second one is 0 times 0. So, A 0 so obviously what is happening if you just take the last row is 0 zeros? So, the rank is certainly not two and hence the system is not observable. So, remember this this this one what you took here and this one what you took here this part is same. So, the same system is actually controllable but, certainly not observable. Observable depends on what you measure actually. Now, you can verify yourself that if instead of measuring just x 1 I mean this. This equation tells me that I am measuring only x 1 if I just put 1 or 2 some some non 0 number. Here that means I am all the, my measurement contains equation partly information from x 1 and partly information from x 2 as well.

Then it will turn out that the system is observable as well so like that actually.

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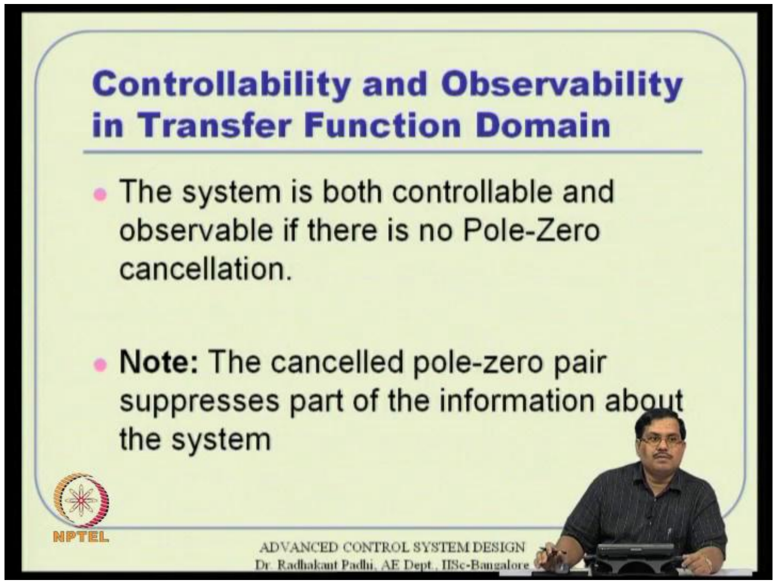

Now, how do you how do you correlate all these results with the transfer function domain? So, people have also thought about that and it all turns out that if the if there is no pole-zero cancellation that means the system is both controllable and observable right but, what if there is a pole-zero cancellation then the the cancelled pole-zero pair suppresses part of the information about the system obviously.

And hence the system is either uncontrollable or unobservable and the direct answer you may not get you have to do some some realization and then only you can talk about actually depends on the, you can take the bigger thing and then talk about the realization and then you can tell you can analyze that whether the system is again either controllable either uncontrollable or unobservable or both either way actually. So, it is I mean the relation that goes to transfer function side is that you should not have any pole-zero cancellation actually.

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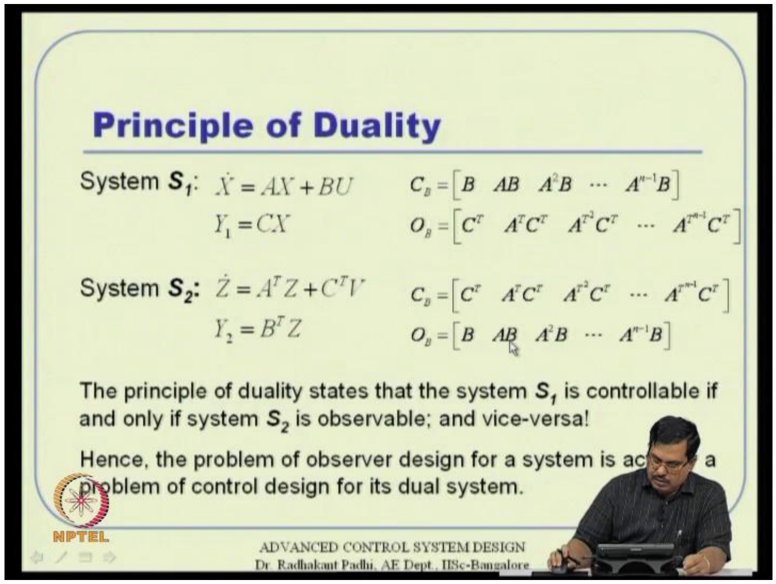

There is a neat concept called principle of duality. Now, which is very heavily exploited in observer design and filter design which is interesting to observe here. So, let us talk about system one which is S 1 and system two which is S 2 system one is our original system what you have AX plus BU and y 1 is CX and system two S 2. I will just artificially create. This is just a parallel artificial creation with this kind of a dynamics actually. I am free to do that anyway this is just a mathematically I am just writing an equation actually.

So, for system two Z is my state V is my control and Y two is my output so what is the controllability matrix for this this is nothing but, B A B A square B like that and the out observabilities C transpose A transpose C transpose like that actually. So, that is what it is now if you do the same thing for the system S 2 the controllability matrix turns out to be like that. Remember this is like B. So, it is B that means C transpose than A B that means A transpose C transpose like that actually.

All right and similarly, observability matrix if you the C transpose C transpose is nothing but, B then A transpose C transpose A transpose is A transpose transpose that means A times B like that actually. So what do you what do you observe here? Whatever is the controllability matrix here is nothing but, the observability matrix here and whatever whatever is the observability matrix here is nothing but, the controllability matrix there. So the, this is what is called as a dual system actually. So, given a system there is always a dual system which is just kind of mirror reflection you can think about that actually.

So, it is, if it is controllable and this one is observable and if it is observable this one is controllable actually. So, for designing a controller I mean designing a controller for system S 1; we will do the procedure in next class probably and then designing an observer for the system S 1. We are not going to do it directly. What you do is you design a controller for this system. It is going to be an observer for that. Actually I mean you as far as math is concerned we will we will think of that we are actually designing an a controller for say for the dual system which will serve serve as an observer for the actual system. I mean that is that is the power of this and there are further ramifications of this concept actually. This is a dual system dual space and things like that are very powerful tools for linear system analysis and design basically.

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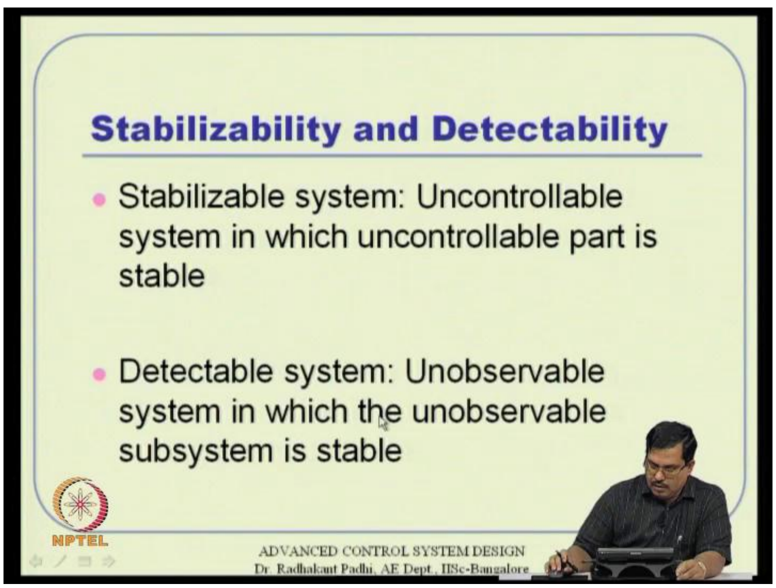

Now, there is another related concept called stabilizability and detectability. So, the stabilizability means remember your system can be uncontrollable but, the uncontrollable part is stable. So, I mean if the, if a system satisfy that kind of a property then uncontrollable parts are not really creating such a big nuances. I mean they will not take your system unstable and things like that so you can kind of safely forget about them. I mean you cannot do anything about them but, they are not creating harm to you actually.

So these are like stabilizability property and these are there is no direct results as such but, I will given an example to demonstrate, how do you conclude about that and all that actually you have to do this Eigen spectrum analysis and and then transfer your system and things like that actually. Anyway similar concept is detectable system that means the system can be unobservable. However, that the unobservable subsystem turns out to be stable. So, you so you may not be able to directly see those states but, those states are if you even if you do not see those states. Those are actually stable. That means stable means what in all linear system they will all equate to 0 basically.

So, they will not create such a big trouble for us actually. So, these are the concept of stabilizability and detectability. Stabilizibility is related to controllability and detectability is related to observability. So, if the system is not controllable but, the uncontrollable subsystem is stable then it is stabilizable and similarly, if it is unobservable but, unobservable subsystem is stable then it is detectable actually.

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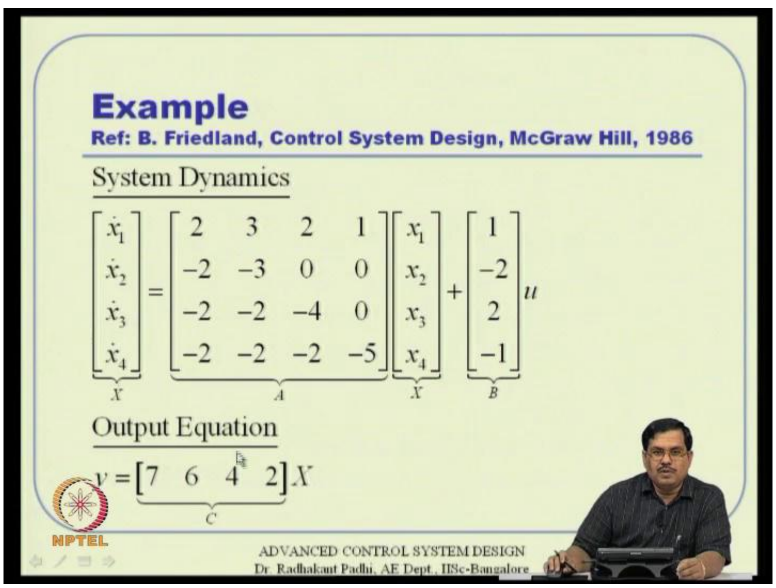

Let us see a nice example. This example I have taken from this nice book actually. It is also a good book for linear system design and all that.

So, this example this is your A matrix this your B matrix by looking at it feels as if it is a fourth order system. I mean the matrix is very dense here. There no direct kind of multiplication of one column to another column things like that. I cannot see that directly when I see this actually but, let us consider this output equation whatever output equation here.

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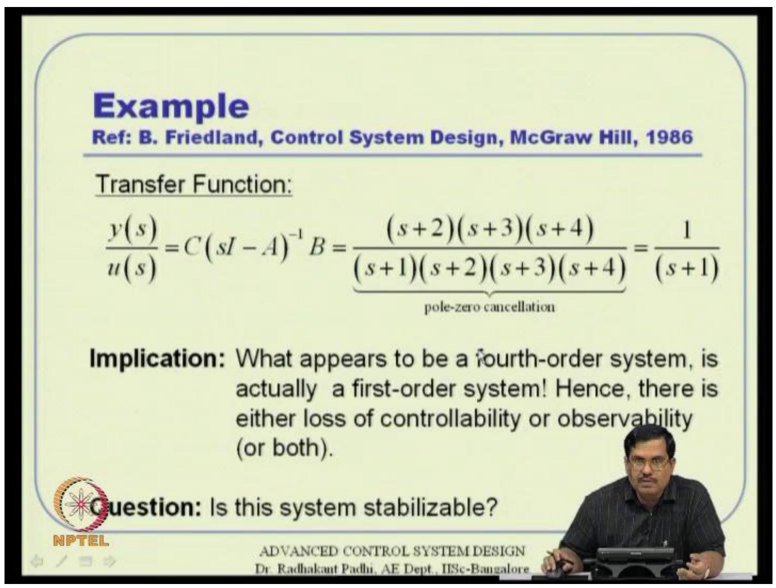

And then you talk about a transfer function and this transfer function once you do a nicely turns turns out that all this s plus 2 s plus 3 s plus 4 are nothing but, they are I mean there is a there is a pole-zero cancelation and that happens; not once it happens thrice actually it happens at minus 2 minus 3 and minus 4 these are poles and they are zeros as well.

So, what appears to be a first order system and fourth order system and in this input output sense? It turns out to be just a first order system actually. So, what you what happens here? Obviously if the if the system is stabilizable because it is certainly not controllable. The pole-zero cancellation basically so we want to answer whether the system is stabilizable.

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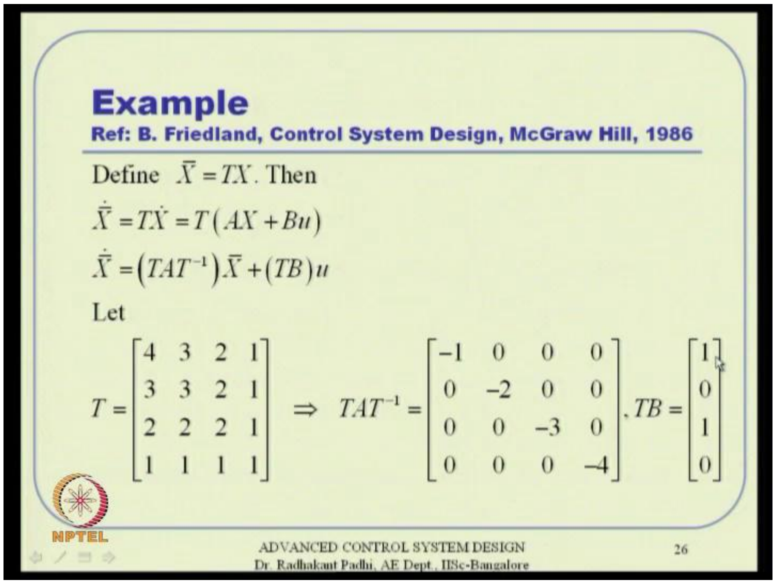

Now, let us see we define a change of variable here X bar is TX where T I'll I will take it that way and then if I take X bar is T X then X bar dot is nothing but, T times X bar T is a T times X dot but, T is a constant matrix so that that is how it is.

And X dot is nothing but, AX plus BU and then X bar dot if I if I if I substitute X is nothing but, t inverse X bar here. So, I substitute it back multiply and expand and then T is like that. So, I can compute all these matrices TAT inverse and TB is like this and then I can formulate this transformed dynamics turns out to be like that. Remember this is the dynamics that I am talking actually. So, this matrix is nothing but, that matrix it is a nice diagonal matrix now and then TB turns out to be like that. So, if I substitute it that X bar one dot is nothing but, minus X bar plus U like that.

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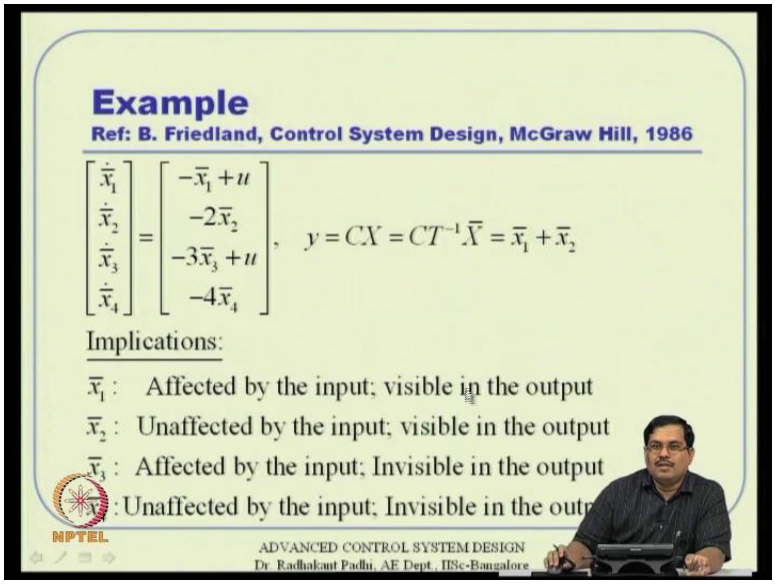

So, I am writing that one actually so what you see here you see here that this particular thing first of all it is very neat this x bar 4 X X 4 bar dot is nothing but, just that so it is just not influenced by any any control actually and then output dynamic is also you can see y equal to CX that is what you are talking y equal to CX and X is nothing but, T inverse X bar and if you if you plug in again CT inverse and all that it turns out to be that.

So, X bar x 4 bar is neither in y and x 4 bar is not directly influenced by U at all actually these are all decoupled completely decoupled from x 1 x 2 x 3. By the way, so it is X x four bar is unaffected by the input and invisible in the output as well. But, x 1 what about x 1 bar? It is affected by the input directly it is there in the output also, certainly X bar is both kind of there effected by the input and visible by the output actually.

X 2 bar, what about that? Certainly it is not effected by input but, certainly it is there in the output and similar x 3 bar it is affected by the input somewhere it is there input is there. But, it is not affected by the I mean it is invisible in the output. So, all this 4 whatever we are talking about it is all there in this x 1 x 2 x 3 x 4 all this combinations what you see actually.

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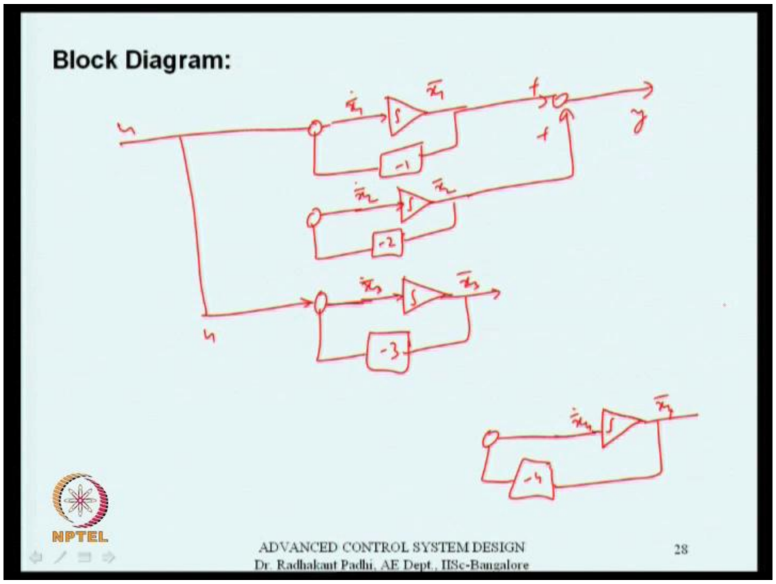

And in a pictorial sense, block diagram sense also it is easy to see this is. If you, I mean can I let me attempt to quickly do this so, this is what about x 4 x 4 is nothing but, some some simple integral so it is x 4 bar dot it is going like x 4 bar. So, it is not talking to anybody. So, it is just coming there and then you have a gain factor multiply by minus four probably and then just just operates that way. So, it is not not talking to anyone actually this is an integral actually.

What about other things actually? So, suppose you talk about let us say x 3 bar. So, this is also like integral you can put x 3 bar dot and then x 3 bar and then it is going to multiply by gain minus three. So, it is going to be there and remember x 3 bar is nothing but, minus three is there but, it is also there plus C U basically. So, if I have U component somewhere it is coming somewhere actually. So, let me just take it as A U component basically so this is like A U component. Now, if you what about let us say x 2 bar I mean see x x 2 bar is nothing but, minus 2 x 2.

So, it does not, there is no direct control influences. So, if I can substitute again I can do a parallel thing this is nothing but, x 2 bar then again it multiplies with a gain minus 2 and then it is there it is not influenced by any control actually what about x 1 bar x 1 bar is again there and there is a x 1 bar dot out here and there is certainly a gain multiplication again x time minus 1 and its and x 1 bar is also plus U. That means there is a control term here so I can I can talk about the control U and then I can that is the same U which will operate here.

And then, what about the output equation? Output equation is x 1 bar plus x 2 bar so x 1 bar is there plus and x 2 bar is plus. So, this is my output so what do you see here again the same concept what you what you what you are earlier visible. it I mean we just analyze from the equation it is also happen here so the output output contains only x 1 and x 2 x one bar and x 2 bar and x 3 bar is influenced by the control but, it is not there in the output and x 4 is not talking to anybody actually.

However, it is all stable. The Eigen values are stable the solution is the e to the power minus 4 t into x bar 0. So, it is going to decay and this is also going to decay basically it is not going to stabilize there actually. So, these are all the four properties. These are all nicely given in that is that is example that I wanted to talk actually. So, just a careful manner if you want try this is that is what actually.

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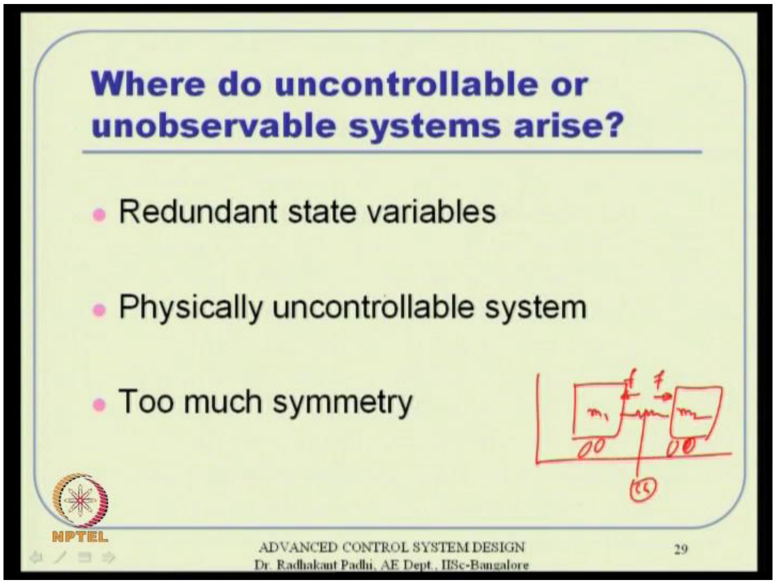

So, we will move on further. Now, question is where do this uncontrollable and unobservable systems are mean the system is controllable and observable well and good but, if it is not controllable not observable.

What is the reason? I mean why where does it come from actually? And it turns out there are main reasons like this first is redundant state variables. That means as system engineers the anytime you want to see a component. Probably you want to write a differential equation which is also not a very good because if you keep on writing differential equation keep on writing states and then it may lead to more than number of states that are required for the system.

So, in that sense it is introduce redundant state variables and you can see nice analysis in friedland's book which will tell you why it happens and then where the uncontrollable thing comes actually its formally feasible to show also mathematically that if you write more number of equations than necessary, then the system becomes uncontrollable and probably unobservable also. Actually then there are concept called physically uncontrollable that means suppose you are like designing a speed control mechanism for a automobile and then you are talking about a temperature regulation inside the vehicle. These two are completely decoupled. So, you cannot talk about controlling the speed of the vehicle by manipulating the AC knob of the of the car I mean these are not possible to do that.

So, they are physically separate and hence physically not controllable. Another example is also very easy to see also. There in the, this book suppose you have this let us say two mass system sort of thing and these are they are related obviously somewhere and you have this let us say you are something like that and you have a system which will which is like the force is coming here and the same force is coming here but, in a opposite direction actually.

If it happens then then mass m 1 and m 2 as far as the C G is concerned C G is somewhere here you cannot talk about controlling the C G by this force f. Actually the force, the moment you same force remember that and same kind of properties and all that actually. So, if you have this system then whatever is the C G wherever it is then you cannot disturb the C G location relative C G location with respect to the two mass basically by applying this force. Actually so that is that is also I mean kind of analytically proven this Friedland's book actually.

And there is also another surprise result which tells us that it if there is too much of symmetry that also is not good. It is not going to help us actually. In other words if there is too much symmetry and specially this is related to electrical circuit, it is easy to see actually suppose what is called as balanced circuit. For example, this this balanced circuit is are typically those those properties. Let us see that let me draw a circuit diagram here which tells us well.

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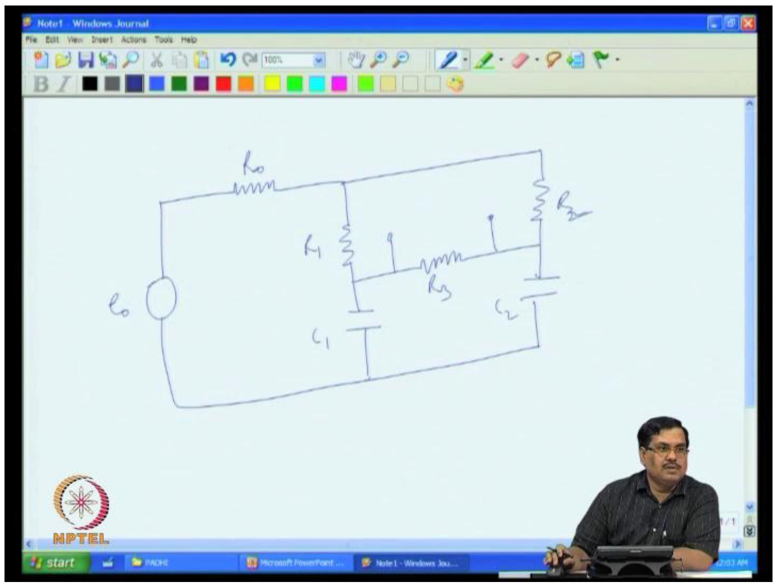

So, let me draw a circle here. This a source let us say e 0 and there is something like this a I have a register like this and I something there and then I have a capacitor and then the circuit closes and then I have another branch of it which is like a another another thing which is like this and the circuit closes as well.

And there is another another register that I want to do actually. So, this is probably the and this you can then R 1 and then R 2 and then R 2 and then r three and then probably R 0 something like that actually. Now, if you have similar similar I mean values of R 1 R 2 and things like that so the, what you want to do suppose you want to control the voltage across R 3 by manipulating e 0. It is just not feasible actually because the voltage drop will this voltage what you see here is remains same as that one actually. Why it is happening because there is too much of symmetry actually across this register R 3.

That is the, that results in that and it is also true for mechanical systems and all that actually. So, where symmetry is good in many applications symmetric systems are typically not controllable also. There is a controllability issue for that actually. So, there are various reasons for system being not controllable and non observable things like that actually and lot more nice examples and all you can read from many linear system books including Ogata and then Friedland and many other books which will give you lot of insight into what is happening for controllability and obsevability. And as I told before these non-linear system controllability and observability are also being studied and specially if you have a control a fine system. What is control a fine system?

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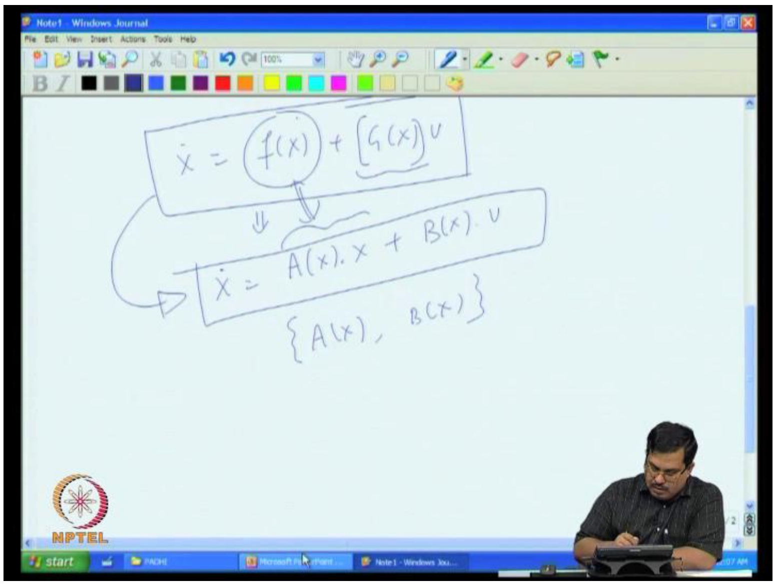

By the way a probably I will, if you have a control a fine system that means you have X dot equal to f of X plus G of X times C U. That means that is remember f of X is a vector whereas, G of X is A matrix. So, if you have a system like this then there are lot of results available for which we can talk about f of X and G of X properties or their smoothness and then differentiability things like that where the results available under sufficiency conditions. That means if those results are there then the systems are supposed to be controllable like that sufficiency condition but, there is certainly there are necessary conditions are not there.

So, in limited sense they are they are not a in many cases it is simply area of research actually and also remember there is a concept called point wise controllability that means we will study that I mean I mean I do not know whether we will study that or as used in this s D r e method and all which is related to l k r control l k r you are going to study in this course. Anyway, so this this system dynamics I can write it locally something like f of X. I will write it something like this a of X times X and let me redefine this z for notational simplicity and I write it is B of X times U.

Then, if I pick up any X at any point of time if I pick up my X value then a is A is a matrix at any point of time and B of X is a matrix so the question is this pair controllable at any point basically. So, if it happens the then it is called point wise controllable this this non-linear system that I am talking here X dot equal to a times X into X plus B times X into B. This is actually a non-linear system but, then any point of time we are interpreting that as a possible approximation to the linear system and how do you get this f of X to a of X into X that itself is a big question actually.

There are various ways possible and then I mean it may not be possible in some cases things like that actually but, once it is and there is no unique way certainly actually. So, if you select a way of doing that then is the system whatever way you have selected from f of X to X A times X into X whatever way you have done that is something called state dependent coefficient form. So, if you do whatever way select your coefficient form it may not satisfy point wise controllability actually. Then this this whatever way you have done it and attempt to do these are linear control systems for non-linear system it may not be valid actually.

This is a non-linear system. By doing that we are we are hoping that we can bring in the linear systems theory and try to build non-linear controllers for this and it is possible to a limited extent if the system satisfies point wise controllability at least actually. So, there are many concepts like this and this is an important concepts controllability and observability all relation I mean all controller and observer designs are possible once the system satisfies these properties. If they do not satisfy these properties; this is futile to do anything. So, it is probably you need to go back and try to see what is the reason why the system is uncontrollable or unobservable.

Some of the reasons may be like this and we want to address these problems first before going to the controller or observer design actually. So, more on that you can write this book; this two book are good description about these concepts especially for linear system actually and lot of theoretical analysis and all if you are interested the Thomas (( )) book is very good in that. But, it will have lot of mathematical rigorous analysis actually linear systems by Thomas (( )) actually.

With this much analysis I mean this much discussion I think I will stop for this class. Thank you.