Advanced Control System Design
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Lecture No. # 19
Stability of Linear Time Invariant Systems

Hello everyone. We will continue with our lecture series. Last lecture we have seen solution of linear system especially LTI system linear time invariant systems. So we will continue our discussion further in this lecture. We will study something we will use those solutions what we derived last class to study the properties like stability, controllability and observability of linear systems. When I tell linear systems in this particular course, we will confine ourselves to LTI systems actually and before I proceed further also remember that stability, controllability and observability are three different properties of linear system they are nothing related to each other in a way.

In our LTI systems, if the system is unstable it can still be controllable and it can still be observable, if the system is probably like not observable, it can still be stable and thing like that actually. So just remember the three independent properties and they are very much useful before we design controller observer actually. So, let us study that in detail. So first we will study the stability, then we will study controllability and towards the end of this lecture we will study observability, so stability of linear time invariant system.

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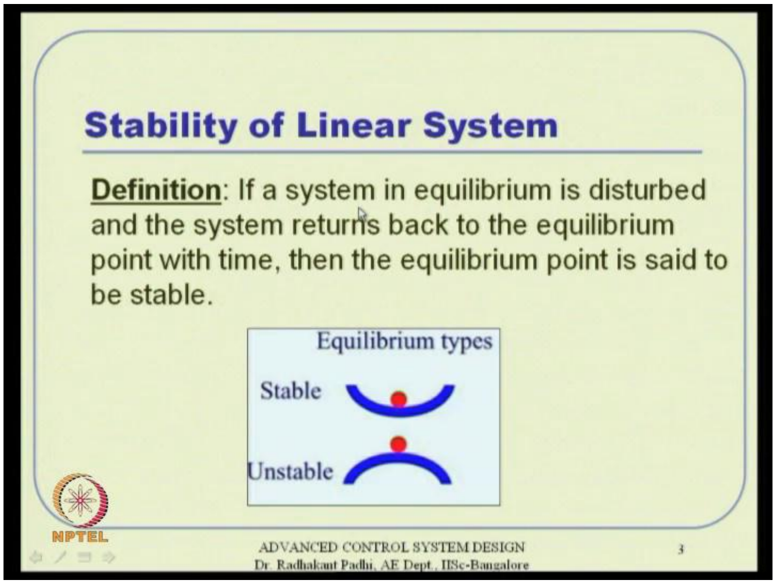

So first of all, what we mean by stability. We all know that a system, I mean when we talk about stability, we normally talk about stability of equilibrium points. And if the system is, I mean if a system in equilibrium is disturbed and the system returns back to the equilibrium point with time, then the equilibrium is has to be stable we all probably know this actually.

Pictorially speaking, if these kinds of a thing happen then the ball is at rest here also ball is rest ball is at rest theoretically, but the moment you disturb a little bit here, it will never come back where as it if you disturb a little bit here then it will suppose to come back and stabilize here. So, these are all very standardized to concepts that we have I mean it is very clear to us actually.

However, also remember that when you discuss about stability of non-linear systems, things need not be as simple as that. There are various notions of stability and we will study those in detail when we talk about lyapunov stability theory, but in linear systems the system is kind of either stable or unstable. There may be a very bottom line case which is marginally stable and all that. Though in other words if you assume like a flat surface here, then the ball if you just keeps on rolling, it will stabilize wherever it goes actually. So that is another that is the concept of marginal stability. Normally we visualize the system is either stable or unstable actually. So let us see, how this concept is tied up mathematically actually.

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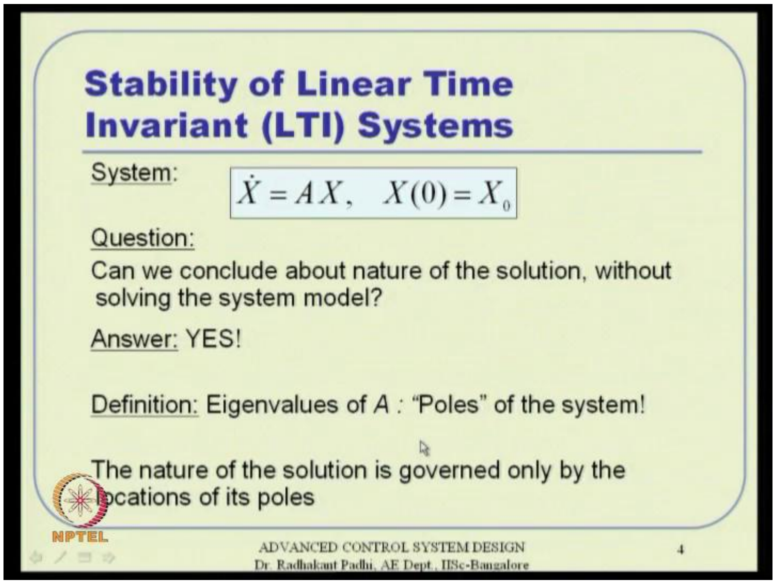

So what we discuss here is we have a system X dot equal to X, also the just this also tells us the stability is nothing to do with plus BU part. And when you talk system stability we normally talk about homogenous system thus the stability behavior actually. So BU plus AX plus BU; that BU part is the forcing function part that is something to do with control ability of the system, but not stability actually.

So when you call that the system is naturally stable, that means we have to analyze only the AX part of it. X dot equal to AX and if it is a non-linear system it will be X dot equal to F of X actually. So control input typically we will not study as part of this natural stability of the system actually.

Anyway coming back, we know this is the system dynamics and this is initial condition X. X of zero is X naught. Now the question is can we conclude about the nature of the solution without actually solving the system order? I mean we already have it, but we all want to know the solution nature, we are not interested in that particular solution starting from this particular initial condition what will happen and thing like that, but we just want to know the stability nature of the system in other word the nature of the solution, will it ever converts to 0? Or will it go to infinity? That is that is the study actually and also remember that we need to discuss X dot equal to X. These are normally delta AX dot equal to a times delta AX that is what the linearization theory actually.

So when you discuss stability of linear system especially, the aim here is to take delta AX to 0 that means X has gone to 0. So the question that we are asking is will this solution starting from this initial condition? What about may be the initial condition X 0 here, will it go to 0 or will it go to infinity actually? The two questions that you are asking a part of the stability analysis and the answer turns fortunately turns out to be yes. And it all has to do with the poles of the system and the poles by definition are, nothing but the Eigen values of the A matrix.

So if you just know the A matrix which is, we are assuming it is a time invariant matrix obviously it is like, constant matrix actually. So, we can valued the Eigen values and the system stability nature is all hidden in the Eigen values of the A matrix actually. And these are all also called poles of the system there actually if you derive, if you take this input output relationship and thing like that when you do this Laplace transform convert it to transfer function method I mean in the state space transfer function. Then you then you see that determinant of s i minus a will call the denominator. So that is how these poles and all will be defined actually in that way. So Eigen values of A are nothing but poles of the system and the nature of the solution is primarily governed by the location of the poles we will see how it is rare and thing like that.

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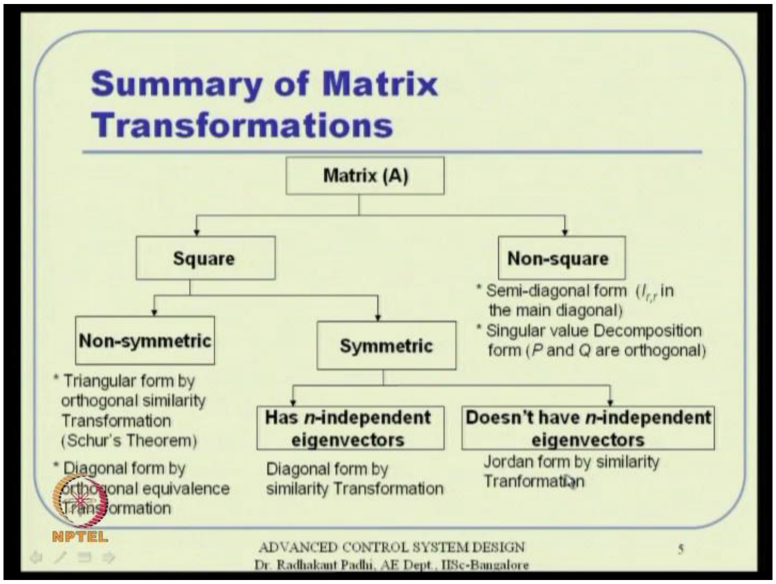

Before we do that we just recall there was a there was a matrix transformation, that we discuss sometime back in one of the previous lectures. The various transformations are possible in a matrix and normally, when you have this kind of a system, X is n dimensional and X dot is also n dimensional. X dot is term by term differentiation that is the by definition. So this is also n by 1 that is also n by 1. So obviously A has to be n by n so it is a certainly a square matrix actually. The plus BU the B matrix need not be square matrix, but A is certainly a square matrix.

So if you see this table we are not typically interested in this non square part of it, we are interested in square part of it. Now this square matrix may or may not be symmetric, and if it is and I mean in this linear system analysis, it need not be symmetric either I mean we are not we are not assuming anything here.

However, you can also see that if it is a symmetric matrix, this it I mean which has it can have independent Eigen value vectors it may not have independent Eigenvectors n independent Eigenvectors. So in general so it is like that you can reduce the symmetric matrix, so it would two diagonal forms by similarity transformation and it can be done for the Jordan I mean if it does not have this n independent Eigenvectors it can reduce it to like Jordan form. And the matrix need not be really symmetric to do that, we can start with any matrix and then carry on with that actually, but if it is symmetric it has 2 possibilities that are all this table tells actually. But in general any square matrix you can really reduce it to like either a diagonal form or a Jordan form by carrying out similarity transformation.

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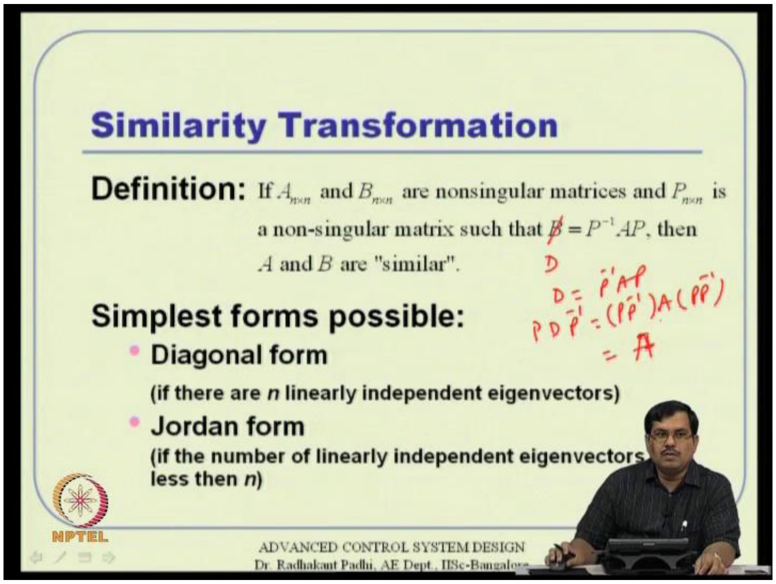

And how do you do similarity transformation like given a A matrix n by n, you can we can construct this n by n matrix P which is supposed to be non singular such that this P inverse AP is, nothing but B matrix and then this n B matrix are supposed to be similar actually. So what are the two forms either it is Diagonal or it is Jordan. So if there are n linearly independent Eigenvectors, then it is Diagonal form or it is Jordan form and all that actually we discussed that. We discuss those things in our matrix review class before and all so.

So let us consider our stability analysis here, first we do first we will consider that the system is let us say it has that the systematic A has in n linearly independent Eigen vectors actually it does have actually. Then what we can what we have here, is that A is nothing but PDP inverse that is all I can write. So this is like B is nothing but D now. So if B is P inverse AP. So obviously, like well this B becomes D here, so then what we have D equal to P inverse AP. So if I take PDP inverse then I am pre multiplying this side by P and post multiplying by P inverse then it is PP inverse A and P A P inverse actually.

So this will be identity that is identity so that is my A matrix. So A is nothing but PDP inverse actually.

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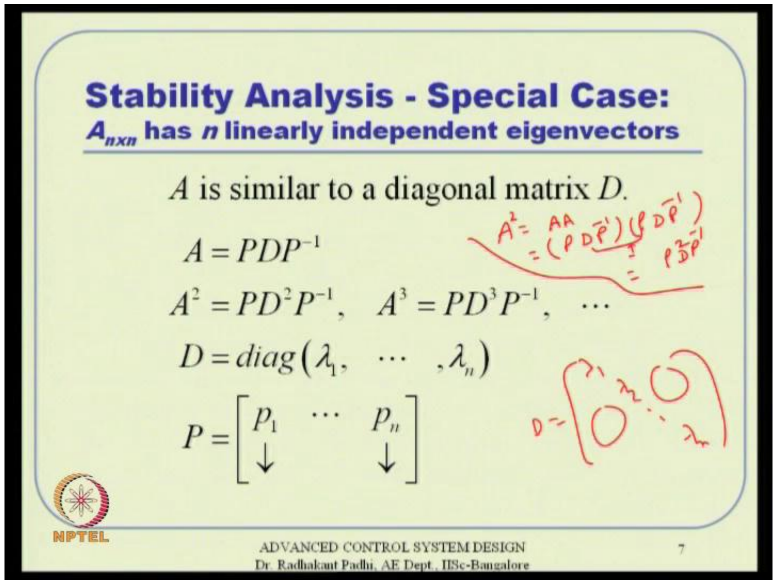

So that is what you are having here, so A is PDP inverse the nice thing about this particular form is when I take A square, that A square becomes it is easy to see. So A square is, nothing but A times A, so A times A is PDP inverse times PDP inverse and these 2 will come in to identity so what you are left out with is PD square P inverse like I do not know.

So similarly, if you do this algebra one more time, you will get a cube is, nothing but PD cube P inverse actually and similarly, a fourth is PD 4 P inverse like that it will continue actually. Where diagonal we know that is n independently I mean this has n linearly independent Eigenvectors, so obviously D this particular matrix D that we are talking about is, nothing but a diagonal matrix where diagonal elements are just Eigen values actually.

So that means D takes the form of lambda 1, lambda 2, up to lambda n; this is just written in a compact form out there actually. So that is what it is written in the form and the P is, nothing but Eigenvectors P one is, nothing but the Eigenvector of the A matrix P 2 is second eigenvector of the A matrix and we have n linearly independent Eigenvectors. So we can write that P 1 is nothing but first Eigen vectors, P 2 is second eigenvector like that actually.

So corresponding to P 1, corresponding to correspond to lambda 1; P 2 corresponds to lambda 2; and P n corresponds to lambda n, like that actually. So we know that I mean this is very clear to us now how do you make use of that let us say. So we know that this particular solution of the system what we started with, and A is your constant matrix the time invariant system so obviously, the solution takes the form of this way e to the power At times X 0.

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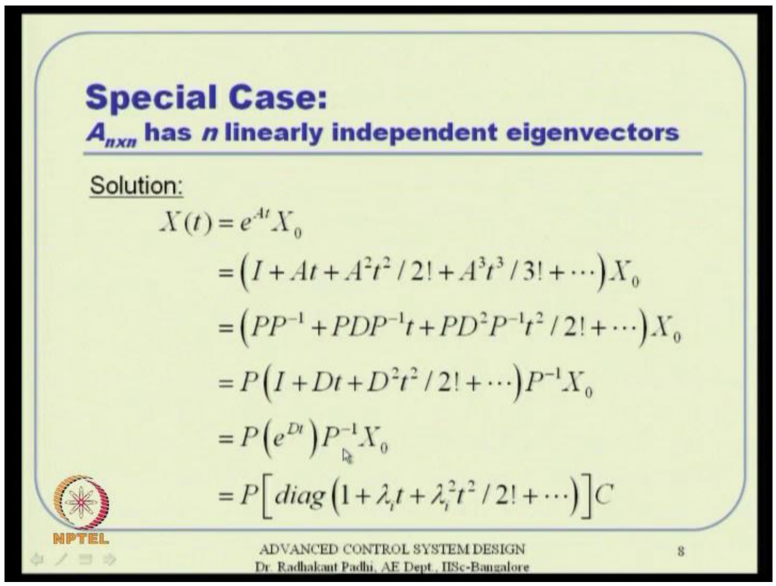

Now e to the power At, I can expand it that way that is the definition of e to the power At. We have discussed that in previous class, we started that as a definition and then formally showed that this particular form when you take e to the power At times X 0, is nothing but a solution of the system and hence the solution and then initial condition got satisfied with respect to X 0 that way we discuss all that in previous class actually.

Now, I will this I whatever we have here the second row what we are doing here is we are just introducing this term that I equal to P times P inverse I just write it that way and then A is nothing but PDP inverse that just that we did that A is PDP inverse A square is PDP square P inverse like that. So we substitute that A is PDP inverse A square is PD square P inverse and then t square by 2 factorial say like that it will continue basically.

So what is nice thing, if you see the t square by 2 factorial all that are scalar quantity they are not vector matrix anything like that. So I can put that anywhere I want so what I do is this t I will I just take it and put it here that means I take it from the right and just put next to D and then D square by 2 factorial I take it from here and put next to D square by 2 factorial I mean D square next to D square. So it become D square times t square by 2 factorial like that actually.

So what I have after that I have an entire sequence of this matrices that is coming up. It is all getting pre multiplied by P and post multiplied by P inverse. This entire term what you see in bracket is everything is pre multiplied every term by term is pre multiplied by P, and post multiplied by P inverse actually. So I can take everything P pre multiplication common P inverse post multiplication common, and then I can then I can have this one actually.

So obviously, by definition again so this is e to the power At so obviously, this is nothing but e to the power Dt and when you have e to the power Dt obviously, these are these are like diagonal of that so let us see that very quickly.

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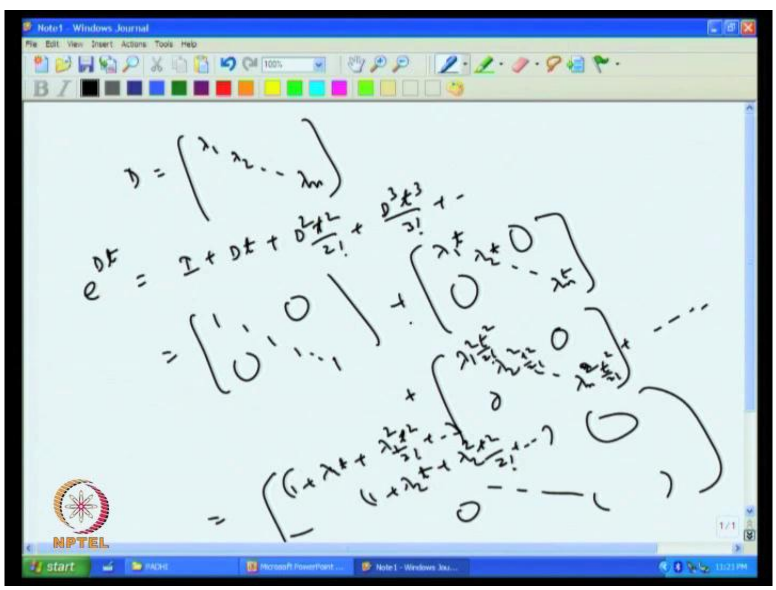

So we have D nothing but lambda 1, lambda 2, like that remember this is like a diagonal matrix. So what you are doing is e to the power Dt. So e to the power Dt nothing but I plus

Dt plus D square t square by 2 factorial like that actually D cube t cube by 3 factorial like that is my definition.

So what I have here I is nothing but 1, 1, 1, 1 in the diagonal plus D, D is nothing but lambda 1, lambda 2, up to lambda n. And then this is t here so it will get multiplied with t everything else is 0. And the second term is something like we have this D square D square remembers this D is a diagonal matrix lambda 1, lambda 2, lambda n and all that. So D square is nothing but lambda 1 square, lambda 2 square, up to lambda n square in the diagonal of diagonal nothing and then we have t square by 2 factorial. So I can multiply or divide by t square by 2 factorial so, it is t square by 2 factorial like that, t square by 2 factorial and the other terms will be like that.

So if you see all these matrices are just diagonal matrices actually. So what I have here if I write one term by term here the first term will be 1 plus lambda 1, t plus lambda 1 square t square by 2 factorial like that, but remember this is just first element of the matrix second element will be 1 plus lambda second diagonal element will be 1 plus lambda 2 t plus lambda 2 square t square by 2 factorial like that and everything we will continue in the diagonal way of diagonal will all be 0 actually.

So what I am doing here I am just taking out these taking this D matrix, I mean this D out here and then I am just substituting that e to the power Dt by definition whatever this is then I just put I is nothing but 1, 1, 1 in the diagonal Dt is lambda 1; lambda D is nothing but lambda 1, lambda 2, up to lambda n in the diagonal, so that is all there multiplied by t so every all other elements are zero anyway.

Similarly, D square t square by 2 factorial again I do and it series will continue actually so this so, and this series results in this 1 plus lambda, 1 plus lambda 1, square t square by 2 factorial in the first diagonal element second diagonal element and thing like that it continues that way actually. So essentially that is what is done here and so e to the power Dt is nothing but this diagonal matrix actually, the symbol transfer a diagonal matrix actually.

And this P inverse X 0 this whatever you see here, this P this P inverse X 0 I am defining that something like C, sum because remember X 0 is a vector and P inverse is n by 1 matrix actually. So the P inverse X 0 is just a vector on that vector I am defining at a C actually.

So I have a square matrix multiplied with another square matrix multiplied with a vector actually. So ultimately the result is a vector which is also I mean X of t is a vector in the left hand side, so that is it is that actually. So then what we do here, so if you see this entire series what you see here, remember P is also nothing but this diagonal this way this size P 1, P 2, P 3, like P is also nothing but P 1, P 2, P 3 like that actually. I mean these are all vectors P 1 is a vector; P 2 is a vector these are Eigenvectors of a matrix, but still they are vectors actually.

So, what we have here the entire solution what you looking at is a series of column vectors multiplied with a diagonal element matrix multiplied with a vector c 1, c 2 up to c n actually. so that entire thing I can rewrite this system in this form actually. So if you think a little and probably I will encourage you to do it yourself may be. So this P 1, P 2 is column vector this is a diagonal matrix and there is a c. So that and remember what is this actually, by the way this series what you see here, what is this? This is the, this series is nothing but e to the power lambda I t obviously. So that is the beauty of having a Eigen I mean this diagonal matrix and all that.

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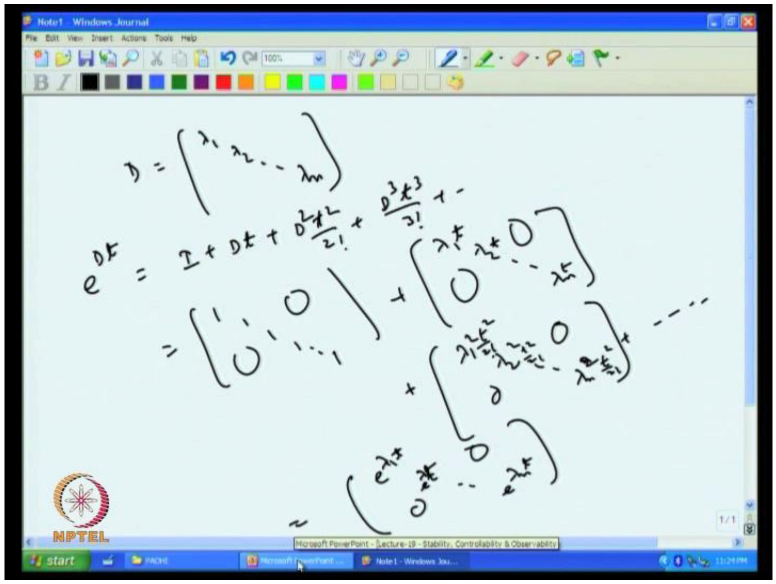

So this entire series will result in something like e to the power lambda 1t; here e to the power lambda 2 t; here and things like that, e to the power lambda n t here that is all it will result in actually.

Because all the series again 1 plus lambda 1; t plus lambda 1 square; t square by 2 factorial plus lambda 1 q, t q by 3 factorial all that result in e to the power lambda 1t like that actually. So that is the beauty there actually.

So I can essentially write this, this matrix that I am talking here as P times this one this diagonal matrix times c 1, c 2 and all that you can just see that actually.

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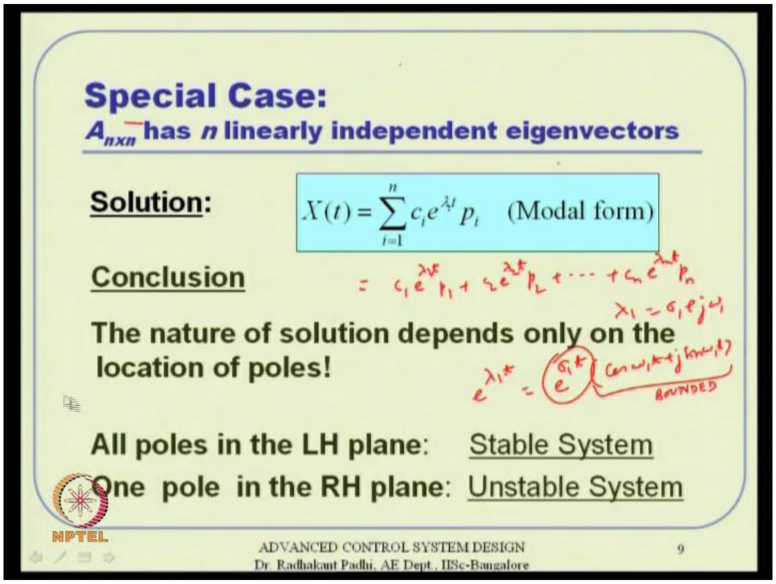

So, what you observe here is also called some this is the same solution whatever we had here instead of writing that way we have been successfully I mean, we are successfully in writing it this way, in the form of remember c. I also come from some P inverse X 0 where initial condition is invariant, actually initial condition goes there.

P inverse is A matrix where it may or may not have direct meaning of that it is a P inverse of this P matrix. Where otherwise it is e to the power lambda I t where lambdas are directly Eigen values and P i is a directly eigenvectors. So this is actually this c i embeds initial conditions inverse matrix together sort of thing. And then after that these constant aside you have the solution in the form of Eigen values and Eigenvectors actually.

So this solution is something called modal form of solution because Eigen value eigenvector pairs are, nothing but modes of the system Eigenvectors are essentially modes of the system. If you have P Eigenvector 1, Eigenvector 2 thing like that. They are essentially system modes actually and the energy associated with that is given by Eigen values actually.

Eigen value did not directly the energy, but they I mean they signify there is energy content actually there by the way. Especially if you have a symmetric matrix then the then we know that Eigen values are real and in that sense it gives even more meaning actually, energies are typically positive quantity. So lambdas have to be positive for that, so system matrix has to be positive definite also the things will start from there will not digress so much on that actually. The whole idea is this Eigenvectors are, nothing but modes of the system and hence we have this X of t given in this form is called modal form actually.

So what you conclude from here, c is a constant thing every c i what you see c is just a constant in fact, c i is a scalar; c is a vector ultimately c i; every element of that vector is a scalar and then P i is a vector obviously, but that is still a constant vector Eigenvectors as for the constant matrix are they themselves are constant. So the time varying part is all there actually. If the power lambda et and what e to the power lambda i t, it is a summation series remember that first term will consist of e to the power lambda 1 t the second term plus term it will contain lambda 2 t and thing like that actually.

So what does it give us? If I can probably write it here itself, so this is essentially c 1 e to the power lambda 1 t P 1 plus c 2; e to the power lambda 2 t P 2 like that actually, c n e to the power lambda n t P n that is that is what you are looking at actually and e to the power lambda 1 t what we know actually, e to the power lambda 1 t is essentially e to the power sigma 1 t. Sigma Plus J omega if I write lambda 1; lambda 1 is, nothing but sigma 1 plus J omega 1, then if I write that then it is e to the power lambda 1 t nothing but e to the power lambda, I mean sigma 1 t into cos omega 1 t plus J sin omega t omega 1 t we know that.

And this one essentially you can write it in a different form, we can write it in a phase form and thing like that I mean, so what you have here this particular thing is always bounded it is never going to be I mean it is never going to infinity or neither it is going to decay that we are sin cosine terms actually.

So whether the solution will go to infinity or go to 0 is all embedded here, e to the power sigma t basically. So if it the if any of the sigma thing like for example, sigma 1, sigma 2, sigma 3 up to sigma n, you have and any one of that is destabilizing in other words any one of that is positive quantity then the solution will certainly go to infinity. So the solution to decay to 0 each of the term has to decay to 0 and for that all these exponential terms that you see here should necessarily decay to 0 actually. So that is that is what we mean actually.

So what it tells you, if the solution nature depends only on the location of the poles and it tells else it essentially gives us that if the poles are in the left half plane, then the system is stable and one pole in the right half plane the system is certainly unstable. Because that part will go to infinity and everything else will go to infinity because of that.

So this all happens when, n has n linearly independent Eigenvectors, so the question is what happens otherwise does it still have that meaning or no actually. So let us see that.

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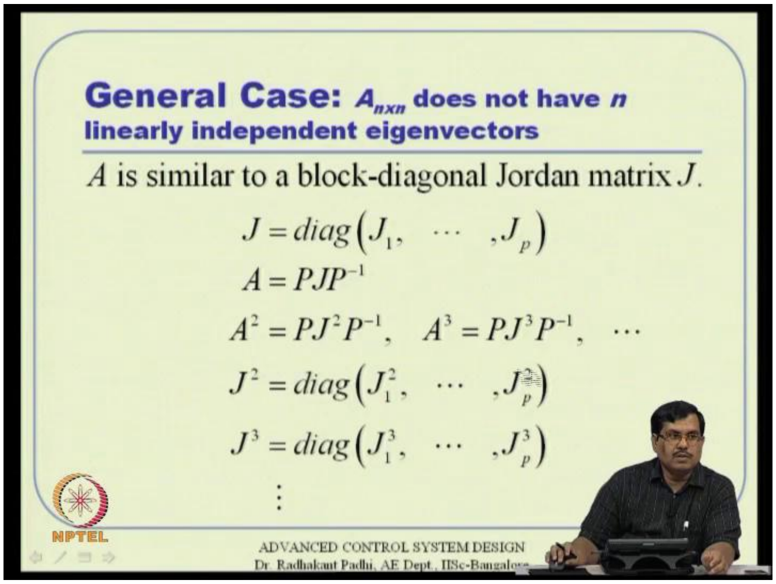

So if it is if it does not have linearly independent Eigenvectors n of them then also what we what we really have, we can reduce this A matrix to a Jordan matrix is actually a block-diagonal matrix actually.

So what happens instead of D we should be able to write in the form of J where J is not diagonal or it is not a purely diagonal matrix, but it is a block-diagonal matrix where each of the blocks are J 1, J 2 up to J P actually. Remember this need not be J n, P is different from n because the blocks I mean, if P becomes n then obviously this J 1, J 2 is nothing but simply Eigen one by one, Eigen value matrix actually that way but otherwise it is J 1, J 2 up to J P.

So a is nothing but P J P inverse almost very parallel to what we did here all that thing what we did here instead of D we have to write in terms of J basically. J is nothing but the Jordan matrix. So a square is P J square P inverse A cube is P J cube P inverse like that but J square J cube will all satisfy this equation remember that instead of lambda 1 square lambda 2 square we should be able to write in terms of J 1 square up to J P square whereas, remember that this matrices are actually block I mean, they are all this J is nothing but block-diagonal matrix. That means these are not necessarily scalar quantity these are small dimensional matrices actually each of them can be a matrix by itself actually.

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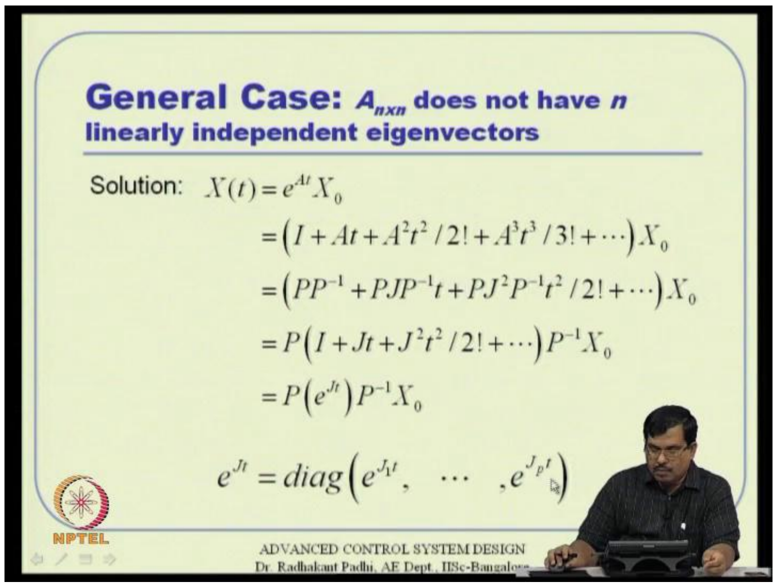

So is, but you can you will be able to here write J square J cube like that and then the analysis also goes in parallel here e to the power At times X 0. So we just do the same exercise what we did before I s b P inverse is A is J P inverse this time and thing like that then we take P left common P inverse right common and then again define P inverse X 0 something like c actually, we will do that later. And then this entire solution what you see this is actually a matrix now. This is not a diagonal element by element will not be able to reduce it further, but we should be able to tell that this is, nothing but e to the power Jt.

And e to the power J t is, nothing but diagonal of e to the power J 1 t e to the power J 2 t like that actually. What you what you what you see here e to the power J t is a big matrix actually. That big matrix will also consist of diagonal matrix matrices now, and each of the diagonal matrixes will be e to the power J 1 t e to the power J 2 t like that actually. Now we have to carefully analyze what is this e to the power J 1 t and e to the power J 2 t thing like that up to this the analysis is fairly same actually.

How do you do that? Now let us say instead of doing every individual thing I will pluck out one particular J. Let us say J 1,, J 2 up to J P I have let me pluck out one particular J and call that a J hat actually.

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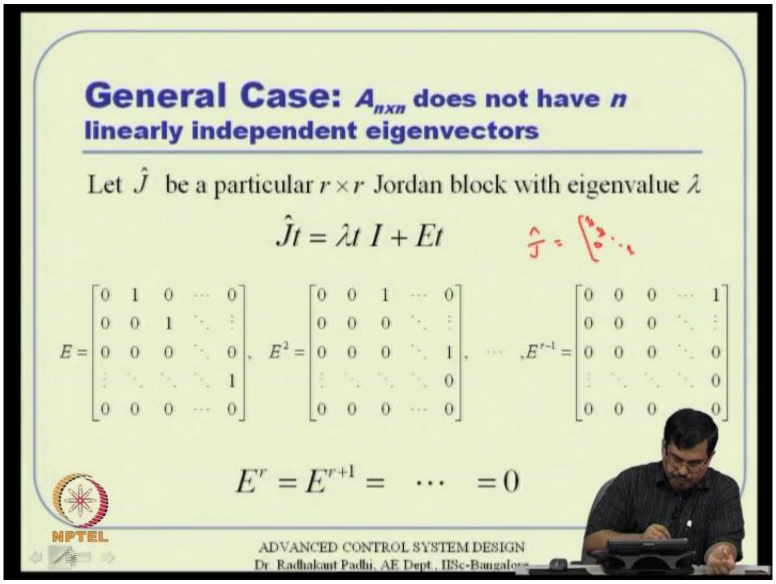

And this J hat can be of r by your r Jordan block remember it depends on how many times this particular Eigen value was repeated actually that represents that r represents that, this J hat that I am picking out from here will that is associated with this particular Eigen value lambda let us say. Where the lambda is repeated r times actually. So the J hat will be a particular r by r block Jordan block actually.

So this J hat times t what I have here J 1 t J 2 t like that, so J hat times t I can fortunately write in the form of this one actually. So 1th you see what is what is J what J by the way any J is, any J hat let us say. This consist of this particular thing we have Jordan form so this is 1 1 1 all are one in the diagonal of let me write it a bigger way let me write it in a bigger way. So J hat that is all we are talking about.

J hat is that particular element.

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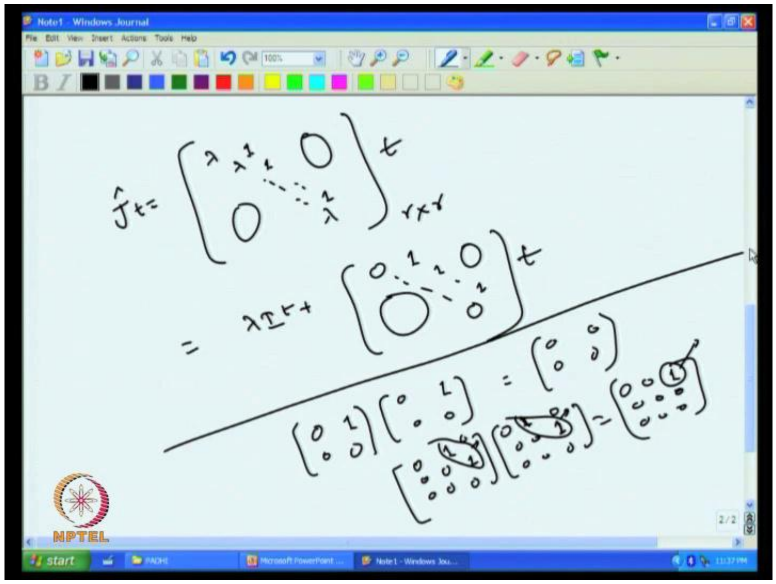

So J hat is, nothing but one here of the all we have to sorry not 0, but one these are all ones here, and then you have lambdas just next to them lambda like that, lambda no sorry I am the I have done a mistake here. So this that is not the thing actually. This is going to be lambda lambda lambda up to up to r times it is repeated so this is going to be lambda lambda.

And then very next to that is 1 1 1 actually one here, one here like that actually, one here everything else is 0, that is the particular thing this J hat that we are talking about actually. So this particular thing I can write it as lambda times I plus this element which is this 0 0 0 now everywhere 0 and then this is 1 1 1 here. That is all I am doing there actually.

So obviously, if I have a t term associated with that then it goes to there and this will go there this is all go to there I mean that is the t term, if I have I mean this t is a scalar point it and so and helps me actually anyway. So that is what is going on here, so I can write J hat of t as lambda t times i plus e t where e is defined as that way.

Now the beauty part of it is this e matrix what we have here it is defined as something like I mean this 0 0 0 0 there and 1 1 1 1 here. So that means this is something called idempotent matrix actually. That means at some power of this it will go to I mean, it will some power of it, it will vanish actually and if it is r by r matrix then e to the power r will be 0 and we can very quickly see that again probably. So let us see that when 2 by 2 case let me just solve that.

So suppose we have 0 0 and then 1 0 here, you just multiply with 0 1 0 0 again this will happens to be all like 0 time 0 plus 1 time 0 is 0 1 times 0 times 1 plus 1 times 0 is 0 0 times 0 0 I mean 0 times 0 plus 0 times 0 is 0 and 0 times 1 plus 0 times 0 is also 0 and if you happens to be 3 by 3 let us say. So what you are talking 0 0 0 and then you have one here, one here. So you have this 0 0 0 and then multiply with 0 1 0 and then 0 0 1 0 0 0 you multiply that you will again get something like 0, I mean you can verify this it will happen to be 0 0 1 0 0 0, 0 0 0 so that what does it tell us actually.

If I have some 1 1 element here, this one will get shifted to one actually it will go to that corner thing. So this will result in one. So then there is nothing to shift anymore so that the next one will become 0 actually. So that is a 1 1 1 shift it will happen n of diagonal sense and it will ultimately would 0 that is the nice part of it actually.

so that this is what here we are observing so e is like this, if e is like this, so I mean this J hat of t is lambda times t plus i plus e t and then e is, nothing but that. So e square is just one element of that is what I just told you this one what you see what you saw here this particular row. Well let me draw that in a better way this particular off-diagonal this gets shifted by one, that is what it happens this is the, remember this is diagonal here, I mean this is diagonal this is diagonal this is one gone now you are having the next one actually.

Now similarly, this process continues up to e to the power r minus 1 and e to the power r will happen to be 0 and hence e to the power r r plus 1 r plus 2 everything will become 0 actually e to the power r plus 1 is nothing but e to the power r into e. So if this is 0 this is already 0 then everything is 0 anyway actually. So we are left out with this J hat of t is in this way.

So what are you having we are primarily having a diagonal matrix which will not which will never decay and we have one more term one more matrix which will certainly decay to a finite series actually. This is not going to be an infinite series this is this happens to one particular element one particular e to the power J t remember that. This can happen this what you are having here is a block-diagonal matrix form of e to the power J 1 t e to the power J 2 t like that actually.

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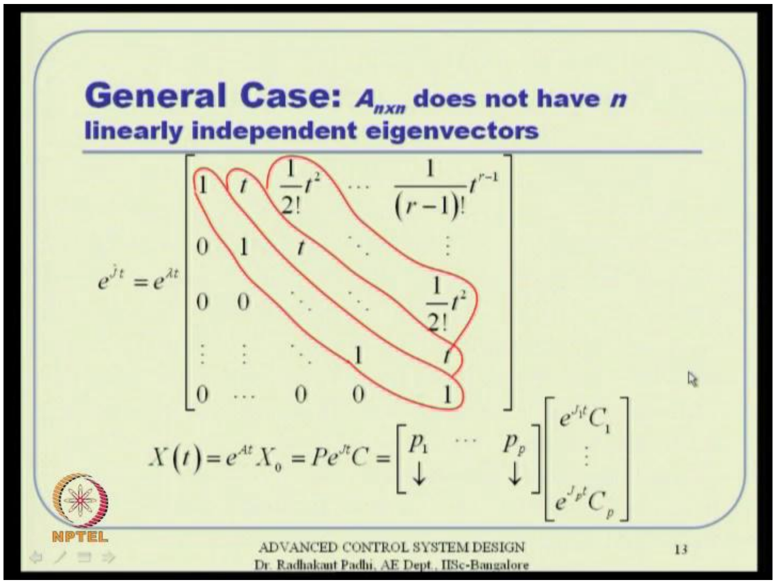

So what this is what it is so what you have here now. Now e to the power J hat of t that is what we are analyzing this is not you see, nothing but what is another thing actually e to the power J hat t e to the power J hat t here is nothing but e to the power this fellow lambda i t plus e t and this is like e to the power A Plus e to the b. So obviously it is e to the power lambda i t into e to the power e t so that is what we are doing here anyway.

All right, so what you are having here e to the power J hat t is e to the power lambda t which is one part of it lambda I t is nothing but e to the power lambda t also you can multiplied by I is obviously, but I is not necessary because there is another big matrix which is getting multiplied anyway here. So this particular thing what we have here e to the power lambda t into e to the power e t and e to the power e t will now turn out from this fellow. This particular series and this series happen to be 1 plus t plus t square by 2 factorial remember that will result in t basically, e t remember e t. So e will multiply get this particular element will get multiplied by t ultimately. This will be multiplied by t square b 2 factorial this series will be multiplied by t to the power r, I do not know this series will be multiplied by t to the power r minus 1 by r minus 1 factorial.

So if you see the first row it is all getting one from here that means it is one element is coming from here one times t. So that is t the second element is from t square by 2 factorial coming from here like that actually, So the first element what you see here is faster fast is one anyway the second one is t the third one is t square by 2 factorial like that actually.

So there is that series is actually I mean the this is actually like a triangular matrix slightly the elements are bad replace. So this essentially like what you have here is actually a diagonal is this is the diagonal entries and then the next one is t t t the next one is t square by 2 factorial like that actually. So what you see here is next one, next one is t t t t all that next one is t square by 2 factorial; that way and the series continues actually.

So what we are having ultimately here after all these analysis actually I can certainly write X of X of t is nothing but e to the power At times X 0 is that that one we analyze before that is what we analyze actually and then e to the power J t we have tried we have seen that this can be written like that, so that is that we have written that way and then we talked about I mean, we this series truncates after e to the power r. So we have to only consider up to e to the power r minus 1 that is where the series is over basically.

And if I put them this e to the power J hat t is nothing but that is e to the power this fellow plus that fellow so this e to the power this into e to the power that, so that is e to the power lambda t into e to the power e At whatever happens there and e to the power e t turns out to be that series that matrix actually. The first row will be that way the second one truncates thing like that actually.

So what are we left out is if you just put them together try to put them and then this P is nothing but again this m eigenvectors and remember some of these are essentially now see this some of them are essentially like generalize Eigenvector now, while we are talking about Jordan form now. So some of them may be Eigenvectors and then for the repeated ones you may or may not have independent Eigenvector. If you have independent eigenvector it does not the C does not arise anyway.

So you certainly are talking about a case where you do not have independent Eigenvectors so we are actually taking the help of I mean generalize eigenvectors actually. So this is what you are putting here P e to the power J t into C. So that I again I put it back here and then this Cs n all that whatever this I mean this particular thing that we are defining as C here this is nothing but C.

What c if the moment if the what you see here c 1, c 2 and all that here I cannot write them as really the elements of the C matrix, I mean these are all also like partition vectors actually. C can be a big vector C 1, C 2 to up to C P are like partition C vectors actually, but anyway this is like up to P 1, P 2 up to P P and then results in that way actually.

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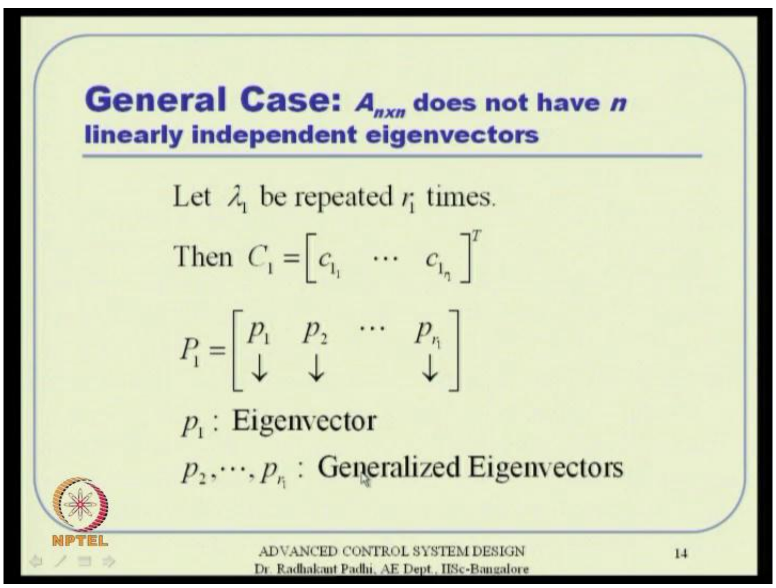

So what you what you have essentially, so this P 1 that we are talking here P 2 up to that is like define that way so if it is r is I mean lambda 1 is repeated r 1 times then C 1 what you see here will consist of c 1 1 and c 1 2, c 1 3 up to c 1 r 1 the lambda 1 is repeated to repeated like r 1 times basically. So P 1 is nothing but P 1 vector P 2 vector up to P r vector where P 1 is really the Eigenvector, but P 2 to P r 1 are nothing but generalize Eigenvector that is how it is constructed actually.

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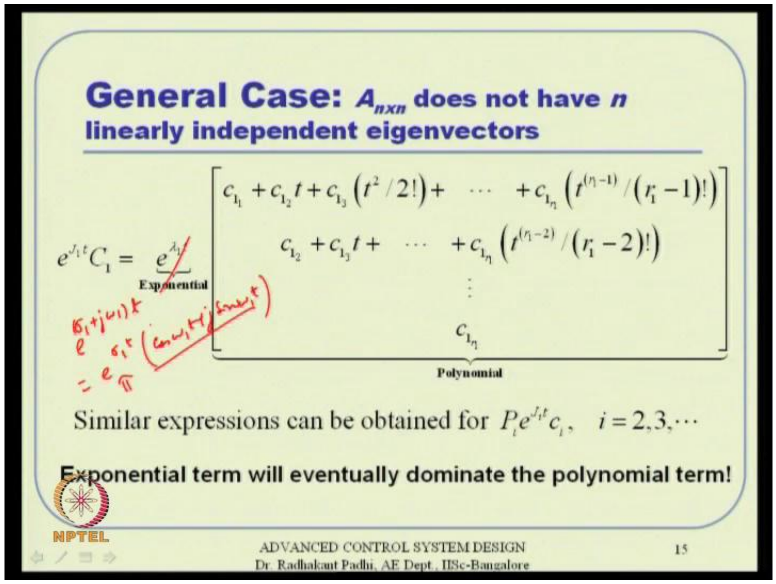

So all these mathematical complexity we have to take care about them, but the point is ultimately I will be able to write it this way C is the power lambda 1 t C 1, is nothing but e to the power lambda 1 t which is exponential again and then this is a vector this large vector what you saw here. This essentially all these vectors I can I mean I can expand that and then put it in this form actually that essentially come from here basically.

So this is actually a vector the first one is a largest polynomial the one the second one is one out of lesser like that the last one is zero-th order I mean this only a constant value actually, but essentially this is just a vector. So earlier we simply had a nice vector basically, where these are like I mean so polynomial terms actually now, but the point here is the point I mean you can do similar expressions this is like for J 1 basically what we are done so J 2 ,J 3 all the thing we will continue that way.

And what message tells us the one is that we have an exponential term and the other one is polynomial term and we all know that the exponential term will eventually dominate the polynomial term. So polynomial can be initially powerful, but exponential is will eventually dominant any polynomial term and that is what we are analyzing when we talk about system stability, we want to find out the system solution as t goes to infinity. So we are not particular about this some particular finite time and all that actually the moment anything that goes towards infinity is our concern here actually really.

So what we are telling here again e to the power that exponential term is predominant and hence, this m dissolved wholes good actually. So e to the power lambda 1 t again I can write that as this e to the power sigma 1 t plus J omega 1 t and then I can write that e to the power sigma 1 t into that cos omega 1 t plus J sin omega t and thing like that actually. So this will not decay, but this will eventually decay if the system is stabilizing in other words, if sigma if the sigma values all the sigma values are negative that means we do not really care about the omega part of it the complex part of I mean the imaginary part of it is does not play role here the sigma is the this real part of it as long as the real value is negative that means the system is going to be stable that is all it tell.

But all of the all of the roots should satisfy that if one of them is 0, I mean one of them is in the right hand side then that particular part of the solution will go to infinity and hence I everything will go to infinity like that.

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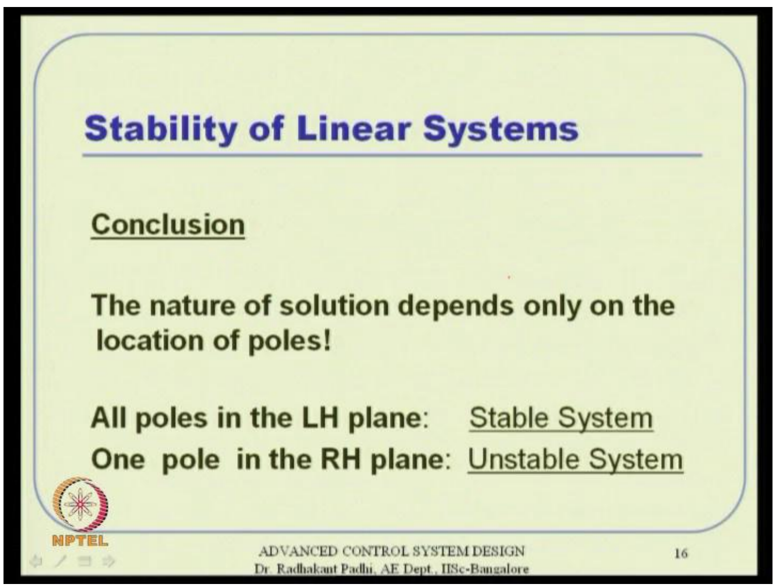

So that is what it is so the conclusion part of it, so what we are telling here if the nature of the solution depends only on the location of the poles. And all poles in the left hand side the system is stable if one pole in the right hand side the system is really unstable that is all we are telling here actually.

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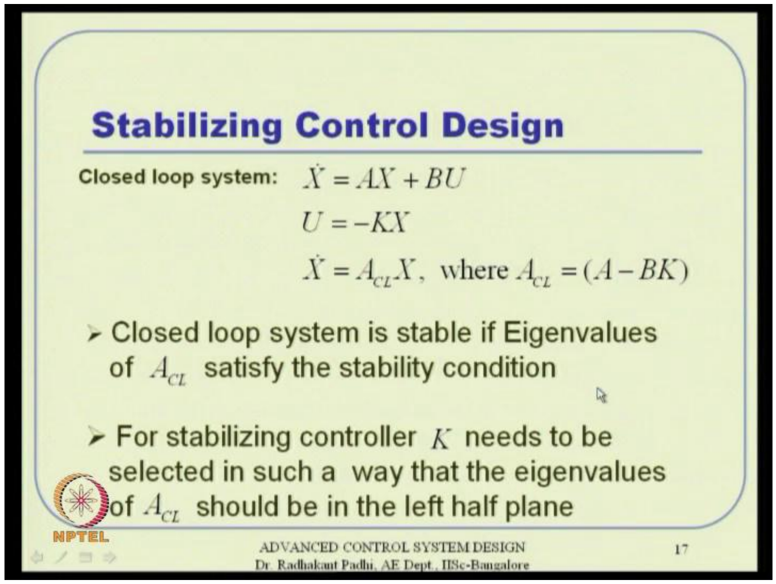

So now I mean going well the next question is if you really know this is the system, this is the conclusion it does not really matter whether you have n independent Eigenvectors or not or thing like that once it relatively simpler analysis the other one is a little bit complicated on this is rather, but ultimately the results are same the results tell us that I do not have to care about the Eigen spectrum really. I just have to do like Eigen values and if all the Eigen values are strictly in the left hand side the system is certainly exponentially stable. That means the error is certainly going to go to 0 as t goes to infinity so that is the result.

Now if that is the result that we know then obviously this is a control design problem see up to it is analysis we have analyzed the problem and then let us say we want to design a control for the linear system now. So obviously, what you one way to look at the problem is like this I have X dot equal to X plus BU this is the system dynamics that I have, what I have here and then I can assume this U to be minus K X actually let us say let me assume that actually.

So simply I if I substitute it back here then I get X dot equal to A CL X where A CL is nothing but A minus BK that is the CL turns for close loop. So the close loop the if I have a control design in the state field platform that way then this force system in the close loop is behaving like an like a homogeneous system where the close loop system matrix is given like that.

So if I see the Eigen values of a matrix that is my open loop poles, but if I have the Eigen values of A CL matrix these that is the close loop poles actually. So even if my open loop poles are I mean some of them are destabilizing and thing like that, if I can if I am able to design AProper K, so that the Eigen values of A CL are all in the left hand side then I'm done actually. So I really do not have to worry about whether Eigen values of a are really destabilizing actually. All of them are stabilizing or not actually they all of them need not be, but if I am little intelligent, the I mean if the and the system allows me to do that obviously, then I can design a control U equal to minus K X where K is, nothing but a gain matrix that that we will see just some of the next I mean subsequent classes we will see how to design a K matrix that is our design part of it.

But once you design a K matrix properly, so that this close loop Eigen values are all in the left hand side then the we have designed a stabilizing control system actually that is the that is the philosophy. So we are given a linear time invariant system we just have to design e equal to minus KX. So we just essentially have to design a K matrix then we know e equal to minus KX actually that is the formula.

So that is the whole idea of a stabilizing control design and we will see that see some of these control design ideas in the subsequent classes. Anyway so let me summarize some of these in this class probably I do not have too much time in this class to study about controllability observability we will see that next class.

And then this is where we started the system equilibrium stable unstable all that and then we stopped ask the question whether the system is stabilize stable or not? And without solving this we really do not have to pick up these a matrix and then try to slog in there and then try to plot it and then try to see and we are not interested in doing that we just asking the question can we infer the nature of the solution without actually solving this and essentially the answer turns out to be yes and this all given by the Eigen values of the A matrix whether it is it can be diagonalised or not it does not matter it has to be only the Eigen value location matters.

Anyway so we carried out this we had took the help of this singularity transformation and it discussed that any A B any A matrix we can actually convert it to either a diagonal matrix or a Jordan diagonal I mean block diagonal Jordan matrix, I mean that is the way actually. So we first took whether we can talk about I mean if it is possible to diagonalizable A matrix then what is the analysis when we started with A equal to PD inverse PDP inverse then A square is like that and thing like that.

So then we carried out this algebra and tell X of t is e to the power At times X 0. So let me substitute e to the power A t and then I plug it a equal to I mean I equal to P P inverse I equal to PDP inverse like that, and then I took out this P common and P inverse common and P inverse X 0 I define it a C and then I carried out if it is like this then this is, nothing but P times a diagonal matrix where a diagonal elements are nothing but e to the power lambda e t time C. And then that essentially resulted in a model form of the system solution from which we are able to conclude that there is an exponential term here e to the power lambda, but lambda can be complex in general.

So there is a C there is a real part which will be like this, there is a complex part which will give me like that and this particular thing I can this is never neither it goes to infinity nor it goes to 0 this is sinusoidal fluctuating all the time, so the whether the system goes to infinity or not is all given by e to the power lambda t 1 that is all we are telling I mean this so the we are telling all if all poles are in the left half plane then the system is stable if one plane in the right half plane the system is unstable. Similar analysis holds good for even if there is no n linearly independent eigenvectors. We just carry out this almost the parallel analysis up to a Point from where we have to depart and then this is like J is, nothing but block diagonal matrix now.

And then each of the diagonal blocks are, nothing but Jordan blocks which has a specific form and then we carried out the similar algebra and then we tell here is the form that we that is very parallel to what we did before, but e to the power J t we have to analyze it carefully. So e to the power J t we root it that way and picked out one particular J out of that, and then it all the that is a case that J hat is r by r plus r by r Jordan block let us say, that I can decompose it the into this form because one I mean this lambda lambda lambda is there in the diagonal let me pick up pick it out separately, plus e t our e is in idempotent matrix of order r actually, so e to the power r will go to 0.

And subsequently, this the series truncates and e to the power A plus B is nothing but e to the power A into e to the power B. So e to the power A is this one and e to the power B is that one. So this particular thing has helped us in writing this form e to the power J hat t is e to the power lambda t into whole all that, and then we discussed if that is the case then X of t I can write it in this form actually. Where c 1, c 2 up to c p are nothing but a specific structured I mean these vectors and all c 1 can consist of that way where to the first element is, nothing but Eigenvector and the all the rest of the things are, nothing but generalized Eigenvectors actually.

So then I will be able to write it in that way and essentially it all gives us the idea that it consists of one component which is exponential. And the other component is all about polynomial so eventually exponential will dominant the polynomial series and hence, I have the similar result that tells me that if all poles are in the left hand side then I have got the stable system if one of them in the right hand side then I have got a unstable system. Then we just well how do you design a stabilizing controller and all that actually.

So probably with that I will stop it, but before I stop we also let us say discuss a little bit about some of the topic here, let us say last class we discussed let us say e to the power At there are various ways of evaluating it actually.

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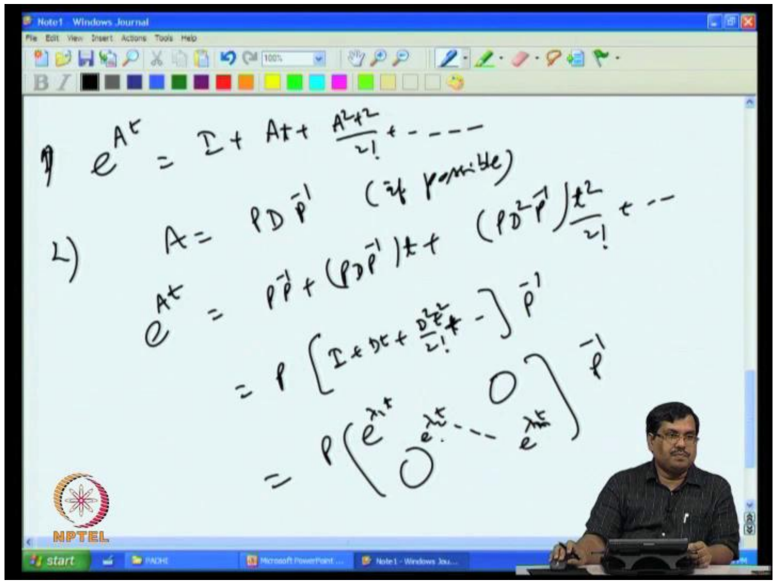

So first thing is e to the power A t the first one is, nothing but I plus A t plus A square t square by 2 factorial. So that is the series that is by definition actually.

So if your A matrix happens to be like idempotent at some point of time this the series truncates then essentially, it gives you the accurate result otherwise this is always an approximate result. So we normally do not want to do that we never know when it will terminate or unless otherwise it is an idempotent matrix of that particular structure we never know whether it is really idempotent or not actually I do not know standard way of verifying that at least to my knowledge actually.

So e to the power A t evaluating that in a series way first of all we do not we never know whether it is very accurate or not the second thing is it may be computationally intensive also. We do not know how long you have to keep on doing this A A square A cube all that are matrix multiplications actually.

So the second term what we do is second approach let us say A is reducible to this PDP inverse sort of thing so obviously this e to the power At if it is if possible. This is in general it is not possible, but D is a diagonal matrix actually so if it is n independent eigenvectors then it is possible actually. So e to the power A t is, nothing but e to the power well PDP inverse sort of things. So well I just write it that way so this is I, so that is nothing but well let me substitute that as AP times P inverse again we can the same analysis actually what we what we did there. So PDP inverse t plus P D square P inverse t square by 2 factorial like that actually. So I can again take out P here and P inverse there then I all that I have is something like I plus D plus D square by 2 factorial like that, and these are t actually obviously D square t square by 2 factorial like that actually.

So what I have here it is, nothing but P e to the power this diagonal e to the power, so e to the power lambda 1 t e to the power lambda 2 t all the way t to the power lambda n t in the diagonal times P inverse. So if I have this matrix which is diagonalizable then I can actually compute e to the power A t that way also where each of that are, nothing but scalar exponentials actually.

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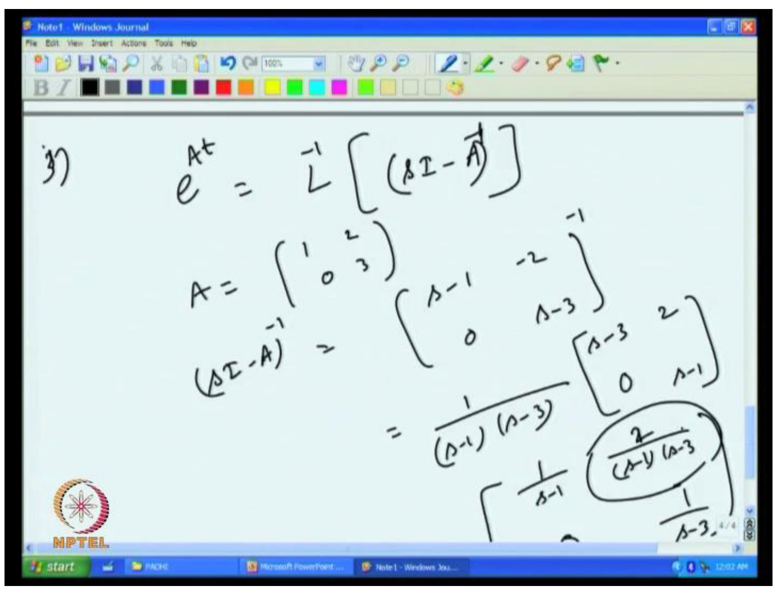

Now we will talk about let us say talk about another one the so what we discussed here is 2 methods e to the first is series the second one is that one the third one is e to the power A t is also Laplace inverse of s I minus A inverse. We have derived that previous class actually. So if you take simple example sort of thing let us say A equal to 1 I do not know two probably, and 1 3 and you want to have this e to the power A t remember this series may or may not truncate if you just directly put it there actually.

So what you need to do is you just s I minus A you calculate that way. That is s minus 1 and s minus 3 and minus 2 here, and then you talk about inverse of that so inverse of that and 2 by 2 inverse is rather simple so it is like determinant in the denominator. So s minus 1 into s minus 3 and then I write is plus 0 is there so it will go actually and then this is change of elements in the diagonal so that means s minus 3 here s minus 1 here and change of sign in the off-diagonal so that is the Eigen matrix actually so that is what it is.

So you are left out with these terms so that is like s minus 3 will cancel from here so I have I got s minus 1 here s minus 1 will cancel from here. So I have got s minus 3 here in the diagonal actually. If I multiply element by element then I'm left out with 1 by sorry 2 by s minus 1 into s minus 3.

So now I have to take inverse of these matrices actually I mean these elements. So I take e to the power At is, nothing but Laplace inverse of this matrix whatever matrix I have same matrix here. So I mean these elements are the first diagonal elements are simpler in this case, this particular thing you have to go for partial fraction actually decomposition and all that then we know I mean Laplace inverse and all how to take we know actually so that is another way of evaluating this I mean this matrix exponential e to the power At.

But this is provided you have fairly decent sized dimension matrix actually otherwise up to that is fairly Laplace s I minus A inverse we have also discussed an algorithm how to how to compute s i minus A inverse last class, but taking inverse of that element by element has to be done symbolically at least. So here we can use some symbolic software and thing like that actually if possible.

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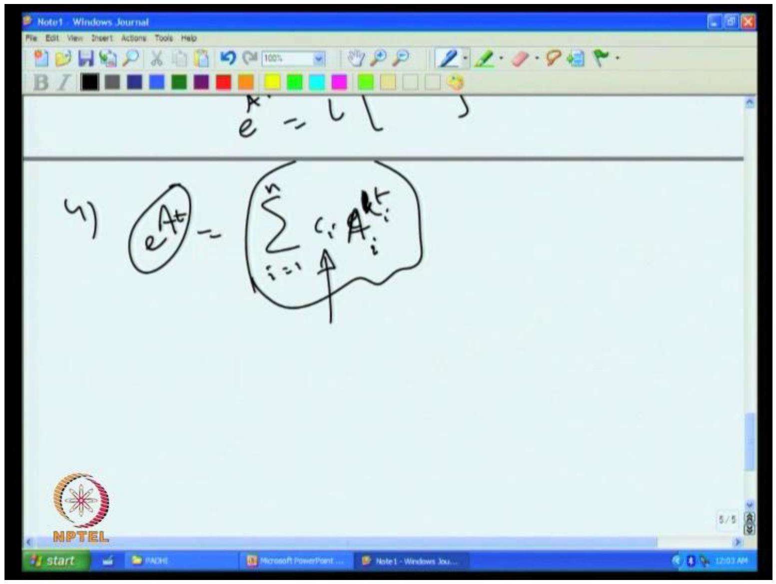

There is also a nice idea which tells us let us say fourth actually that I can actually write it e to the power At I can write it as some summation of some e some c i e to the power some I mean sorry a to the power i t something like that I will tell the exact thing next class probably. So in other words what it tells me actually not this not I there I here.

What it essentially tells me with some coefficient k i or something actually. So what it essentially tells me here is that this e to the power At I can always write it in a finite series provided I compute this coefficients in a good way the exact formulas and all I will probably discuss that next class actually.

So there are various ways of doing that e to the power A t which will also be necessary for analyzing controllability observability like that there we will study all those in the next class actually, thank you.