Advanced Control System Design
Prof. Radhakant Padhi
Department of Aerospace Engineering
Indian Institute of Science, Bangalore

Module No. # 08
Lecture No. # 18
Time Response of Linear Dynamical Systems

We will have we will study this time response of linear dynamical systems in state space form in this lecture actually. So, we are here in lecture number 18 now and previous lecture, we have derive this linearization in other words linear systems about operating point for non-linear systems we discuss that and this particular lecture, we will study this how do we get a time response or the system solutions actually, for the linear systems actually, but before that let me correct a small mistake that we observe actually, what I did in the previous lecture.

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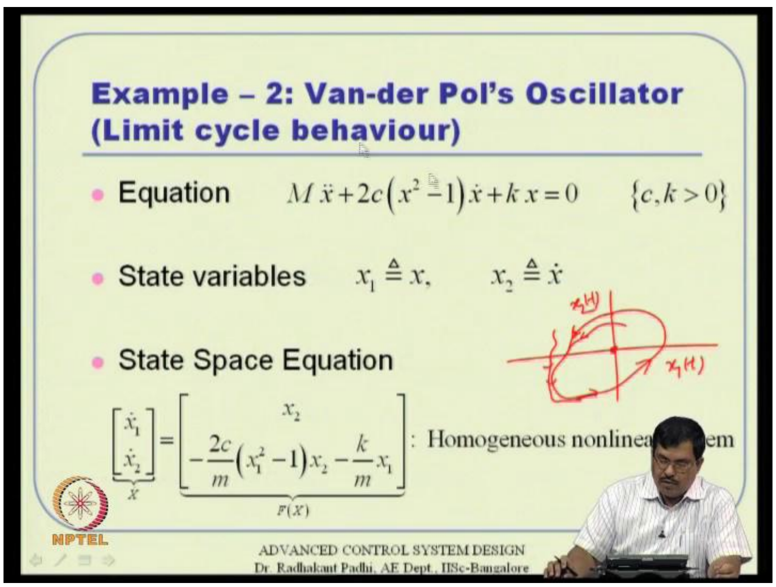

So, this example we discussed in the previous lecture where Van-der Pol's oscillator and this was the non-linear system and if you recall there was this particular term in other words this one this term was taken as 0.

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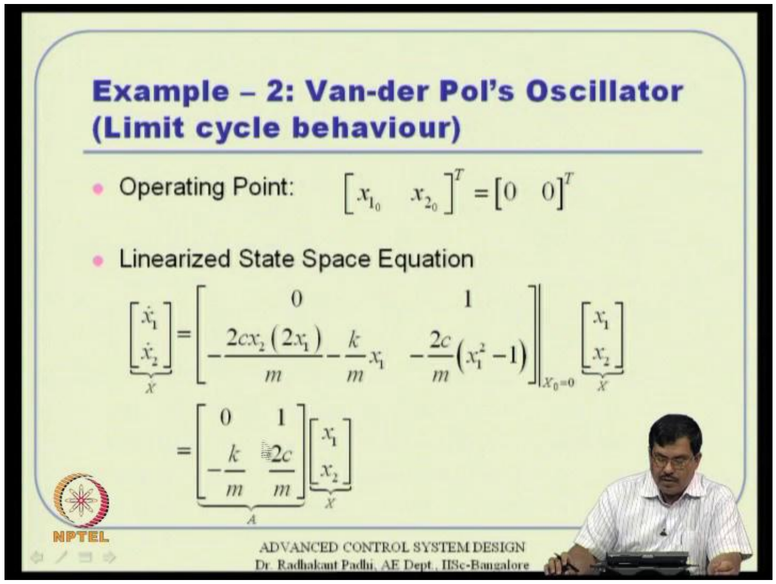

Actually, which is not true because once you have this equation I mean this second equation here and you take partial derivative with respect to x 1, you are having this term which is a constant term there is no not associated with x 1 anymore.

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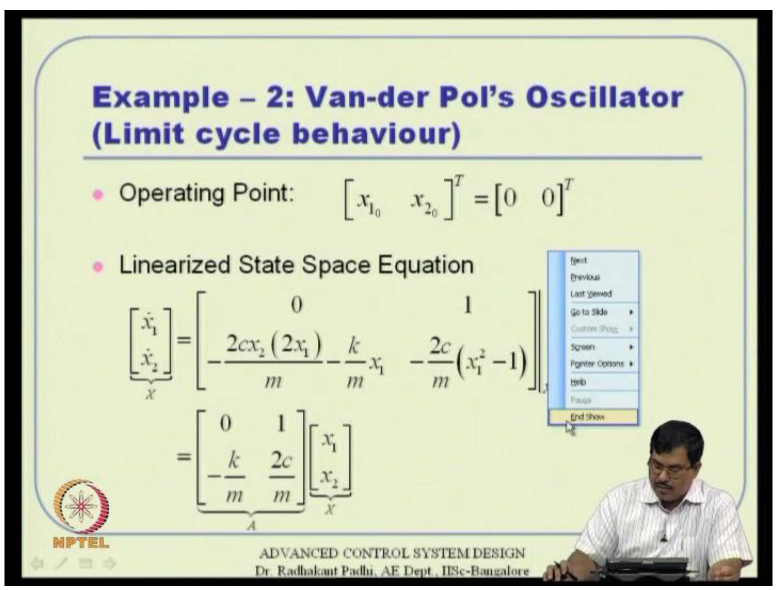

So, that is how this constant term will appear here and the a matrix will turn to be that way so it is not like 0 0 here what we discussed in the previous lecture actually. Sorry for the mistake, but this is the correction what you need to do.

Anyway lets continue with the time response of linear dynamical systems in state space form actually. So, what is I mean very quickly linear systems are very nice because of two major things one is the satisfy this principle of superposition, Otherwise by definition they not necessary linear systems so we are not worried about that.

So, then the uniqueness theorem is also very good because it tells us that there is only one solution for the linear systems that is why great uniqueness theorem actually, means this systems will never admit multiple solution. These are some of these great things why this linear systems theory is very handy to deal with it actually. One is the system stability nature is independent of initial condition that is another one and then this one it talks about the system can have only one solution actually.

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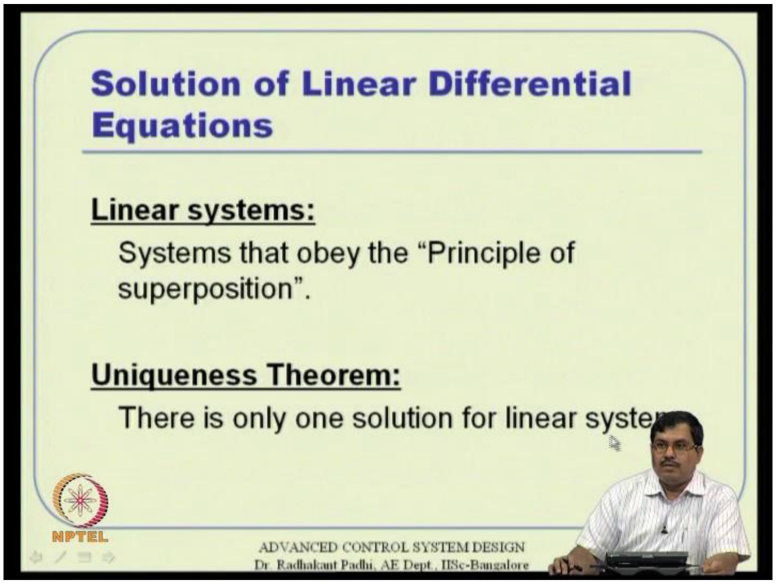

So, let us but we will not prove this I mean what you want to do here is we want to explore this result actually. So, let us start with a small scalar case again and then we see how to build up there actually. So, and this scalar homogeneous linear differential equation I mean in my view nothing can be simpler than this.

We have this x dot equal to ax with initial condition x of t 0 is x 0 so what the solution is x plus two calculus sort of thing plus two algebra you discussed that d x by d t is that. So, d x by x is nothing but, a times d t and then if you take integration both sides then d x by x is l n x and it has to be a times t plus some constant and this constant I am interpreting that is l n of c.

Then this l n I can take it to left hand side and tell l n x minus l n c. So,that is nothing but, l n of x y c is equal to a times c a times t where l n is natural logarithm with base e actually. So, once you have this equation known to us then x by c is nothing but e to the power a t x by c is e to the power A t and hence x equal to e to the power t into c.

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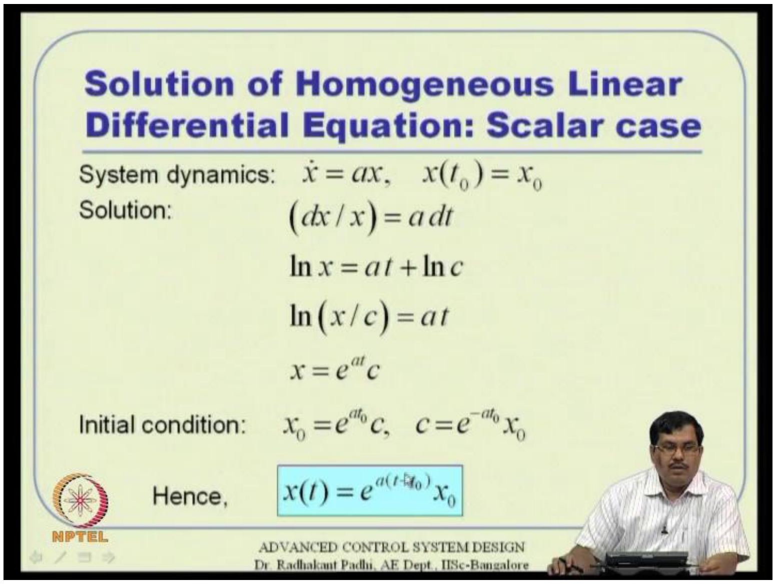

So the, but c is an arbitrary constant. So, we have to evaluate that so, initial condition is known to us so, we will put that h t equal to t 0 x equal to x 0. So, that is x 0 equal to this equation now so c is nothing, but the solution of that so that is if you take this one to the other side and c is nothing but that.

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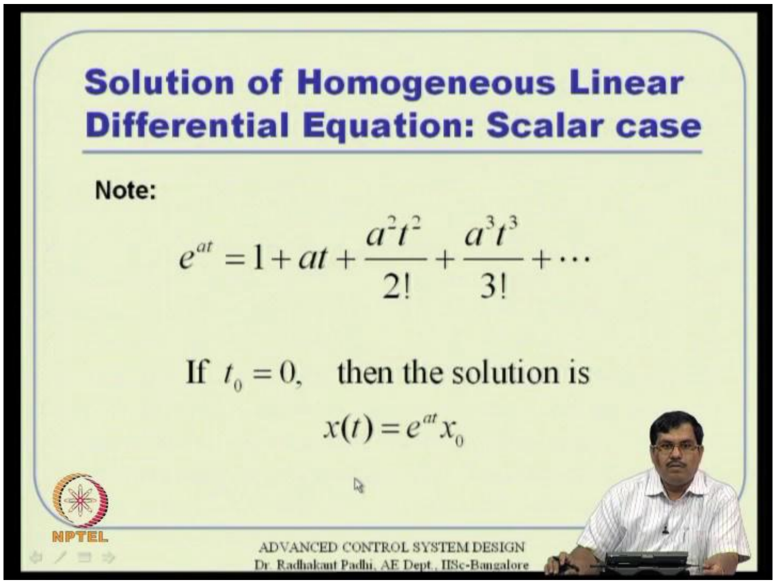

And hence, if you put them together c is c is this e to the power negative exponentially to the a times t 0. So I put it back here and try to combine what I get, I get x of t is nothing but e to the power a times t minus t 0 into x 0, and if some cases we assume this t 0 initial time is 0 then it is simply e to the power a t into x 0 with the assumption that t 0 is 0.

In general t 0 is not 0 then this is the solution actually. And what is e to the power a t e to the power a t if you expand it like an x power series and all this is the form actually, e to the power A t nothing but 1 plus a t plus a square t square by 2 factorial all that actually, and as I told you if t 0 happens to be 0 then the solution is e to the power a t into x 0 because that t 0 is 0 simply.

Now this is a very standard thing that we know already, that from cross algebra actually. Now what you are going to ask is a little bigger question and what if the equation is something like this x dot equal to AX, remember this BU term is not there.

And all these lectures will talk I mean this particular lecture is especially you are concerned about linear systems. So, it is the system dynamics is X dot equal to x plus BU, but the BU term is still not there and that we still talking about homogeneous system linear and what the their vector equations now actually. So, the amount number of equations put together actually. So, X dot equal to AX and then what you want to do we want to find some solution to that with the with the knowledge that we have X of t 0 equal to X 0 that initial condition all is we also have.

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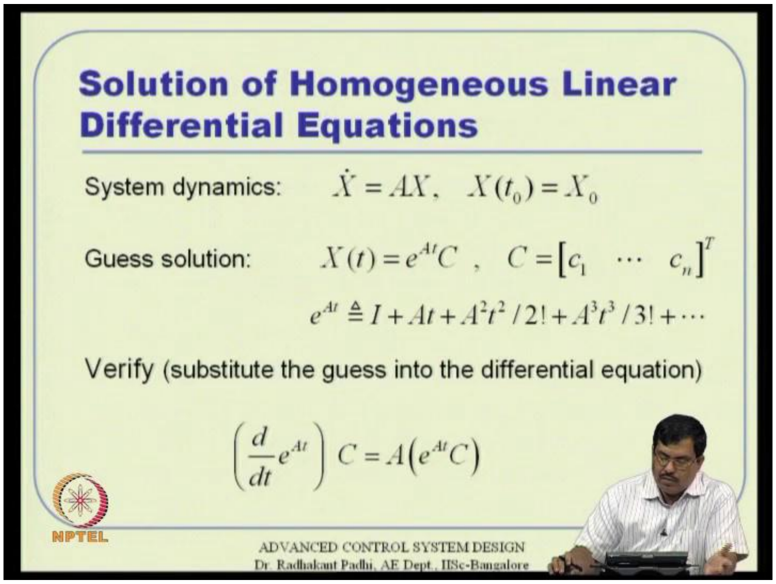

Now how do we do that actually, now let us try to find out some solution by getting inspired from this solution actually, whatever solution we have we want to get inspired from there and then tell can we really talk about a solution in that form actually. So, what I doing here we will we will try to formulate something like this remember the solution here is x equal to e to the power A t into c where e to the power A t is like this.

So, exactly similar way we want to define and tell X of t lets guess a solution that way X of t e to the power A t into C where C is a constant vector and e to the power A t lets define very parallel to what we know already a is a square matrix remember that so a square a q all that thing I define actually.

So, whatever we have here, we will just define a very a very parallel thing to that definition wise it is very much possible. So, we will just define e to the power A t that way and then tell lets guess a solution that way where C is a constant vector. Now why you are interesting? Why you have you do you have to guess a solution? Now this is where this uniqueness theorem will come into picture and tells only there is only one solution actually.

So, in other words you can write the solution I can write it as Cos of 90 minus something like that, but then the results are same anyway. So, that is the greatness of uniqueness theorem. So, what we are interested here is we will guess a solution and try to see whether that is a solution, if that if the if that happens to be a solution then that must be the solution because there is a uniqueness theorem the solution cannot be different actually. So, that is the motivation for doing all this actually. We want to guess a solution and let us try to see if it is a solution really.

If it is a solution then this should be the only solution basically that is that is the that is the logic actually what you are looking to. So, anyway so this is the firm that you are defining and how do you know that this is actually a solution on to so, you can simply verify by substituting as you already know the form of the solutions so whether it is a solution or not simply substitute it then. So, we will substitute to the left hand side remember c is a constant thing so it will come out of that thing differentiation.

So, we have a I mean this, but this is actually remember it is a e power A t is a matrix, so d by d t of a matrix that is what we are talking actually here, but by definition d by d t of a matrix is nothing but element by element differentiation such that that is some of the things we have just discussed while doing matrix theory division actually. We have done some 3 lectures here for the trivial matrix theory you can see some of those towards the end of those classes we discussed about how do you define these differentiation integration and some associated results actually.

So, this is actually matrix and the d by d t of that that means element by element differentiation that is what I mean here and then a times x is nothing but that a times that actually. So, that is why I am just substituting that. Now let us do further things the result is like this if the, so we want to do this actually d by d t of e to the power A t what is that I mean that is what we want to see.

Now e to the power A t by definition is that that is what we have started with so d by d t of e to the power A t is element by element all these actually, so this means if you have a matrix you do not know that we discussed before d by d t of a plus b is equal to d by d t of a plus d by d t of b that is a standard result which will come from simple definition and all that actually.

So, if you if you have this if you use that I mean expand this results and use that further then d by d t of all that is nothing but d by d t of that plus d by d t of that plus d by d t of that like that actually, and then we are left out with an option here that a matrix is a constant matrix actually.

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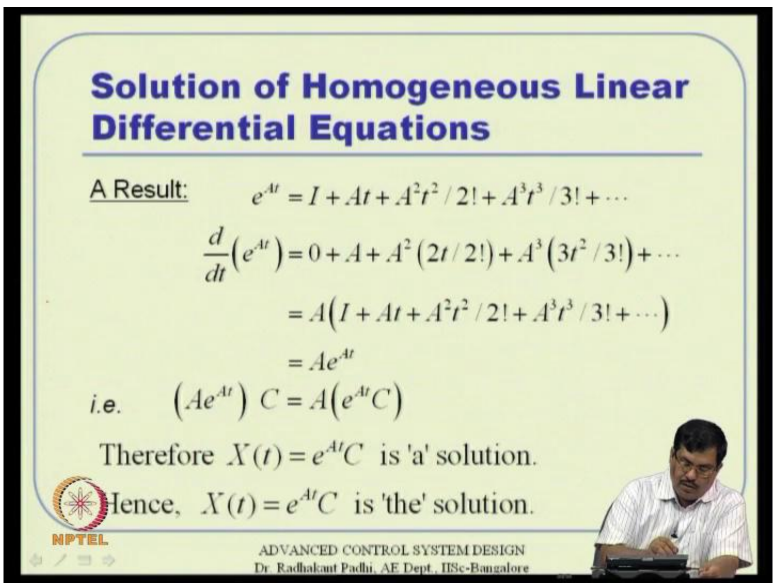

If a matrix is a constant matrix we do not have to talk about differentiation inside and all that is that is where this time varying linear systems in time I mean independent what is that linear time invariant systems are different actually. When you have linear time varying system this e to the power A t solution is no more valid by what you are talking here is linear time invariant system actually.

So, these are constant matrices so we can take out from this differentiation process and tell differentiation of t with respect to t is one so, that is what it is differentiation of t square by 2 factorial is that one and differentiation of t q by 3 factorial is that one and things like that actually.

Now remember this, an infinite series really. So, I have a luxury of taking in a common this is 0 anyways so, I will discard that, and then I will take a common which pre multiplies to everything and then I will be left out with matrix this series actually and fortunately this also happens to be e to the power A t.

So, what you are having here d by d t of e to the power A t is nothing but a times e to the power A t which is very close to what we already know in scalar system d by d t of e to the power A t is nothing but, a e to the power A t which is very close to what we already know for scalar thing that means if a is a scalar then that is what it is actually. So, this is in this definition it is also satisfies that way actually. So, d by d t of e to the power A t is nothing but a times e to the power A t that is what we want to show here.

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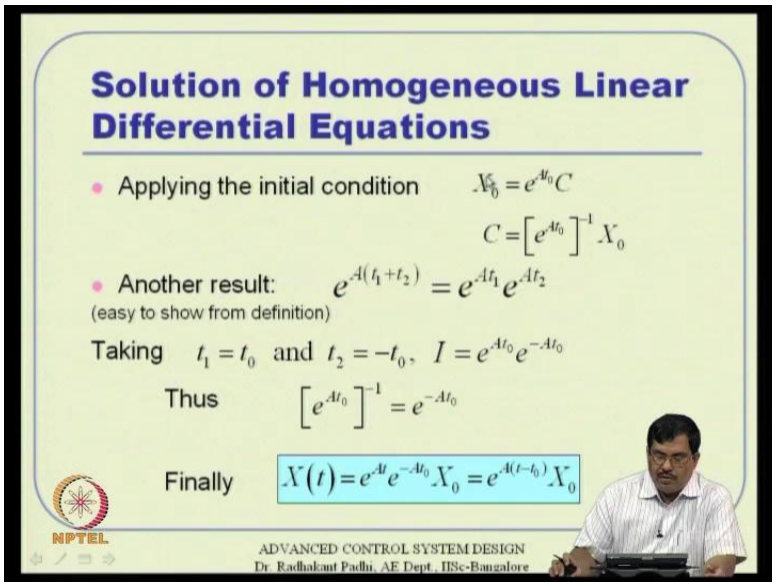

Now what you do here we got something here d by d t of e to the power A t so we will substitute that. So, that is nothing but, a times e to the power A t times C obviously c comes from here is equal to that actually, so that is that is what we write here. So, what is it now if you see this these are same the matrices the matrix multiplications do commute A; B; C is equal to whether you take A B times C or A times B C. So, that you see a times B into C or a times B into C so that.

So, in that sense there are associative law is valid in the multiplication sense for matrix theory. So, that is how it is true and hence what you what you have done here all that you have done is we verified that this guess solution what we started with is actually a solution and hence it is the solution actually.

Now this is not the not the end of the story because this is a form where the C is unknown and thing like that actually. So, we are to kind of do further analysis and find out this C then only we get it complete idea of the solution actually. So let us try to do some of this further processing of this.

So we want to apply the initial condition now. Remember this is a solution so what is the initial condition is that, so we want to substitute it. So x 0 equal to e to the power A t 0 times c, so that is what we are doing here, and hence c is nothing but this inverse actually e to the power A t 0 inverse of that.

Now there is a great theorem in the matrix theory which tells us that e to the power A times t 1 plus t 2 is nothing but e to the power A t 1 into e to the power A t 2. Which is not very difficult to show actually, you can you can show that using this rather easy e to the power A t 1 is we know that formulation A t 1 plus A t 1 square by 2 factorial like that, and if the power A t two is nothing but I plus A t 2 plus A t 2 square by 2 factorial like that actually.

So, if I take e to the power A t 1 plus t 2 that is by definition e to the power A t 1 into e to the power A t 2. So I have to simply multiply things like that. So, you have the you can multiply this two series this series and that series and then simplify this terms and then things like that and then you then you can easily show that that is true actually. E to the power A t 1 into e to the power A t 2 will result in e to the power t 1 plus t 2 terms actually.

So, you have this probably one term I can show you I plus A t 1 let say further things are there, this is I plus A t 2 further things are there so this will result in first one I into I so that is there the second one will result in A t 1 resulting from here. And then it will result in A t two also then it will result in different that what is that a square t 1 t 2 and things like that actually, it results in some of these things.

Now if you consider this first 2 terms this is nothing but A into t 1 plus t 2. So what you what you have here this is this is if you just consider of 2 linear term this is nothing but e to the power A t basically, e to the power A t 1 plus t 2 actually is not it, because this e to the power A t by definition is this 1 x 1 this is nothing but, I plus a into t 1 plus t 2 plus further things, so first order terms I shown you actually, second order terms and all that you can probably derive yourself actually.

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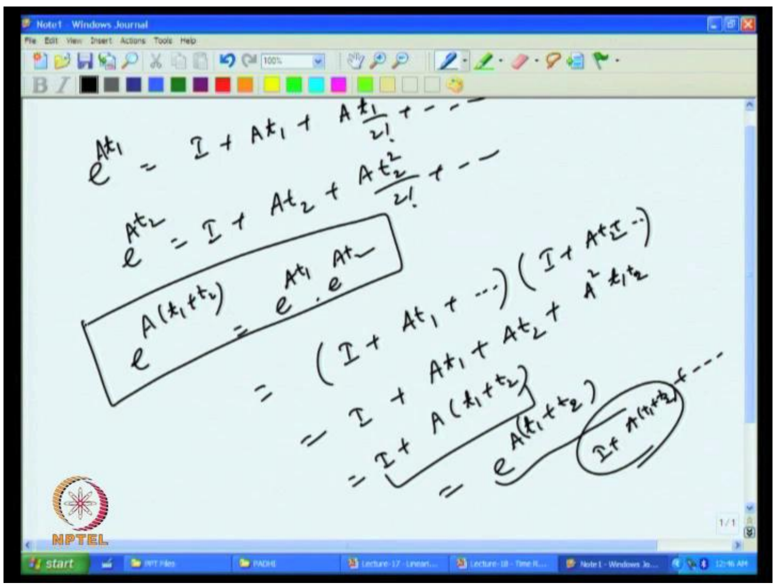

Essentially, just the definition and so basically that way, but also remember that this is true in general this particular thing what you see that is true in general however this also remember is a common sort of thing that suppose you have e to the power A t or let us say a plus b into t.

If you have that that is in general not true to e to the power A t into e to the power b t even though the left hand side is defined hand side is not equal to and this is equal to in general this is in general not true, but e to the a plus b into t is equal to e to power A t into e to the power b t provided a and b commute actually that means A B equal to B A you can show that that the condition that is necessary actually that A B if A B equal to B A then it happen.

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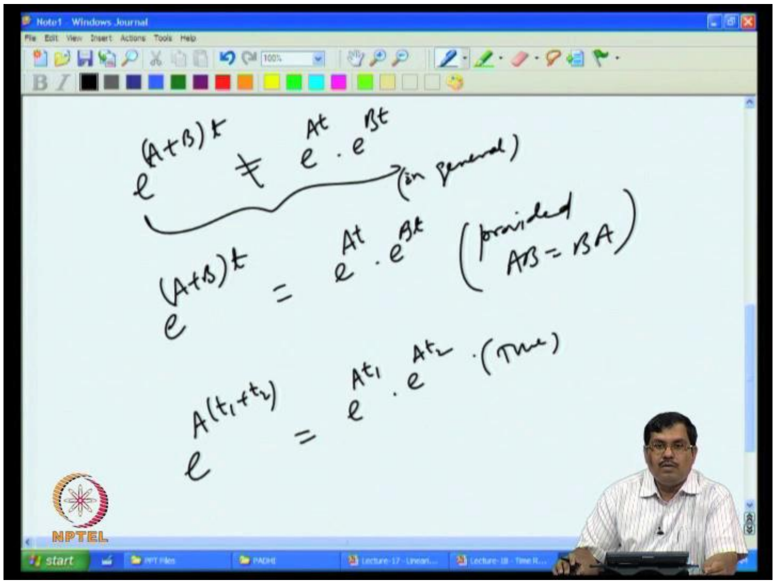

The other case what you discuss is something like this A into t 1 plus t 2 which is actually like e to the power A t 1 into e to the power A t 2 remember t 1 and t 2 are scalar actually, so t 1 t 2 is equal to t 2 t 1. So that is true sort of thing. So, that is true always true basically, this is first one is not necessarily true that, but is true only may be equal to B all right. So where are we just stopped like that so, this result is there, so let us try to exploit this result actually.

So, what you do we take a special case where you assume t 1 is equal to t 0 and t 2 is equal to minus t 0 we just assume that and then this result will show that this is e to the power 0 and e to the power 0 is identity that is matrix is nothing but e to the power A t 0 into e to the power a minus of t 0.

Why is that because this why you are doing because this an inverse matrix what is popping up here, we really do not know what we evaluate actually that is what we want to simplify here. So, using this result what you have here it also very clear that some matrix into some other matrix is identity obviously, this 2 matrices are inverse of each other. So that tells me that e to the power A t 0 inverse what we have here is nothing but e to the power minus A t 0 that is all

So, if I substitute this result now in the in the expression what I already have then x of t is equal to e to the power A t into e to the power what was the result by the way the solution is e to the power A t into C and C is nothing but this term so, e to the power A t into this inverse into X 0. So, e to the power A t into this inverse is nothing but that this e to the power minus A t 0 into X 0 and again I will excite this relationship what you what you already know I will again excite that to combine this actually.

Now e to the power A t into e to the power a minus A t 0 is nothing but e to the power a of t minus t 0. So, I am using this result one time to get evaluate this one and then once I have this results back in I mean put together then I excite this result one more time and combine them together, and tell that is what it is actually that is my solution.

So, finally what you are getting we are getting X of t is nothing but e to the power A of t minus t 0 into x 0 that is the solution actually. So, that that is the solution for homogeneous time invariant linear systems actually and if t 0 happens to be 0 then we know it is e to the power A t into x 0 that is the standard solution anyway.

All now let us go to the next one the next one is non homogeneous system non-linear differential equations I mean sorry linear differential equations time invariant linear system, but it is a non homogeneous systems that means B U term is there and that is our ultimate objective, but here also remember that this u of t we assume that it is known to us as a function of time.

That means somebody has given us a control vector as a function of time or as a tabulated value of numbers something like that. We are not designing a control for say here we are just interpreting the numbers that are available to us as simple time varying numbers actually. Whatever are given to us already actually and these are initial conditions are known to us.

So, we know that this non homogeneous differential equations will admit solution in 2 parts one is the homogeneous solution and the other one is particular solution, and homogeneous solution is the solution without this term that we just derived and then there is a particular solution.

The case here is the differential is the initial condition needs to be satisfied for the total solution, it need not be satisfied with respect to only the homogeneous solutions never actually. Whatever initial conditions are given to us anywhere in the differential equations they must satisfy with respect to the total solution actually so that you should keep it in mind.

So, what you are doing here is we are taking the homogeneous solution as it is lets dump this initial conditions formation here also, and the particular solution we take it this form where C of t will adjust so that the initial condition also gets satisfied for the total solution. We are taking the same initial condition here by the way, but the C of t is still there with us we want to adjust it later so, that the total solution will also satisfy the same initial condition actually that is the thing.

And this method of variation of coefficient of parameter what this called this X of p will assuming this form again and try to show try to derive what is this C of t that is what the idea there actually, and c is not a constant but, it is a time varying vector remember that say it is a coefficient vector we can say that way not a constant vector it is a coefficient vector.

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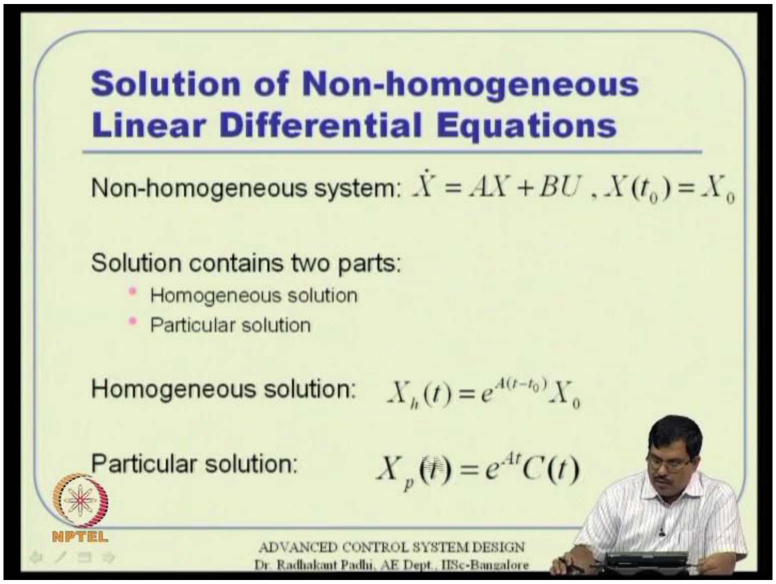

So, now we want to derive what is this X of X p of t, and then we will want to see whether this initial conditions is satisfied for the total equation or not actually, so why I doing here, so X p is a particular solution.

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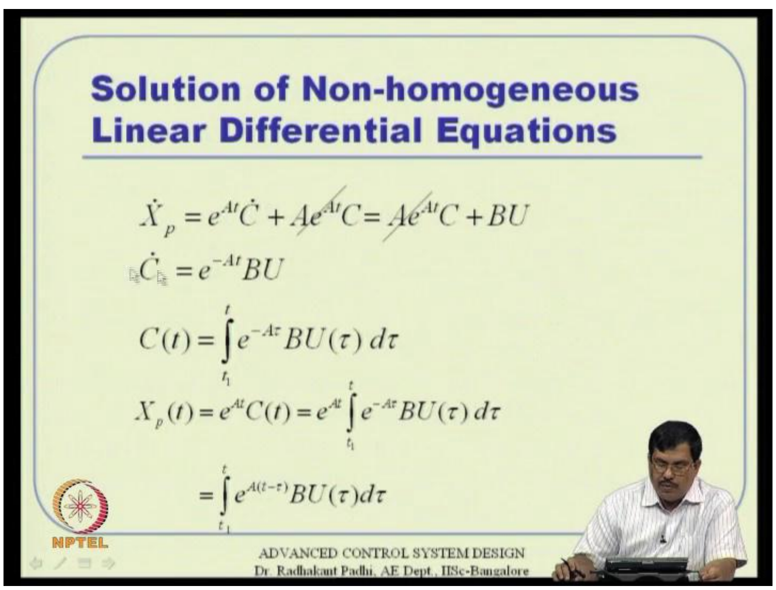

So, it must satisfy the differential equation also, we substitute it back in the differential equation and tell x dot equal to X plus B U let me substitute this term here.

So, if I substitute by this is X dot is nothing but d by d T of this term this entire term and both the terms are term varying remember that so, I cannot take one out of this differentiation process really. So, let us I mean just keep that in mind so substitute back here and try to see what you are getting there actually. So, d by d T of this particular term is nothing but d by d T of one term into that plus d by d plus this term into d by d T of that that is a standard result actually.

So, this result that we discuss in vector matrix theory d by d T of let us say a times b is nothing but d by d T of A into B plus A times d d B by d t you can say that way and the case here is we cannot alter the sequence of operation is very close to the scalar result, but you cannot alter the sequence of operation actually, all right so that is what it is.

So, we apply this here d by d T of this is nothing but, d by d T of that into that C t plus this vector this matrix e to the power A t into d by d T of C t. So that is what we do here and then we also know that that should be equal to x plus B U and this is X dot this left hand side this part is x dot that is also equal to A X plus B U and X is nothing but that actually that e to the power A t into C is nothing but X actually that is what our particular solution tells that so, we substitute that there and then tell very clearly we notice that these 2 terms will cancel out.

Once the terms I cancelled out we are left out with c I mean this term is equal to B U and hence C dot is equal to e to the power A t inverse which is e to the power minus A t that is that is the inverse term into B U all right. So, if this is the term what is the what is the good thing about that remember B is a time invariant matrix that is just a number U is a function of time that is given to us as a function of time actually, we are assuming that and then what you are assuming if the power A t or minus A t is actually a function of time. So, everything that is appearing in the hand side vector ultimately this A vector remember that is a function of time only.

So, here is a vector differentiation the d by d T of C vector is equal to some time varying vector which are all functions of time only the C term does not appear in the hand side. Once you have that that then C of t is nothing but integral of this value the d by d by d T of some vector is equal to some other vector and this is a clear which is only a function of time.

So, if we tell C of t C of t is nothing but integral value of what happens to the hand side actually, and that is how we write here and purposefully we introduce a variable t 1 so, that we need to adjust the t 1 to satisfy the initial condition for the total solution. So, it is integrated over t from t 1 to t where t 1 is a variable now and let us try to fix that actually.

So, what is X p with t 1 intact e to the power A t into C, but C is also given like that. So, the X p is nothing, but e to the power A t into that e to the power A t remember t is just an integral limit actually. So, it is not a variable of integration so, I can push this term inside and then combine this 2 e to the power A t I can pt it inside because as far as integration operation is concerned this t is actually not tau is different than t and this t is just a constant number sort of thing. So, I can put it inside and then I can combine these 2 terms and write that way.

So, X p is nothing but this integral t 1 to t with that term actually. So, now I put them together so I tell x t is nothing but X x plus X p homogeneous solution for plus particular solution homogeneous solution is that plus particular solution is that where t 1 is still of e variable remember that. Now we will put the initial condition which tells us that that t equal to t 0 my X is X 0 this is my solution for any point of time, but t equal to t 0 that is also my solutions so I will put that solution back X of t 0 is X 0.

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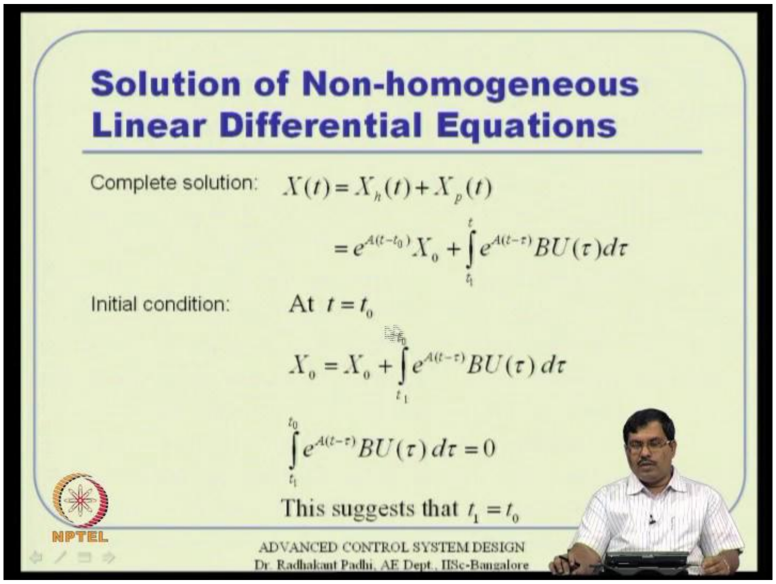

So, that is X 0 is equal to X 0 because remember t equal to t 0 means this is e to the power 0. So e to the power 0 is identity all e to the power 0 is certainly identity this e to the power 0 that means all 0 0 0 everywhere 0 this is only left out with identity anyway.

So, that is what I will substitute I mean I will put it back and tell that is my identity. So, that is I left out with only X 0 term here and then this integral actually. So, what I am getting here at t equal t 0 X 0 is equal to X 0 plus some value where t has become t 0 actually, because I am substituted t equal to t 0.

Now this integral must be 0, because X 0 equal to X 0 so this integral must be 0 no matter whatever is my B whatever is my U whatever it is and the way it should happen to be 0 is, when I take t 1 equal to t 0, so integrate this integration happens to be from t 0 to t 0 then only this integral will be 0 actually. Other than the trivial condition that 1 of the term is always 0 and thing like that we can opt for do not consider those trivial conditions. So we will tell in general what happens is this integral has to be 0 and that integral happens to be 0 only when t 1 is also equal to t 0 that is how we get a value for t one.

So once I get the solution value for t 1 everything is known to me X 0 I know everything is known to me inside actually, so the solution happens to be this form actually. Solution is e to the power A into t minus t 0 into X 0 plus this integral where the lower limit is t 0 to t now, I mean the limits of integration t 0 to t with that term actually.

So, also remember the integral term in the 4 solution is like a convolution integral actually it is actually a convolution integral. E to the power t minus tau into beta and all that actually so that is just a, that means that is some impulse response characteristics and things like that you can visualize that way actually.

But the point to note here is it is like a integral I mean it is like a convolution integral actually. Anyway so we will also see an example how to make use of these equations and thing like that actually. So, we will also make it clear about this solution form.

Now also before you proceed further also remember that most of the time we also design the U as a feedback form that means U equal to minus K; X; U is not really a function of time the moment it becomes a function of time it is a open loop solution really, because you are ignoring everything else you just apply the control with some pre determined function of time actually, so that is how the open loop control is defined.

So, we do not do that most of the time we do state feedback form which is the nicest form ever actually and sometimes you do output feedback also, but we can in general visualize that as state feedback form, and then if it is given to s in a feedback form like that then the life is also becoming 0, because we could say X dot equal to AX plus b. So, B U you replace that as minus K X so, what you get X dot equal to A minus B K into X. So, that essentially if you define a close loop matrix as A minus B K then is essentially very I mean equivalent to a homogeneous system.

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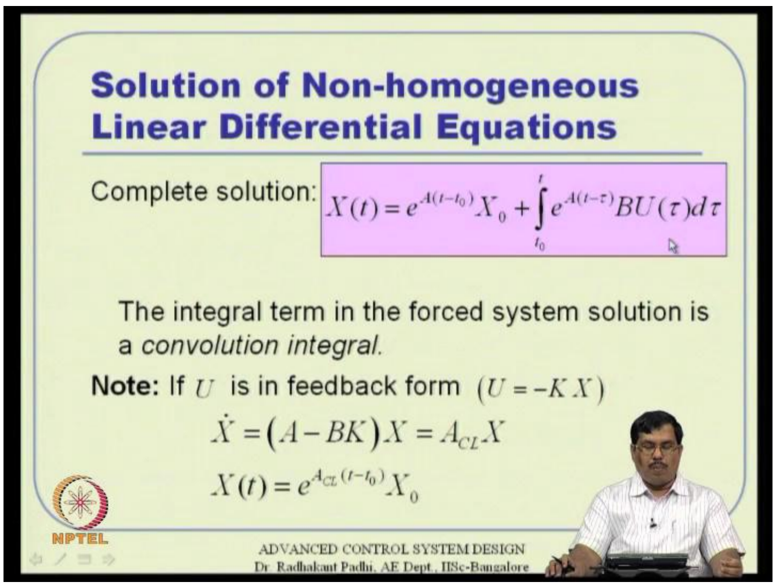

So, this is the close loop system dynamics happens to be a homogeneous system. So, homogeneous system solution you already know basically. So, X of t in that case will be something like this e to power of A minus B K whole into t minus t 0 into X 0 that is how it is actually.

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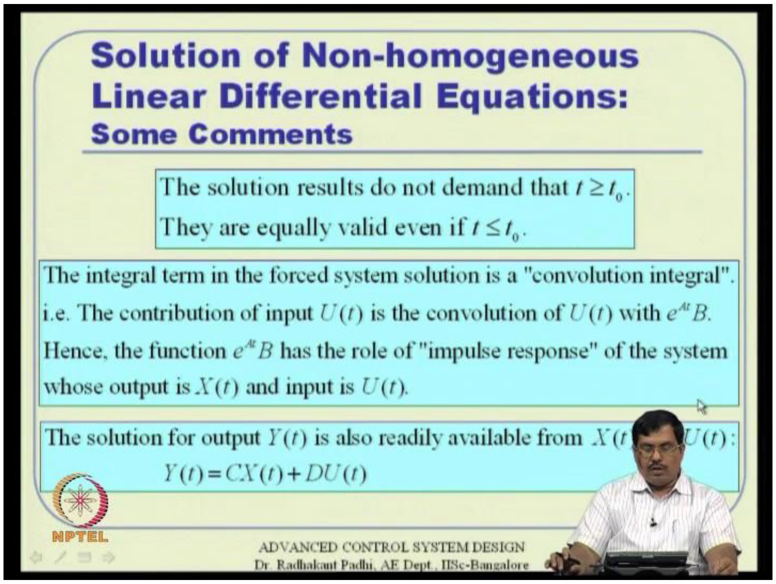

So, this form is essentially irrelevant only when the control is given to us as a function of time nothing else actually it is not given to us in a state feedback form actually that way.

So ,some of the comments before you proceed further, what you derived just now first of all, nowhere we have mentioned that t has to be greater than equal to t 0 t can be less than equal to t 0 also. So that mean the solution is equally valid forward in time as well as backward in time that is what it is and as I told you the integral term in the forced system solution is a convolution integral. So that means contribution of input U t is the convolution of U t with respect to this e to the power A t times B, so what does that.

So, what does that mean this e to the power A t times B has the role of an impulse response of the system whose output is X t and input is U t that is just a comment actually you can see that that way and for output we really do not want to do a 1 more time because you already know X of t now and U of t is given to us actually. So Y of t is nothing but C of X t plus C into X t plus D into U t that really we do not have to do it again actually. So, these are all the small comments that you can probably remember before you go further.

So, here read the example, let us try to use it whatever knowledge we have for a simple problem the simple one dimensional motion of car with a force f t. So, you see simply use this Newton's law it tells us m x double dot is equal to f of t without any friction there is friction no drag nothing actually, just tells us m x double dot is equal to f x t f of t so, x double dot is 1 by m into f of t obviously, and you also assume that mass of the vehicle is constant here that means what are you telling here.

So, if you x double dot is this equation, so first thing to start with is to put there in the state space form first thing to start with so we define v as x dot and then tell that is my if I this is my system dynamics I will write that X dot is nothing, but v and this is 0 1 and then 0 and v dot is nothing but this fellow so, that means this is 0 0 this is 1 by m here. So, this is your a matrix and this is your b matrix actually. And also we know we also assume that the initial condition is known to us x 0 is nothing but x 0 v 0 this is known to us actually. So, we want to find a system a system solution for this with the assumption that f of t is known to us actually, that that is the problem starting from this initial condition of course.

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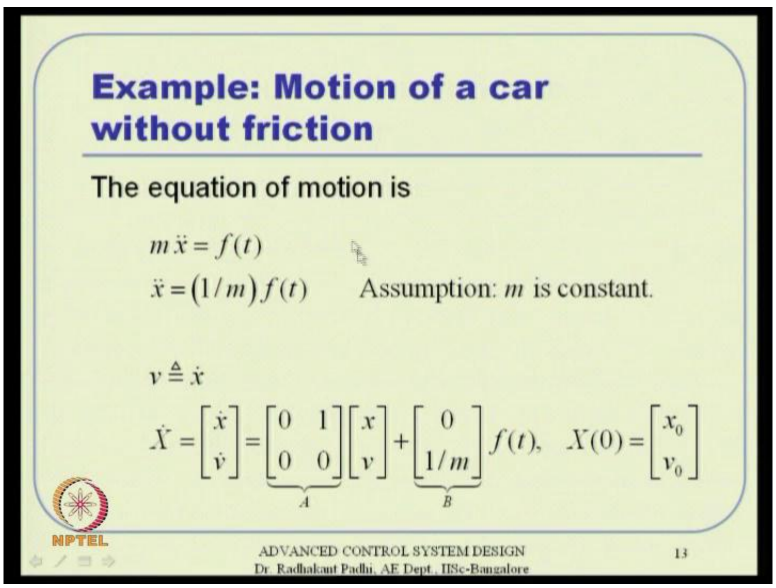

So, let us let us see that so this is the solution that you already know remember this t 0 we are assuming it is 0 basically. So, t 0 is 0. So let me write it properly t 0 is initial time 0 initial states are given like that this is known and mass is assumed to be constant there is no friction this equation tells you that there is no friction that essentially involves no drag actually. No friction no drag mass is constant t 0 is 0 and initial conditions are known to us several assumption actually

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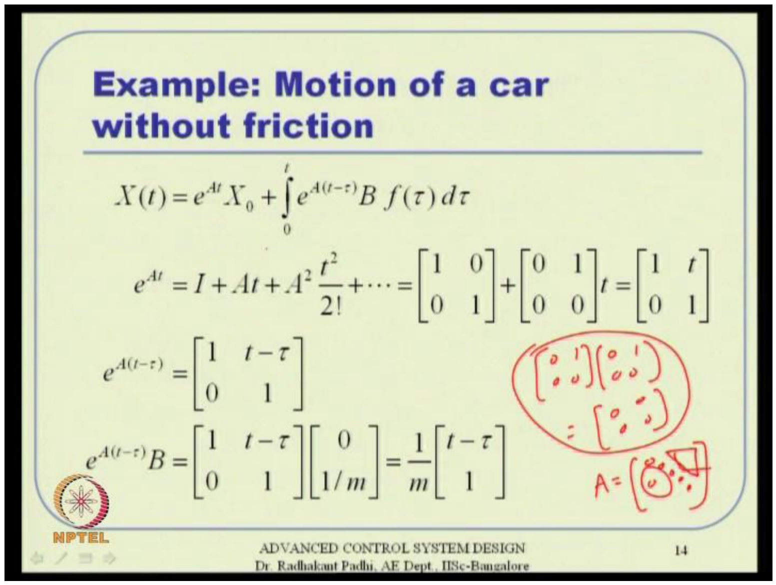

Just a one dimensional problem also this is not coupled with any other direction this is a simpler process simpler problem which results in 2 state system dynamics because the Newton's law is all related to double dots of the position that is that is the reason actually now X of t is like this, so that is what we know actually we just derived that where e to the power A t is that and in this particular case it so happens that this is a ray matrix and then if I substitute that then s square onwards it is all 0 actually.

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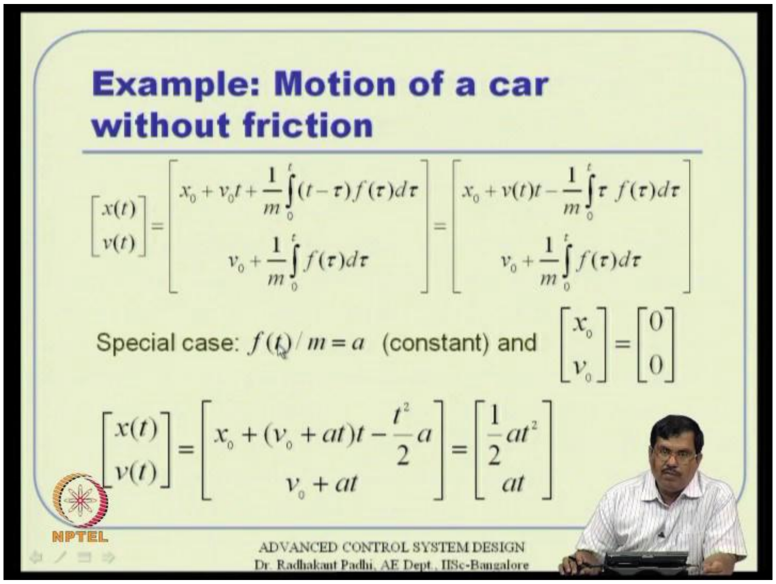

So, it results in I which is identity then a times t plus s square will be 0 if you see of you can do it yourself properly 0 1 0 0 you multiply that with 0 1 0 0 multiply again that with 0 1 0 0, if you see that that is actually nothing but, 0 because 0 time 0 plus 1 time 0 plus again the second one is 0 times I mean this 0 essentially what I mean is 0 0 actually, if you multiply this 0 times 1 and 0 is 1 time 0 0 second one then the third and fourth and all that actually.

And in general this the characteristics accurate just to tie up with matrix there a little bit this is this is something called idempotent matrix actually, that means at some power some finite power of a it is 0 and hence the further powers of a also 0 these are called idempotent matrix and one the thing that satisfies these kind of matrix through the properties all are 0 including diagonal elements being also 0 but off diagonal can that also can be something actually.

Lower diagonals are all 0, I mean this actually a triangular matrix with diagonal elements also being 0. If it happens to be like that then it is a special class of an important matrix somewhere it will happen to be 0 actually. So, you can also see that this a 2 by 1 I mean 2 by 2 matrix where only one was there so this matrix what you are talking here, you have only one here 0 0 here 0 here. So, it is idempotent matrix so it turns out be just this actually.

The beauty of this is the series to expansion is exact now series of expansion is not an approximation anymore we put it here that I 1 t here and then tell it is the power e minus tau what is that e to the power e times e minus tau what to what to do? So, wherever t appears here we just substitute that by t minus tau, so t appears here so I substitute by t minus tau here that is all actually so e to the power A t minus tau is known to us and then the next thing is to multiply that with b vector actually. So this matrix multiplied by that is actually one by that into m actually so that is this matrix 2 by one.

Sorry 2 by 2 multiplied by 2 by 1 is nothing but a 2 by 1 matrix that is known to us actually. So we know all these bits and pieces now because this is the this is the solution that you wanted to do and then B matrix was known to us if the power a minus t we just found out if the power a minus t if the power a times t minus tau into b also we have found out. So we now substitute all that and then see that what it results in actually.

So, it will so happen I think I mean it is very straight forward exercise here onwards. So, it will so happen that x of t and v of t will be given that way actually you just substitute it like that X 0 X 0 is nothing but x 0 and v 0 these are initial conditions. So, you substitute it there substitute it there and then try to simplify it will result in that actually. So, special case if you take that let us say that f of t divided by m is a constant in general that means if you have a car motion probably, is not a rocket motion then m is a constant quantity fairly and then f of t by m is also a constant quantity provided f is a constant quantity that way.

So, if that happens to be constant and initial position and velocities are also 0 0 then I i can really simplify this to that way actually this entire expression what I have here remember x 0 and v 0 are 0 0, So, the all the thing will be are 0 0 here this is A t square minus t square by 2 so it will result in half A t square and v t is nothing but, A t so this is also very compatible with what we know in plus two physics actually.

That means u equal to A t and X equal to half A t square probably that is the thing that you studied may be before and the so that is actually compatible with several assumptions like that, that means all in addition to this assumptions what are you telling you no friction no drugs t 0 is 0 initial conditions are known to us and on top of that if the is a constant quantity and position and velocity initial conditions are both 0 0 not just a number what is 0 0. Then these results are like that actually and the several assumptions actually.

So, what is the utility of the state space system first of all this is a verification and the utilities lot more than this it tells any system dynamics that is of any dimension need not be only 2 dimension and things like that need not have several assumptions like that you can still get a solution actually. So, this solutions what we really know from our early analysis is no more valid whereas, this solution what you are talking here is very much valid.

You can have a time varying force you can have let us say t 0 naught equal to 0 and things like that you can still get a solution using this formula actually then f of t can be arbitrary also need not be a constant quantity f of t in general can be as some function of time function of time whatever it is you can still get a solution and know the position and velocity of the vehicle actually that is that the beauty actually. Now before we conclude this lecture we will see e to the A t evaluation. Remember this liner systems theory you can never escape this reality that e to the power A t comes back to us repeated.

As long as we talk about linear time invariant system e to the power A t is a universal reality actually. So, people have paid a lot of attention to get the to get the quick way of evaluating it the power A t or a or a good way of with let us say I mean less approximation errors so evaluating e to the power A t like that actually. Lot of attentions are being focused like that, so let us see one attention 1 one result that is very useful later actually. So, what is that I mean if you have the solution if you if have a system equation like this X dot equal to AX where X of 0 is actually X 0 .

Then the solution using Laplace transform is like this you use Laplace transform here left hand side s into X of s minus X 0 obviously. Now that is the Laplace transform result is nothing but A times X of s actually. So then you talk about what I can simplify this, so we take X 0 to the and side and this one to the left hand side and tell s I minus a now into X of s is X 0 so, X of s is nothing but s I minus a inverse times X 0 this is my Laplace's transform result.

Then from here if I take inverse Laplace transform then X of t is like that actually. However, the this problem this homogeneous time variant system admits the solution we know that, that is what we just write actually. So, if you equate this 2 it is very obvious that if the is nothing but, this expression Laplace inverse of s I minus a inverse so, you first you have to do s I minus a then you talk about s I minus a inverse that is a symbolic representation of inverse matrix and then you talk element by element Laplace inverse actually then you will get e to the power A t basically.

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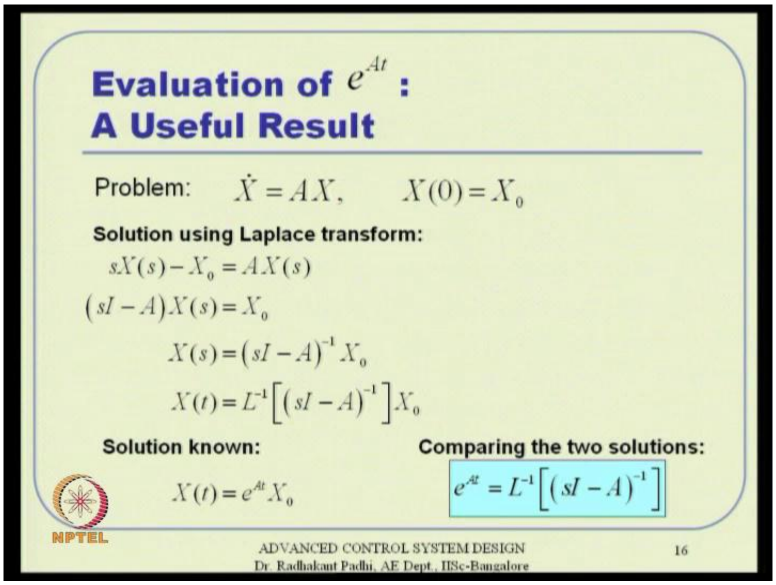

So, how do you get that now in a 2 by 2 problem it may be simple 3 by 3 probably you can do, but there is if the dimension of the matrix increases you cannot do actually. So then people have thought about how do you do that in a good closed form solution way actually how do you come up with in a very efficient manner how do I mean is there any numerical algorithm that can be revised actually, so let us see that quickly.

So, A is something like this is n by n matrix. So, these are elements all over so, s I minus a is obviously that this just an algebra a matrix is that so s I minus is that and s I minus a inverse is nothing but adjoint of this matrix divided by determinant of that matrix, that is from well a very straight forward result actually. Now we need to find out the adjoint of this matrix and the determinant of that in a very efficient manner that is the problem actually. That is all we are talking we are talking about evaluating this s I minus a inverse in a efficient manner can we do that.

Now s I minus a remember s I minus a determinant is just a scalar quantity determinant is always a scalar quantity and s I minus I is that one so, 1 by n matrix remember one time is multiplication of these diagonal elements first term is a multiplication all there, so that will result in n-th order polynomial. So, s I minus a in determinant is nothing but an n-th order polynomial let us write that n-th order polynomial as like this.

So, s to the power n plus some coefficient raise to the power n minus one like that actually, but still remember this is scalar value ultimately. Now adjoint of s I minus a is a matrix, but adjoint means I let us we are talking about adjoint of I mean this matrix adjoint matrix is nothing but co factors of the elements errors in different places obviously, so if I take the co factor of this element that means I forget this and forget that and then talk about that one somewhere.

So, that means this fellow is nothing but n minus 1 third of polynomial because one term is given already. However this is actually a matrix so, the coefficients will be matrix, but the polynomial of each of the elements of that matrix will contain at the maximum n minus 1 third of polynomial. So, you have n minus 1 third of polynomial out here if the power n minus 1 here, so E 1 is nothing but a matrix here E 2 is a matrix.

But the polynomial coefficients are n minus 1 n minus 2 like that actually last element is just E n actually now we have to find out a logic for which we can quickly compute this 1 a up to a n and E 1 E 2 up to E n actually, that is our problem if you do that you are done actually, because we know that actually. Now how do you do that let us do that let us multiply this equation this one is equal to that, so we will multiply that and that is equal to that so that is what you are doing here.

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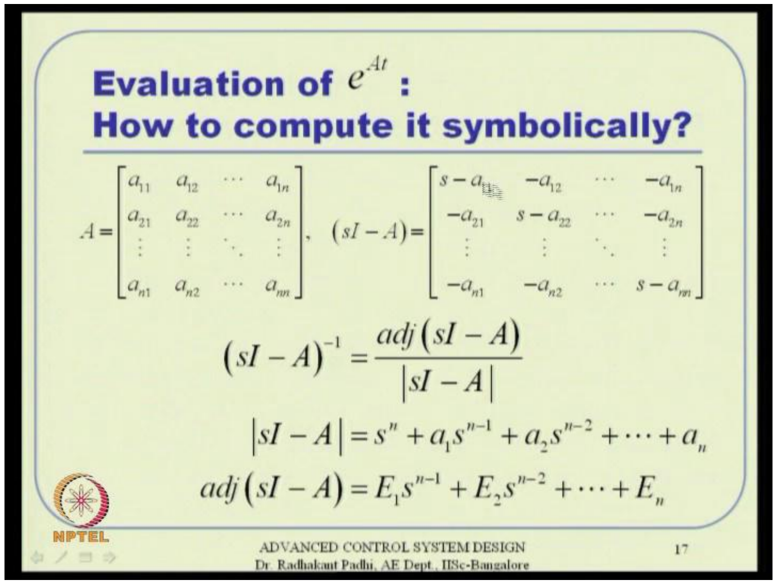

You simply multiply that actually we substitute here that means this adjoint of this we substitute back here determinant we substitute back here and then this s I minus a will multiply both sides actually, that is all we are doing actually here s I minus a into s I minus a in inverse is nothing but s I minus a into adjoint of this is nothing, but that by determinant of that that is what nothing but that that is all you are doing here.

Now this is nothing but identity this matrix is identity obviously. So, I times this one all that is nothing but, this one numerator whatever numerator you have this is identity times this polynomial is equal to what you have in the numerator. Now what I have I can I can expand this polynomial in the left hand side and hence I will get say s to the power n into I plus a 1 times s to the power n minus 1 into I things like that in the left hand side.

And the hand side is all given like this actually, you can simplify that first term is s to the power n second term will be s to the power n minus 1 and I just take common and I will left out with that actually so, what is the beauty of that I have a polynomial of n-th order these are all matrices now with I remember that I is multiplying everywhere so that is all matrices. So I have a polynomial matrix in the left hand side I have a polynomial matrix in the hand side so I can equate the coefficients actually.

So if I equate the coefficients what I am what I am getting e 1 is e 1 is nothing but, identity and this term E 2 minus E 1 is nothing but a 1 I and things like that so E 1 is I e 2 minus E 1 is 1 I and things like that actually so what is there suppose E 1 is identity so E 1 is very clear E 1 is I simply start with I now if I know this a 1 a 2 up to a n all these then E 2 also I know E 2 is nothing but, a E 1 plus a 1 I E 3 is nothing, but a E 2 plus a 2 I things like that.

So, the question is can I know this a 1 I 2 values actually a 1 a 2 a 3 up to a n values and fortunately it so happened that a I can be computed that way this part I will not show actually this is there in this book you can see that actually. So, a I can be computed that way that this suggest a recursive algorithm really that means E 1 I will start with I then I will quickly compute a 1.

A 1 is given like that 1 one into trace of a times E 1 so that is a 1 I will be knowing actually once I know a 1 I can compute E 2 once I compute E 2 I can compute a two once I compute a 2 I can compute E 3 like that actually. So, there is a recursive algorithm which can which you can make I mean make use of actually. Ultimately remember that I am I am done here that means E m is equal to that I am done with.

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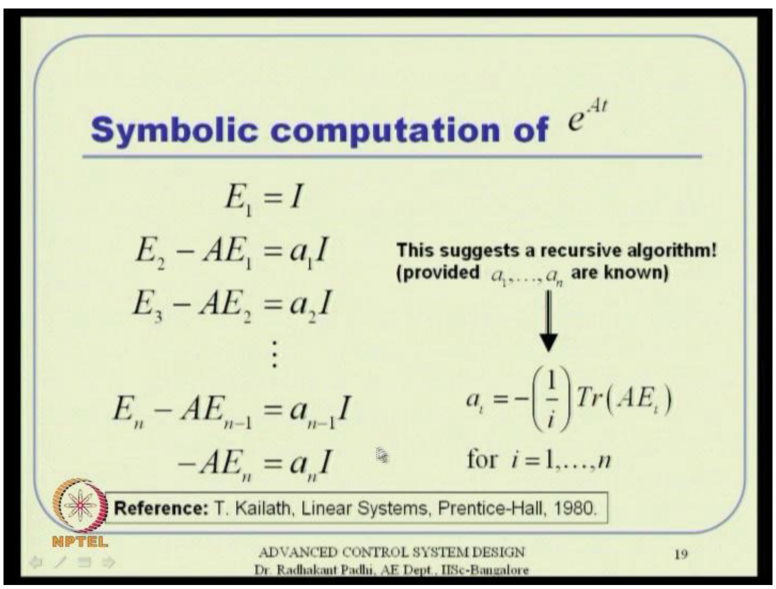

So, the last equation what I have is simply for verification purpose that is all E 1 to E n that is all we require that is all we require if you see that all that we require is E 1 to E n and we are done here last, but 1 equation. So, the last equation is there only for verification purpose actually. So, the ultimate I mean this is just a method of evaluation of operating now let us discuss a little bit about time varying system because that is what you have not discussed before.

And in general we know solution of the linear systems can be time varying as well actually. So, let us quickly discuss about this homogeneous linear systems that means B U part is still not there, but a of t is now a function of time. Now if that happens the solution happens to be in this form where this phi of t tau is nothing but a what you call as state transition matrix actually so, it takes 1 state in a 1 particular point of time and it transits it to a different point of time.

It takes a state at a let us say time tau and it gives a value of state at time t so that is a state transition matrix. It transits this state to this state actually. Now there are various properties that are available for state transition matrix. Remember the state transition matrix has no close form solution in general only if it is time varying I mean time varying thing goes to time invariant thing that means A of t is really a constant matrix then this phi of theta is nothing but, like a exponential t minus tau and things like that.

So, that is so that is only special case otherwise in general this state transition matrix is do not have closed form solution. However that satisfies these nice properties from which you can actually derive a neat a numerical solution very quickly. Now the first equation is nothing, but this 1 which is a very similar to what the state equation is X dot is d x by d t is a time A of t times X so del phi by del t is nothing but, A t times phi times phi.

But this differential equation is a vector differential equation and this differential equation is a matrix differential equation, phi is a matrix remembers that. So, to obtain a vector solution we have to actually get a matrix solution first and that needs to be done in a numerical way in general, but for numerical solution also you need a initial condition I mean what either numerical or closed form whatever it is you still need a numerical I mean initial condition.

And initial condition is fortunately given like that because if you freeze any t not necessarily t 0 t 0 any t then X of t is equal to X of t that means this matrix is nothing but identity. So it is also valid for t 0 basically, so if you have any time substitute phi of t 0 t 0 is identity any way. So, this equation coupled with this equation gives us a complete information, so this is difference matrix differential equation and this is initial condition with that we can integrate that equation numerically because A of t is not a constant number that is the problem actually.

And if you can integrate it in a closed form well and good because A of t if it is time varying in practical system they are not given as nice functions of time not like linear function quadratic function sinusoidal function not like that they will be typically given to us in the form of numbers actually. That is the problem otherwise you can think of numeric I mean closed form integration also actually that way anyway.

For now there is another property which tells us that for any 3 different points of time t 1 t 2 t 3 t t 3 1 phi of t 3 t 1 is also like phi's of t 3 t 2 and t 2 t 1, so this is this is interesting suppose you have this t t axis then you have t 1 here t 2 here and t 3 here suppose you want to know this x of t 3 probably, and then corresponding you can jump from here to here and then there to there this is a similar to what going there actually directly.

So, this is this we will talk about again this results can be easily derived by simply plugging the definition actually. I i suggest that you do some of these derivations yourself the first the second one I have explained the first and last I suggest that you do yourself actually that means it is it is very easy I mean if I talk about x 3 for example, and I try to transit from x 1 to x 3 then the resulting matrix is that so x 3 is this matrix into x 1 and x 2 is this matrix into x 1 if I if I get that and x 3 is like that actually.

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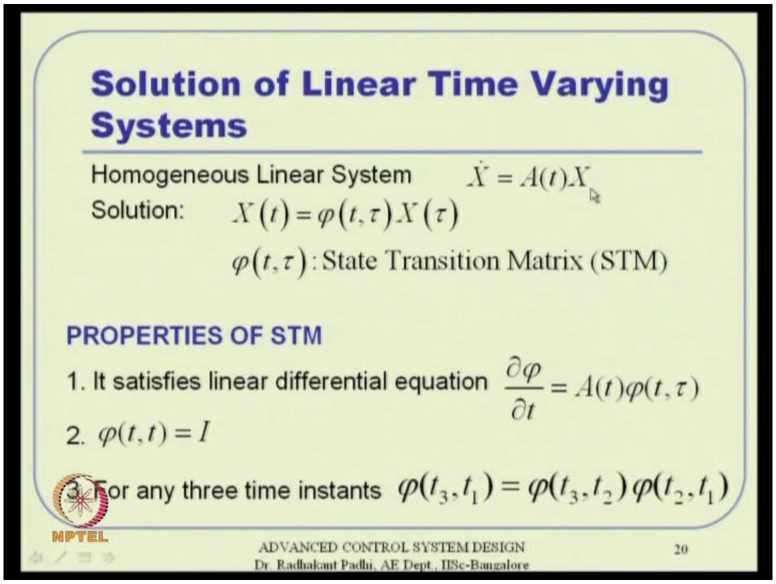

So, just substitute the definition and then try to simplify you will get it very quickly actually. So these results are very standard for state transition matrix. Now this is also another thing that if you want to take inverse of this matrix then you just have to alter the coefficients actually. I mean this tau t is nothing but t tau inverse and therefore, time invariant system these are very neat you do not have to really talk about 2 different variables altogether these 2 variables can be clubbed together in the sense if phi t and phi tau were there is nothing but phi of t plus tau and then phi inverse is also phi of minus t actually.

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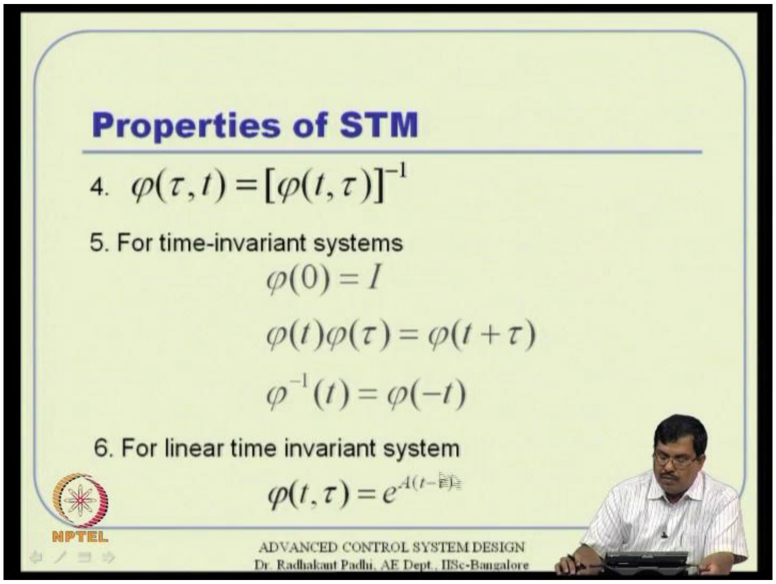

So, for linear term invariant system as I told you before phi of theta is nothing but E to the power A t minus tau, which is very compatible to what results you have already got for l t h systems. Now how to make use of them for in the time varying systems that is what you have to that is what you have studied, I mean the whole motivation is to make use of that for time varying system, so you start with this guess solution again.

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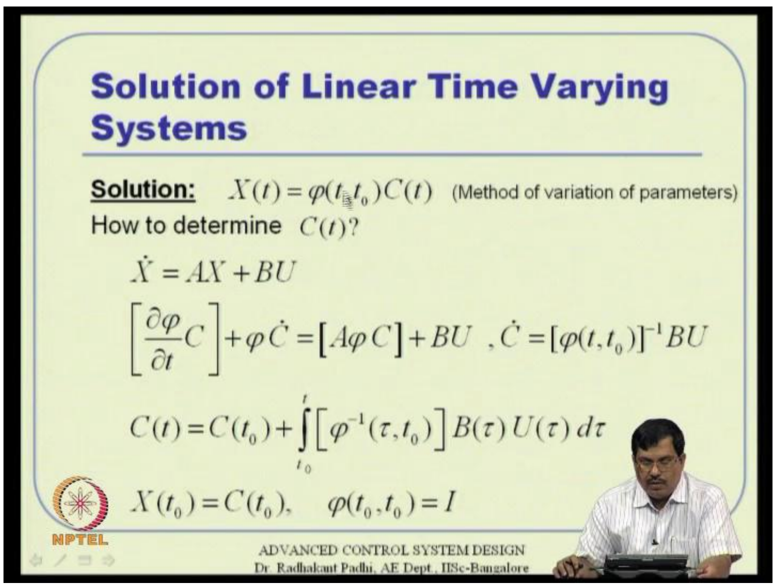

And then tell I will make use of the method of variation of parameters where C vectors contains unknown things and all that exactly similar to what you have done before we proceed only remember that E to the power A t we cannot talk, we have to simply work with this phi matrix e to the power A t is a special matrix for phi, but in general it is phi is phi. So x dot so what is that actually X of t is that so dot is nothing but that del phi by del t into C plus phi times C dot that is what you have done before in the sense of e to the power a t.

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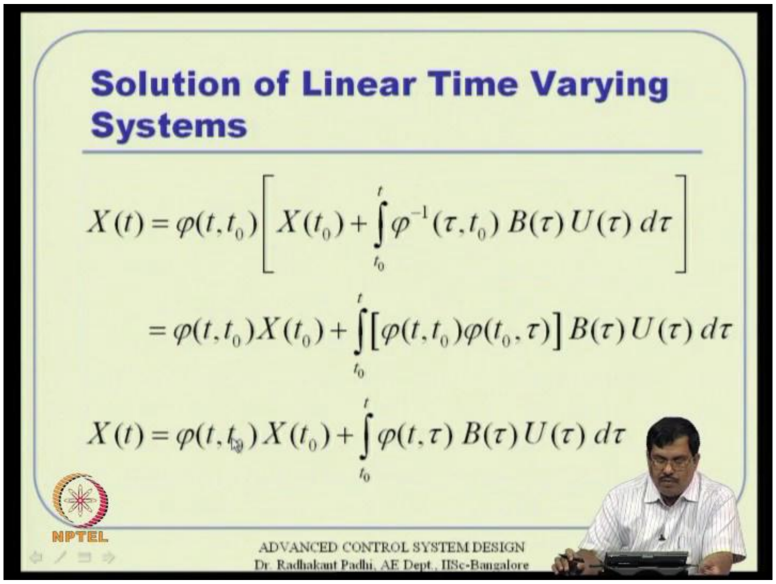

Now we will stop this particular thing I mean we will cannot be simplified for you have to stop here and del it is nothing but del phi by del t into c plus phi times C dot actually. So hand side is also that A times X so X is phi C plus B U so this equation suggest that C dot is nothing but that actually, because this equation this one and that one had cancel out. So, C dot is that so c of t is that exactly similar to what you have done here it is phi inverse now.

So, X of t 0 is that phi of t 0 t 0 is identity that is that is the things that you need to remember. So if you substitute it with then it turns out that X of t is all that and this one first term is that and second term is that I can take this integral inside and then try to club them together, because inverse is altering the coefficient phi inverse of tau t 0 is phi of t 0 tau that is one of the results and then this one goes inside.

And now this is like 3 variables here t into t 0 and then t 0 to tau that means I can interpret that as phi of t tau intermediate variable will go so that is what the solution ultimately. So, X of t is nothing but this one all that is valid provided you have a closed form solution for phi t t 0 this equation is useful provided you get a closed form solution for phi t t 0.

If it is not useful then probably you can do numerical solution, and if you really want to do numerical solution then why is doing all the things I mean you can just even stalk with this one and take numerical solutions directly. So, there is no point in going through the numerical solution of phi and then substituting it by again and then carrying out further integrals and all that to get a solution and all my point is you can directly get the numerical solution like that so. Anyway so, that is all we can need to talk here as we discussed various ways of this time solutions for linear systems both for time invariant system as well as time varying system on the way.

We also saw E to the power A t some algebra with that and as well as how to get a computational way of getting this E to the power A t and here you also remember all that you have done here is to get this coefficients in a in a simple in a in a recursive algorithm sense, once you get the coefficient you still have to do this Laplace inverse of this matrix, which needs to be done element by element yourself. Just keep that in mind and various other ways are also there probably you can see some of that as we go along actually. We will stop there in this class thanks a lot for the attention.