Advanced Control System Design
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Lecture No. # 17
First and Second Order Linear Differential Equation

Lecture today and before we proceed further, let us just revisit very quickly what we discussed, in the first lecture. So, the benefits of advanced control theory we were discussing last time summary motivation and all that very quickly. The benefits include first is MIMO theory that means, multiple input multiple output obviously, it is lesser assumptions and approximations.

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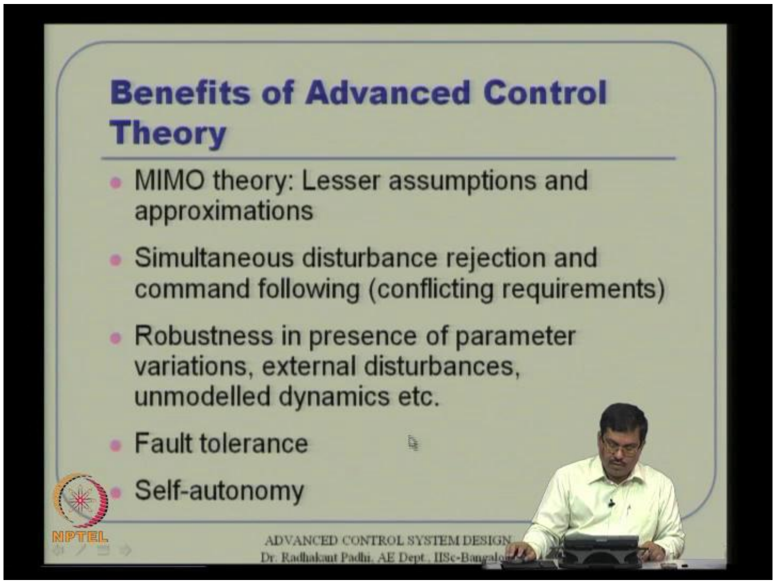

Then we can also design control system, which will take care of simultaneously it will take care of disturbance rejection and command following. So, these are typically conflict in requirements however, we will be able to do that. Then we can also explicitly discuss about robustness of the controller in terms of parameter variations, external disturbance, unmodelled dynamics etcetera. There are various ideas modern ideas like fault tolerance self-autonomy things like that, which can also be I mean, we can handle those problems in the framework of advanced control theory.

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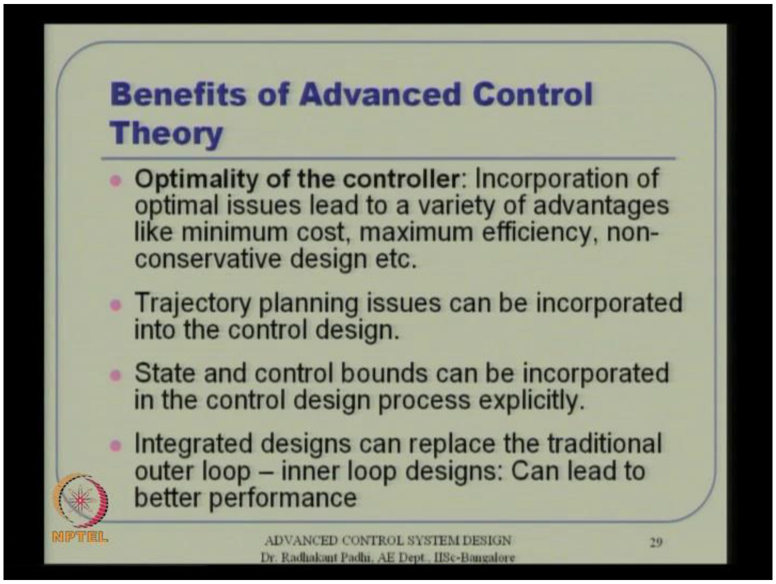

Then we can even talk about optimality of the controller, which traditionally this classical control system do not talk about. And we also can do this explicit trajectory planning issues can be incorporated into the control design in the framework of optimal control. If you talk about and state and control bounds can also be incorporated in the control design process explicitly that means, you do not have to live with the error. You can even design the control system base taking into account those bounds, if you know that a priori basically.

Integrated designs of the they can replace the traditional outer loop inner loop sort of structure that means, you visualize the problem as it is not like parts of system and things like that. And then it can obviously lead to better performance these are called integrated designs essentially or in aerospace industries they integrate in guidance and control.

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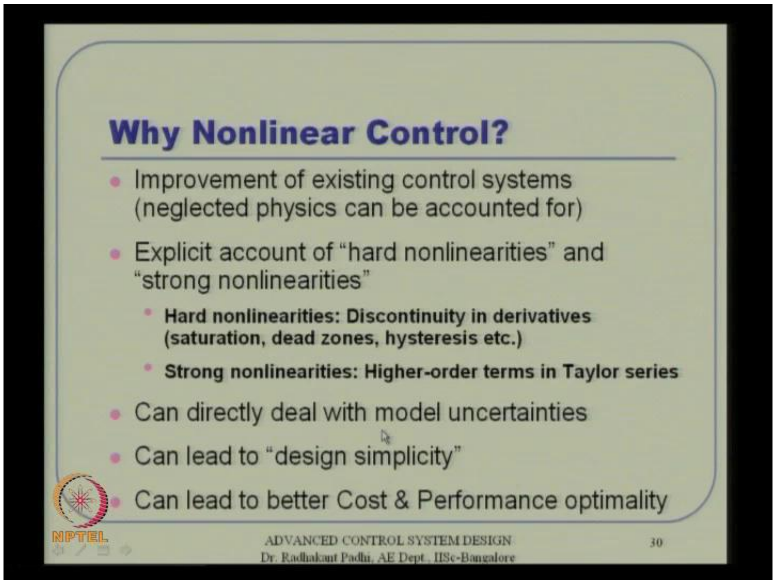

For example, actually, then you discuss, why do we talk about non-linear control and especially compare to even though MIMO system, MIMO you can design control system based on linear theory as well as non-linear theory. So, when you discuss, non-linear controller it one of the obvious reason is we want to improve the existing control system. So, there may be some control system already based on linear systems and you account for nonlinearity and then improve on that. Then we can explicitly account for something call non-linear hard nonlinearities and strong nonlinearities.

We discussed, about that in the last class anyway hard nonlinearities include discontinuity in derivatives that means saturation dead zones hysteresis. And then strong nonlinearities or higher-order Taylor, higher-order terms in the Taylor series essentially. So, those can be explicitly accounted for actually, in the non-linear control framework obviously. And we can even talk about we can directly deal with this model in uncertainties as well. And it can sometimes it can lead to design simplicity, which is little bit counter intuitive the moment we talk non-linear control it feels difficult. But some design approaches are available, which can even which can be simpler than the linear design techniques.

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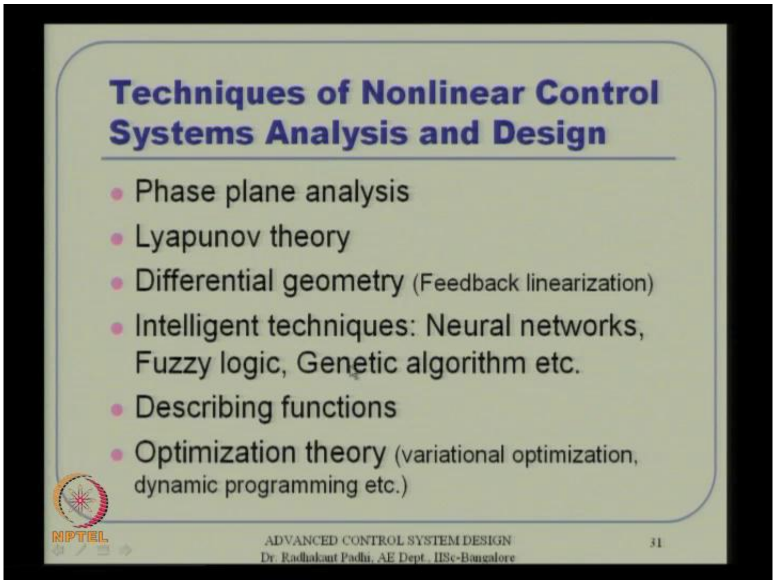

And then there are various techniques we the available phase plane Lyapunov differential geometry. I mean we discussed, about all that and some of these may be not be discussed in this particular course, but many of these we will discuss.

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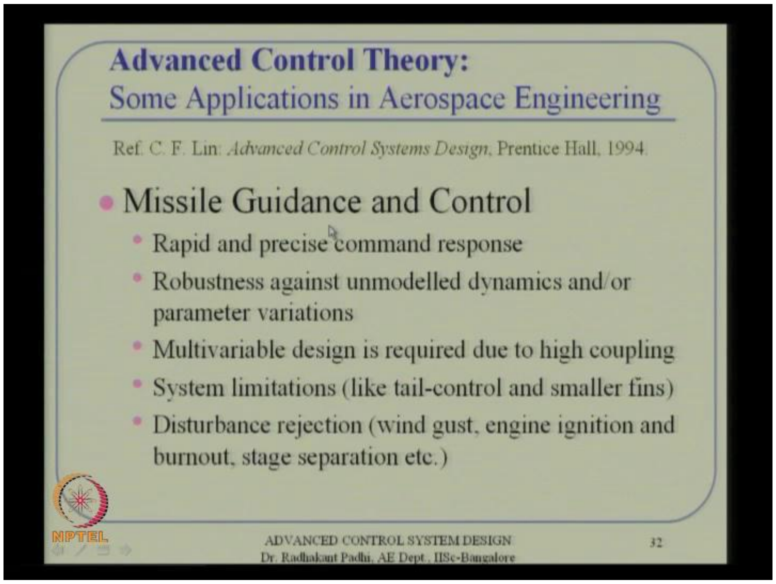

Then advanced control theory applications in aerospace engineering in particular there are various applications, but couple of them, which comes to mind is missile guidance and control. Then we can have variety of advantages compare to classical design. So, especially when missile control application we talk about rapid and precise command following or rapid and precise that means, you want the turn to be as minimum as possible like direct hit sort of thing. So, any approximation there will lead to compromise in the performance.

So, you really do not want to do that actually, that is one way. Robustness is also required against unmodelled dynamics and parameter variations, because typically the parameters do vary and then they cannot be predicted exactly, because weather conditions. For example, you cannot predict exactly before you fly basically, and then multivariable designs they are required due to high coupling system limitations are there disturbance rejections are it is requirements. And there are variety of challenging I mean requirements, because of which we really need advance control theory for missile guidance and control.

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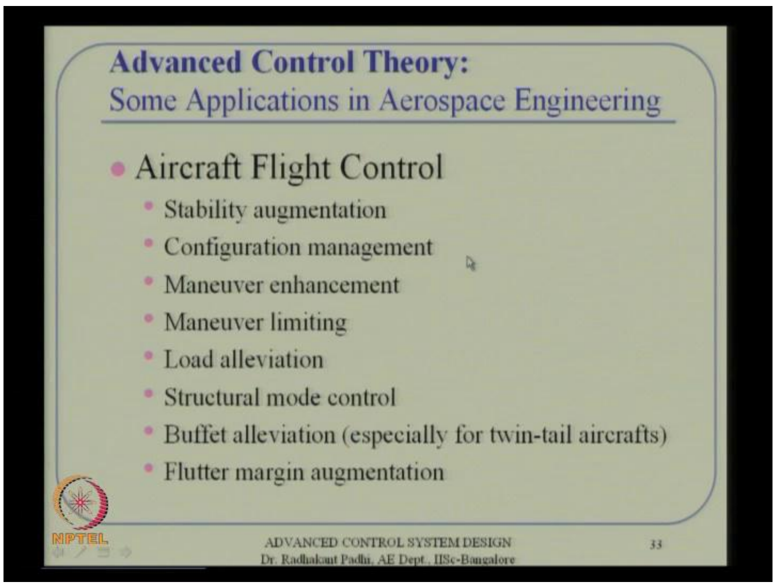

And next, aircraft flight control even though lot of classical designs do exist and many aircraft do fly in that, but still you talk about modern control theory for variety of reasons. One is stability augmentation whatever, stability regions you get you want unlock the domain more. So, you want to see whether you can do that with advanced control theory. Configuration management another issue that means, you already have a base line configuration and suppose, you want to kind of modify the configuration little later. Then entire thing needs to be repeated, if you talk about classical design or linear control design.

So, but if you really, talk about advanced control theory specially the non-linear control theory may not you may not need that. So, there are variety of advantages when you talk about aircraft flight control especially people now talk about reconfigurable control as well. That means on flight suppose, your part of the wing is gone, because of battle damage or something has happened there .Then how do you quickly reconfigure yourself and then take your vehicle aircraft back to the station that is one way. Or if you are talking about a commercial aircraft then at least you should land somewhere safely I mean you should not crash.

So, the those are the things or if possible you can just land it as if nothing happen really. So, those kinds of applications are on the research domain and these are really advanced designs that we can bring into in aircraft flight control actually unmanned were air vehicles variety of applications.

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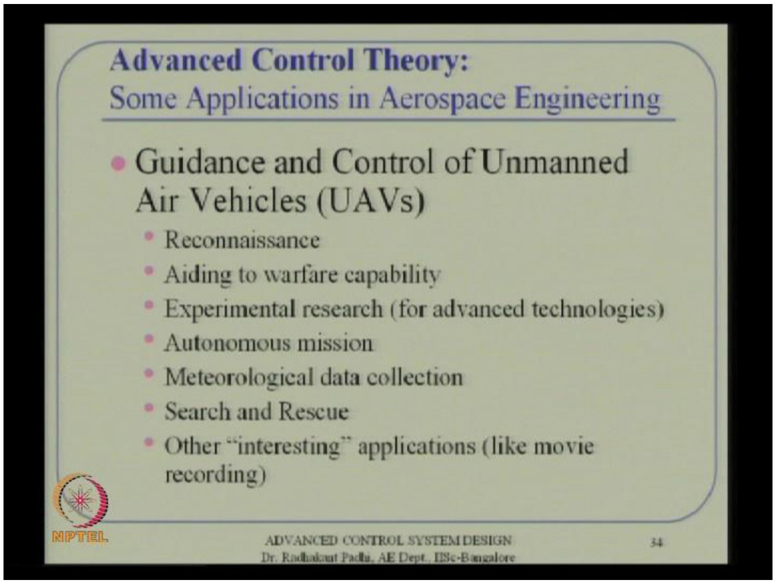

For example, reconnaissance for collecting the information from the in missile warfare capability it will again, give a different tool for you to apply. Then experimental research becomes easy, because it does not cost too much of money to build a big aircraft and you quickly can experiment your conceptual ideas there. And many of the academic research that is the one particular thing that gives us very good tool for experimenting flight control designs actually quickly. Then there are variety of further applications like autonomous missions you do not really have to plan meticulously where to go where to come and things like that.

So, you just I mean very rough way you give that guidance then after that the vehicle takes over and it actually, flies in autonomous mode. So, those are variety of applications that you your unmanned air vehicles are kind of important for modern research and all that. And there again, pitch time applications like search and rescue meteorological data collection even movie recording. I mean those are you can just fly over and with a camera and then record a movie. So, those are variety of applications that we are looking for and aerospace applications not necessarily only three there many other applications as well.

So, with this we thought we will I mean, we had some proper motivation to go further. So, then coming to today's lecture as we discussed last time, we will revisit this some of this classical control issues and all that before we go for modern control really. So, in that sense, this particular lecture today we will discuss, about first and second linear differential equations. So, that is the topic of today's lecture really So, very quickly we will revisit what you already, know probably in under graduate courses and all that.

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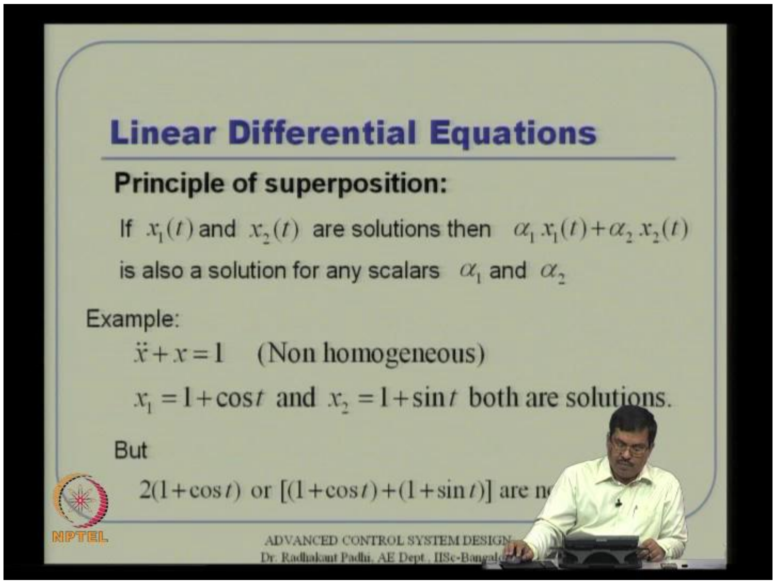

So, when we discuss, about linear differential equations the very first thing that we that comes to mind is principles of super position again. So, if x 1 and x 2 are solutions of some differential equation then this alpha times x 1 plus alpha I mean alpha 1 times x 1 plus alpha 2 times x 2 should also be a solution for any scalar alpha and alpha 1 that is the principle of super position really. For example, if you have x double dot plus x equal to 1 essentially, remember this is a non homogeneous equation, because this is not equal to 0. Then very quickly you can verify that 1 plus cross t and 1 plus sin t both are solutions is easy to I mean easy to verify lets verify quickly.

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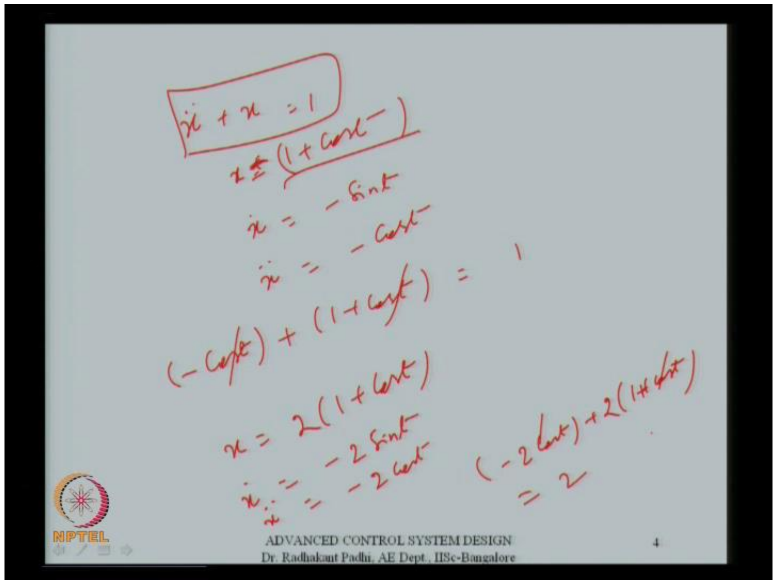

So, we have this x double dot plus x equal to 1 and we are exploring the possibility of whether, x equal to 1 plus equal to 1 plus cos t is a solution or not. So, what you do you put x dot equal to minus sin t and x double dot is equal to minus cos t. So, if you substitute it back then what you get? You get minus cos t plus x, which is 1 plus cos t and that is equal to obviously 1. So, it essentially satisfies the differential equation. So, individually, if I take 1 plus coz t it is actually, a solution of this differential equation anyway and similarly, we can verify that one plus sin t also satisfies that. However, if you put both 1 plus cos t and 1 plus sin t even though they are individually they are solutions they will not satisfy the principle of superposition it is easy to verify.

So, then let us for example, 2 into 1 plus cos t whatever, you have here let us verify whether that is there or not. So, simply I talk x equal to 2 into 1 plus cos t, if you carry out all the algebra there then it will turn out that x double x dot equal to minus 2 sin t and x double dot equal to minus 2 cos t. And, if you substitute it back what you get minus 2 cos t that is the thing and x equal to 2 into 1 plus cos t and that essentially gives not really one but that is two. So, essentially does not satisfy the differential equation really, I mean the combined multiplication really does not satisfy.

So, where is that actually, I mean this equation is actually, very close to linear system anyway. This equation is very close too however, we should also remember that when you apply principle of superposition, you should apply to the homogenous system not to the non homogenous system. If I make it 0 this equal to 0 then it will satisfy you can verify that quickly.

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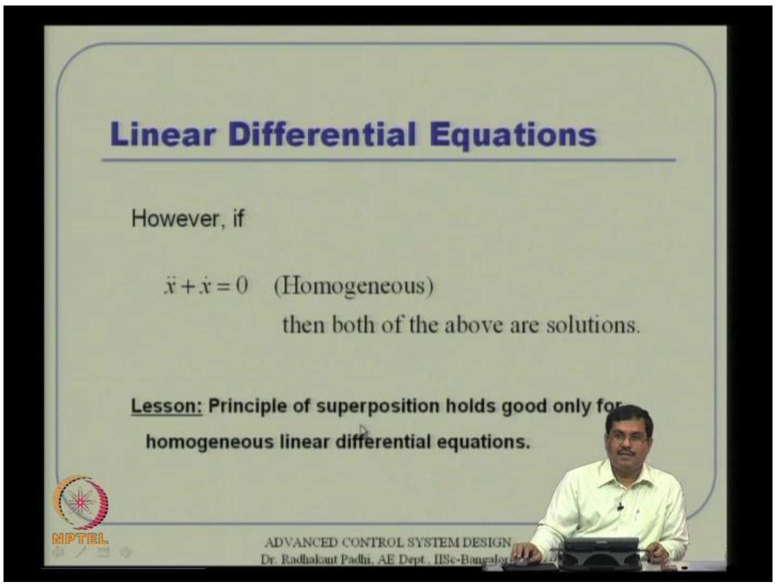

So, that is what I told this if it is x double dot plus x dot equal to 0, then it will very quickly. I mean any solution that individual solution; if you take then it will satisfy the differential equation anyway basically. So, that is even x dot x double dot plus x equal to 0 that is the original thing that we started with just instead of one I put 0 then it will satisfy that. So, the lesson to be remembered is principle of super position even for linear systems holds good only for homogenous part of it. The entire part it may not hold good actually so, that is the lesson to remember.

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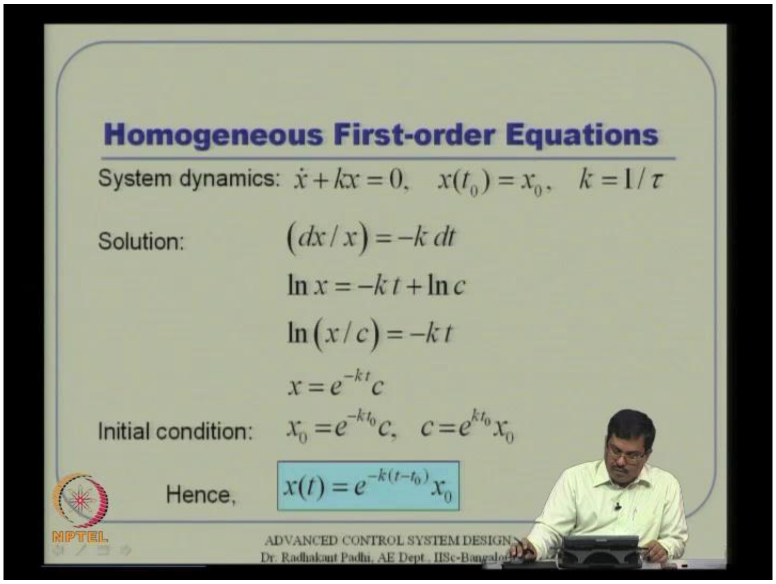

Now, let us go to solving the differential equation now we suppose, somebody gives us the first order linear differential equation then how do we solve that. And we will essentially start with this homogeneous systemic equation that means, homogenous system where the right hand side is actually 0 with initial condition x of t 0 equal to x 0 and this k that is equally 1 over tau, where tau is typically known as time constant. So, this at this part of the solution we are assuming that tau is constant and hence k is constant.

So, how do we go and solve it is essentially we have we know that from two calculus here, that given this differential equation I can very quickly go to that part of it, because x dot equal to minus k x. So, dx by dt is minus k x let us quickly see that. So, this is x dot equal to minus k x. So, this is essentially dx by dt this is dx by dt is equal to minus k x So, that means dx by x equal to minus k dt. So, that is what it is kind of written here this dx by x equal to minus k dt what you got here is written here (No audio from 13:27 to 13:39).

So, let us quickly go to the next part of it. So, once you know that this is dx by I mean this dx by x equal to minus k dt, then you can integrate both sides and then take logarithm of x natural logarithm is equal to minus k times t plus some constant and that constant let me write as some logarithm anyway. When I take this one to the left hand side I will get l n of x over c is equal minus k t and hence, if I take the exponential form x turns out to be e to the power minus k t into c. Now, this initial condition we have to apply, remember without initial conditions differential conditions are not complete so, only differential equation you do not have.

So, now let us try to apply the (( )) I mean this initial condition So, we put t equal to t 0 and then x becomes x 0 So, we get x 0 equal to e to the power minus k t 0 into c. So, c becomes solution of that, which is essentially e to the power k t 0 into x 0 very clearly from this equation. You can see and then, if you substitute this substitute it back in this equation this c will substitute it back here. And then make it complete and then it becomes x of t equal to e to the power minus k into t minus t 0 into x 0 that is the standard form that we know very well 1 plus 12 I mean from our previous analyze. Now, let us see what other implications of that when you discuss, about e to the power something there is also exponential form available to us.

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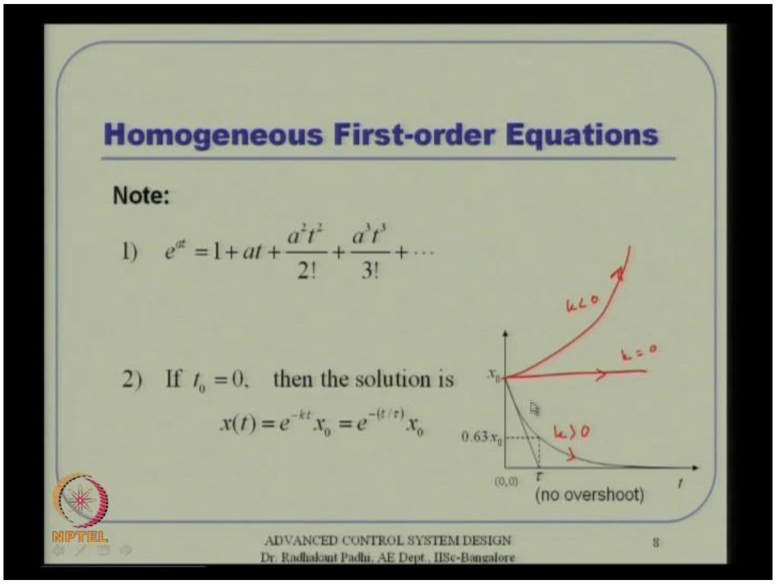

How do you evaluate e to the power a t 1 way of evaluating that is using this exponential form I mean this probably power series you can say that you expand it in terms of power series. Actually, this is essentially an infinite series unless at some point of time it truncates I mean whether it truncates or not I mean it depends on how this series evolves and all that. But in general it will not truncate the moment a is a constant it will keep on staying there.

So, after, if s is probably less than 1 then after some power we typically truncate telling that it is no meaning after that.

However, it will never become exactly 0 and this power series form I have explicitly put here, because this is the form. We will need it later for MIMO system for modern control things and all there is also a notation of e to the power a t, where s is a matrix is not a scalar. So, at that point of time we will need this exponential I mean this series form to evaluate e to the power a t and all that is one way of evaluating that So, we will see that later as we go along. However, if a t 0 is 0 for this specific case or special case suppose, we just start to visualize the solution at some point of time and initially the time is 0 at that point of time.

Then this t minus t 0 will go from here and essentially we will be living with a much simpler form, it looks slightly simpler compare to the previous solution essentially. So, we have this x of t is e to the power minus k t into x 0 that is the form that we will typically use all the time under the assumption that t 0 x of t 0 is I mean x 0, where t 0 is 0 basically, under the assumption the simplest assumption. Typically, the first order equations also admit some solution like that provided this k is positive, if k is really the tau is positive. So, k is positive that means minus k is negative. So, then, if you see this solution and all the solution will evolve like that actually.

So, essentially there are two, if other ways of course, it also a it will evolve like another way. Suppose, if it is other that k is essentially negative then it will easy unstable, which will go like that, k is positive it will evolve like this and then it will go towards 0 obviously, if k is 0 then it starts some value and stays here So, that x dot is 0. So, this is valid for k greater than 0 this is valid for k less than 0 and this is valid for k equal to sort of thing. And if it is stable there are notions of like, if I go one time constant away that means, my t evolves by one time constant then my value of that x naught will decay to all most 63 percent of what I had initially. These are all very standard results is very easy to I mean analyze and see ourselves.

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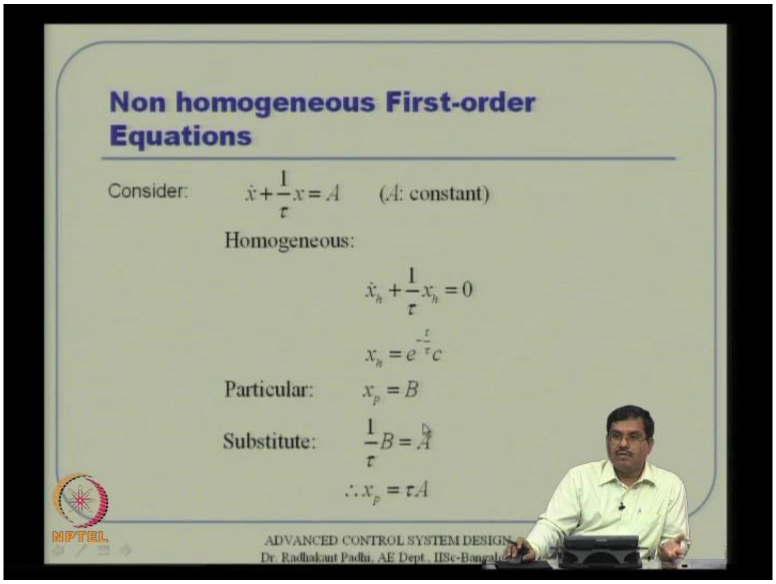

Next, we will discuss, about non homogenous equations that is about all about homogenous part of it. Suppose, you do not have 0 here but let us start with the simple thing you have some other constant A then what happen? Then the solution evolves in two parts one is homogeneous part, where this is equal to 0 and there will be a particular solution the total solution will be part like homogenous plus particular solution. So, we have homogeneous solution coming as it is that we discussed, just now. So, the homogeneous part will turn out to be like that remember, I have all ready written k equal to 1 over tau, because that is the standard form anyway here.

Now, particular solution depends on the form that you have here and if you have a which is a constant here in the particular solution will also become another constant B lets say. So, that is how we start with then we substitute this solution back into the differential equation and try to see, what is their particular solution that is the approach? So, we start with this particular solution a constant B then this is 0 anyway derivative is 0 and hence, put this constant back in there 1 over tau B is equal to A. So, that means, x p the particular solution becomes tau into A, because remember B equal to tau into A this is the particular solution. So, what is the total solution? Total solution is this homogeneous part of it plus this particular part.

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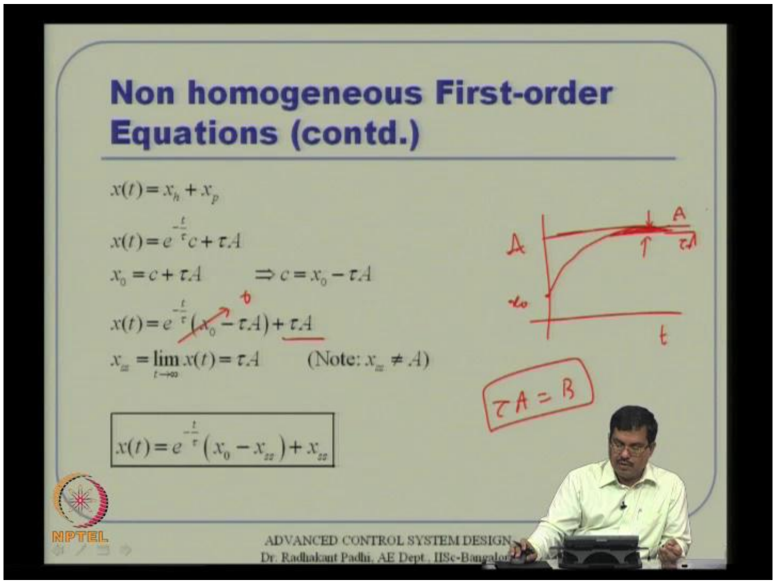

So, what is written here? So, total solution is homogeneous solution plus particular solution. So, takes that form. And now, remember that if you have a non homogeneous differential equation we must apply the initial conditions for the total solution not for any homogeneous part only basically. So, we have to apply the initial condition for the total solution here. So, we have a total solution here, then we apply the initial condition here So, at t equal to well t equal to 0 I mean t 0, which is we implicitly assume t 0 is 0 here So, x 0 that is equal to c, because this becomes one here plus tau. So, if you solve it then c becomes x 0 minus tau a here and then x of t you can go back to that and that is this part of it, because you all ready have a c known to you by now.

So, you substitute it back and tell this is my total solution. So, that is the total solution that you really, have and here remember this part will go to 0 as t goes to infinity this part will go to 0. So, their x of t will really approach tau times a, it need not approach a, so, there will be some sort of a steady state error basically. Because if you really have this let us say, if you put tau and this is your a value and you start with some initial condition x 0. You need not really go to that particular solution as it is there will be steady state errors actually unless tau A is equal to B. However, we have already, seen that B is tau a anyway here that is that what it is So, it will actually essentially go to this. The steady state error I mean is that is the analysis part of it, how do I say how do I analyze that actually.

So, that is I mean but sorry no there is a small error here probably So, what is what will happen is it will go to tau a unless tau a is equal to b that will not that will not have that part of it actually. So, what you what I essentially mean is this part will go to 0 as t goes to infinity. So, we will be left with this tau a part of it and that is what you see the steady state value is essentially tau a. So, if this is a, this will go to some other value this a this tau times a depending on what value you have.

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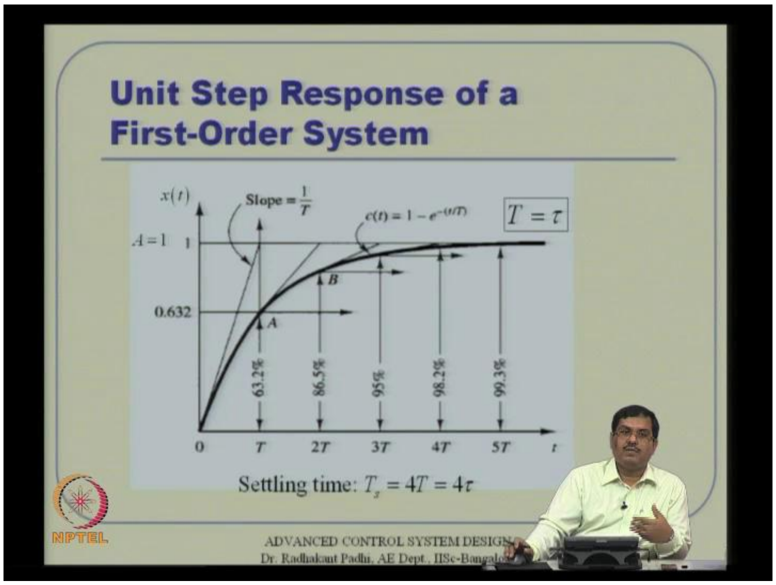

So, this is what is written here I mean, unity step response and all that if you talk about that is a very standard analysis now, what happens, if I put a equal to really one here. Then we discuss, about like a very standard form of solution it will go towards that anyway. So, it is the response plot will be something like this and the this is remember this capital t is tau here this is in this plot. So, we first tau we will have 63.2 percent there and then 2 tau we will approach something like 86.5 percent 3 tau 95 percent 4 tau 98.2 percent things like that.

So, if you it will never be exactly equal to that anyway. So, what the idea here is let us talk about 4 tau being very close to the value that we really want 98.2 percent already. So, that is the type of definition that we bring in that we tell this is the settling time actually, because once it goes there it cannot come out anyway that is the nature of the thing it will just remain within that bound. So, we will take 98.2 percent very standard definition sort of thing. So, then we will tell that is the time what is by which the solution really settles.

So, that means, it is actually settling time actually, sometimes people use 3 t or 3 tau as settling time also assuming that 95 percent is good enough. But by default, if nothing to us then we will take 98.2 as our standard definition actually, settling time is (( )). So, this is all about first order systems, if you have different type of inputs like sinusoidal and all we can still talk about different solution and all that is not very difficult to derive. So, once you know the particular solution there are form available you substitute it back and derive it.

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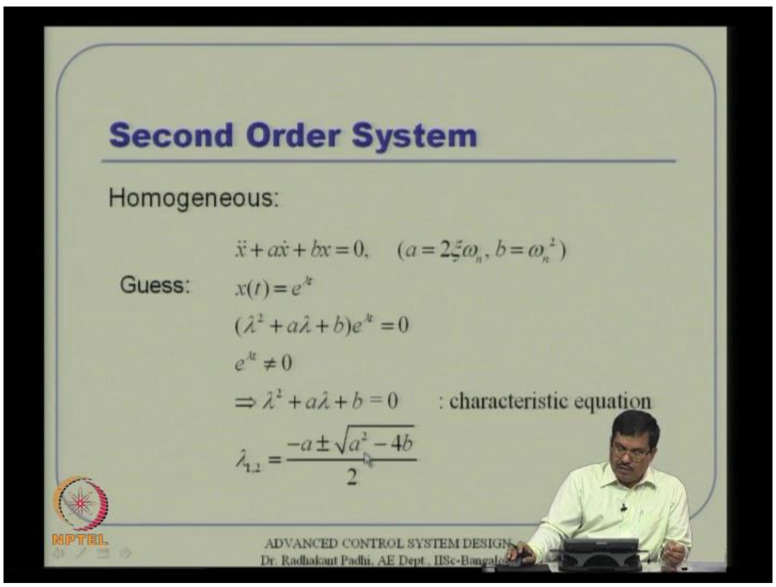

So, let us move on to the second order thing, because remember this is also a kind of a review class. So, we will not talk too much detail about that. Homogeneous systems first we will start then second order system as well and remember this standard form this a what you see here is taken as 2 zeta times omega n and b is equal to omega n square. So, we now what is the approach I mean we cannot really directly go and solve it. So, what we do is there is a kind of some sort of a theorem available to us for linear system especially, for linear system that is valid in general, which talks about uniqueness. That means, if you get one solution that is the solution that means, there cannot be any other different solution anywhere.

So, we let us start with a guess solution and see whether that becomes a solution and if it really becomes a solution then we are done, because that is the only solution that we can have actually. So, we essentially get inspired by this solution form what we really had before that this is the form actually I mean x of t is e to the e to the power something times t into x 0. So, whether that kind of solution really satisfies the second order form or not let us start guessing that is the motivation that is the inspiration for guessing that solution also. Lamda is arbitrary constant this point of time and we are simply guessing that that can be a solution for x.

So, let me explore that now next step is just to substitute it back in there, because the form is already known to us. So, once we substitute it back we will get this kind of a equation and since we know that e to the power lamda t is never equal to 0 there will be always some sort of inequality sort of thing So, it is never equal to 0. But we still want this equation to be equal to 0. So, the only way that can happen is the coefficient is 0 and then the coefficient is going to 0 is something called characteristic equation. So, if I am able to find a solution satisfying this equation this quadratic equation then I am essentially satisfying this equation and hence, I am satisfying that equation anyway.

So, let us find out what values of lambdas are satisfying this particular characteristic equation and obviously this is a quadratic equation. So, I will get a solution in this from and hence, this gives me some sort of a dilemma here that whether this solution this roots what you are getting are they repeated or are they real or are they complex conjugate. Remember, if it is a polynomial with real coefficients then the solutions from in this form should have complex conjugate pair, if it all they are complex, if they are not complex that is fine. But if are if they are complex they must be conjugate pairs actually provided this polynomial has real coefficient that that is the thing. So, assuming a and b are real numbers that has to that is where it will lead to actually.

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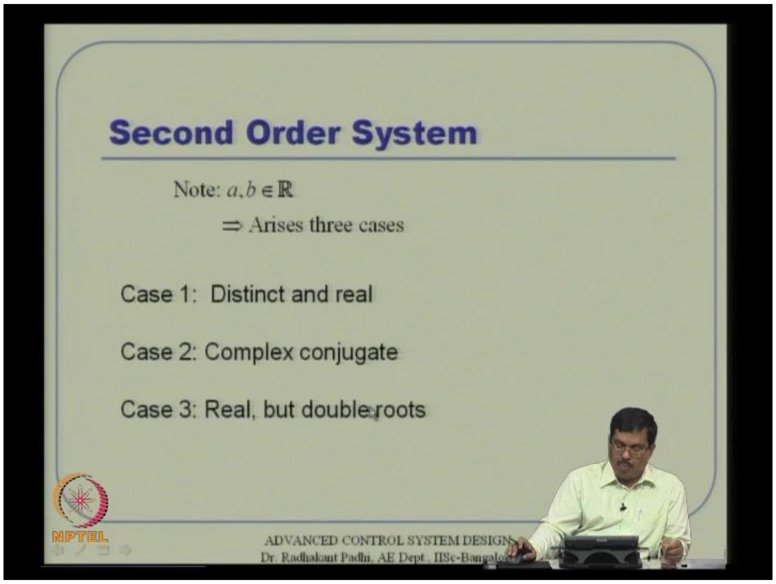

So, let us analyze the case by case that means, if a and b are real then there are just kind of three cases and the first is real. And I mean distinct and real roots that is that will happen when this fellow is actually positive So, this square root of that is also, some sort of a positive number actually. And then there will be complex conjugate that means inside the square root we have a negative quantity. So, then it will arise to some sort of a complex conjugate pair. And then, if it is there can be a third case, where it is real but double roots that means, if this a square is equal to four b then this part is not there. So, we have a solution, which is actually double roots actually.

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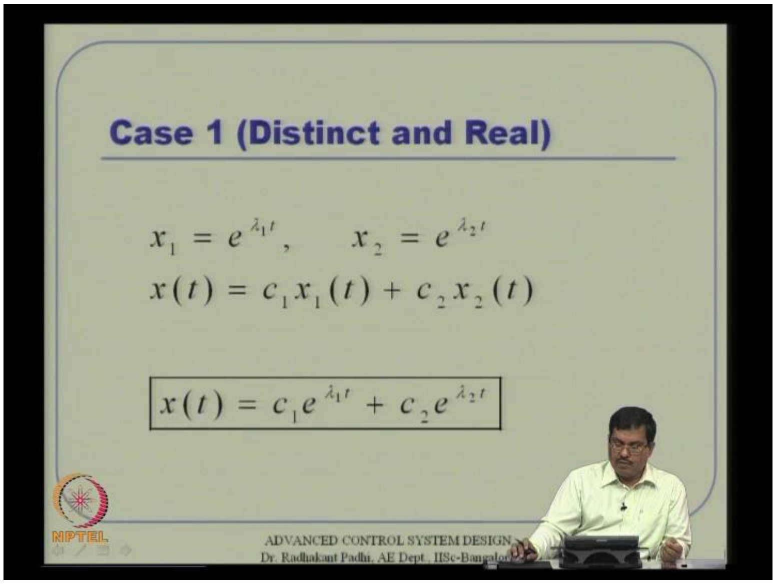

So, let us, talk about case by case here So, we have a distinct and real case first, So, we have x 1 taken as lamda 1 t and x 2 is taken as lamda 2 t, because they are these are distinct and real anyway. So, these are something like best line solutions or the (( )) to the solution sort of thing. So, in general we will have c 1 times x 1 t plus c 2 times x 2 x 2 t, you can when this is actually, a kind of principle or superposition. We are taking advantage in an indirect way so we have two basic solutions. So, we want to compose some sort of a function, which is c 1 x 1 plus c 2 x 2 sort of thing, where c 1 and c 2 are constants. So, that is the general form of solution for the homogeneous part of it.

So, we have a solution, which is c 1 times e to the power lamda 1 t plus e to the c 2 times e to the power lamda 2 t. So, that is our that is our solution in general. Now, remember this is a special case, where you discuss, about distinct and real roots and if you have initial conditions known to you properly then you will substitute it back. Remember this is a second order differential equation that means you need two initial condition both x and x dot. So, we substitute it back and then find out c 1 c 2 that is the standard way.

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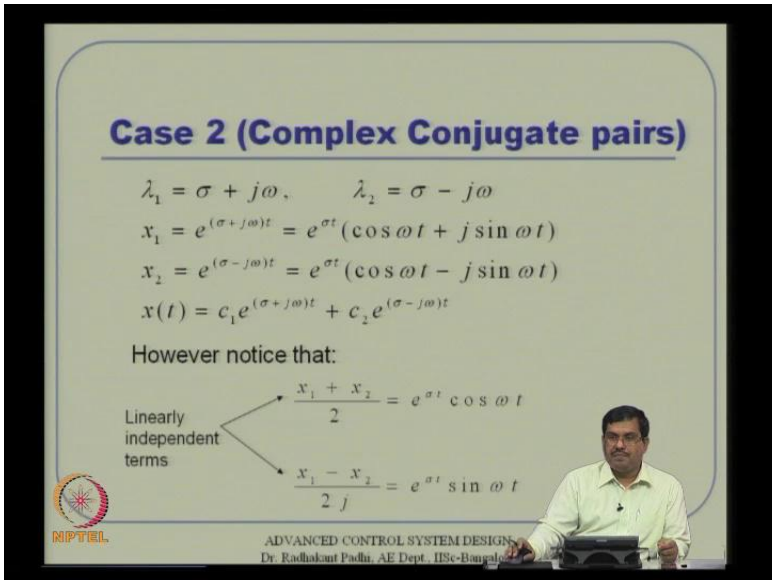

And then we have the second case, where it is complex conjugate pairs then what you do? So, if it is complex conjugate I will be able to express lamda 1 as sigma plus j omega and lamda 2 is sigma minus j omega that is the complex conjugate pair. So, I go back to my solution basic solution sort of thing so x 1 is this part of it. So, I will be able to decompose this telling that, if this e to the power sigma t plus this demoivre's theorem is there in complex a complex analysis. So, e to the power j omega t is cos omega t plus j sin omega t I will be able to write that. And similarly, x 2 is e to the power sigma minus j omega t so, that I will be also able to write it in that form.

So, essentially I am getting a solution of that form, which is very similar to the form that we already have here for the distinct and real. However, the basic solutions are kind of containing these exponential terms, which is these complex numbers in there. That means, in this form the solution looks as though it is a complex solution really. And typically, I will not be very comfortable to deal with that. So, here we will do a little more algebra telling that, if these are this is in general form if x 1 and x 2 are actually, some basics solutions then any linearly independent sort of combination that will also become a basic solution.

So, I am forming something like linearly independent terms, where x 1 plus x 2 by 2 and x 1 minus x 2 by 2 j, if I do that these two are actually, linearly independent terms anyway. So, that will essentially lead to that if you do this algebra x 1 plus x 2 means this term will go and x 1 I mean minus x 2 by 2 j means j part will also go there. So, you can do this simplify simplification of the algebra yourself then tell this is e to the power sigma t cos omega t and this will e to the power sigma t sin omega t. So, instead of taking this complex solutions as it is I will I am just forming some sort of a linear combination, but they are linearly independent terms and then taking those terms as kind of my basic solutions around, which I will form a general solution.

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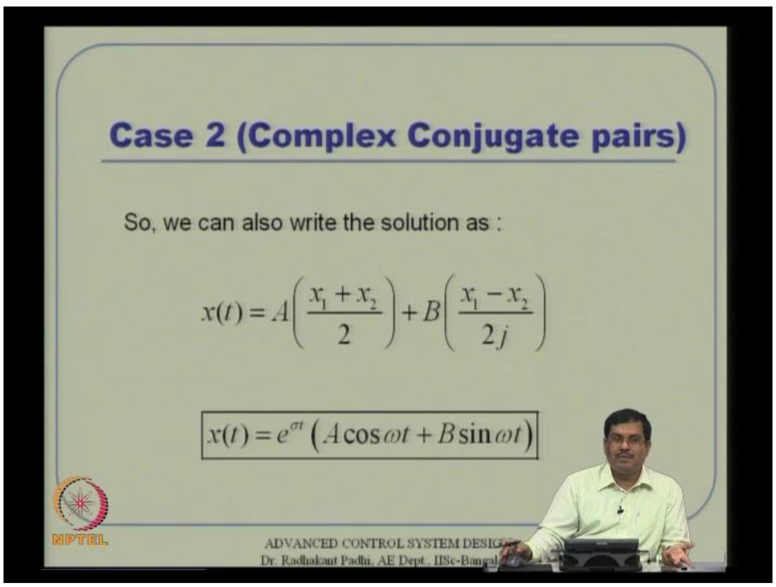

So, then I am ready with that, so, what I am telling now my general solution is a times this basic solution plus B times this basic solution. So, and essentially what I am telling is this is my general solution x of t e to the power sigma 2, I mean sigma t whole into a cos omega t plus B sin omega t that is what I am interpreting. So, that is how but this is actually, very real solution that there is no complex term out here actually. The solution that we really get are actually they are solutions even though we have some complex numbers in this form of the solution So, that is the motivation of getting out of that.

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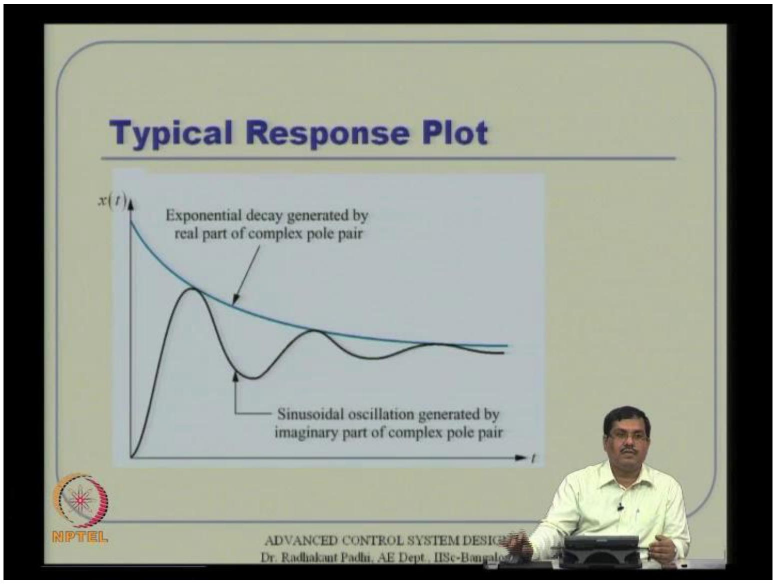

Now, some sort of analysis of this response, if you really have this kind of a solution there then it has a exponential term and there is a sinusoidal term. And we all know the if there is no there is for a moment, if we consider this sigma is 0, then it essentially a cos omega t plus b sin omega t. So, that is actually, undammed solution, which will keep on oscillating like a sinusoidal wave. And then this e to the power sigma t will serve some sort of a envelope for that means, this sinusoidal component will lead to this oscillatory component. But this oscillatory component is suppose, to decay and that is that decay is given by e to the power sigma t. Suppose, sigma is negative then this entire solution will eventually go to 0 no matter what kind of what kind of oscillations you have actually.

So, these are analysis of the solution that you already have. So, this is this part is like sinusoidal oscillation generated by imaginary part of the complex pole pair. And this is the exponential decay, what is dictating? That the decay of the pick value actually, if you just join the pick values then it will turn out to be a nice decay function.

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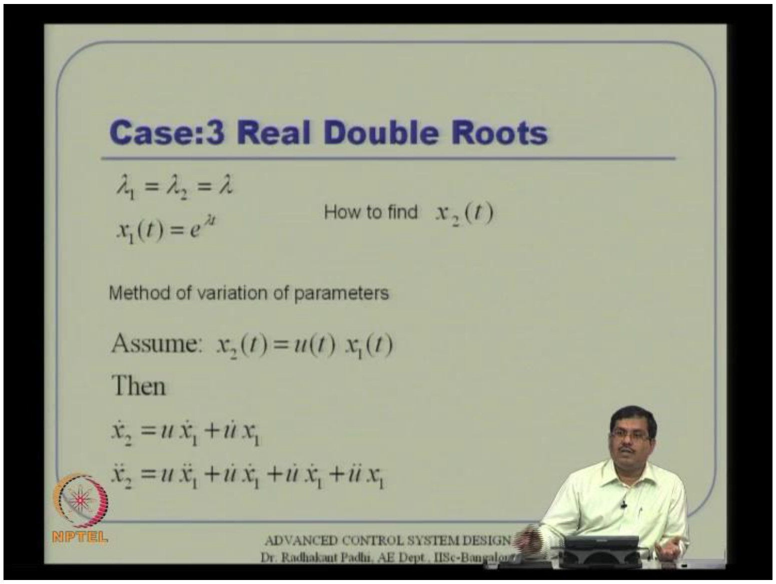

Now, we will move on to case three, where you really have a double root, then what you do? That is the last one that is left out probably. So, in this case, we have lamda 1 equal to lamda 2 equal to lamda that means we have only one solution. So, obviously, we have one basic solution but this remember this is second order differential equation. So, essentially we need two basic solutions actually So, how do you find out that this is this x 2 of t basically, then one method is method of variation of parameters that means we essentially take help of the x 1 to find x 2 I mean that is that is the our idea basically.

So, we have x 1 available to us already and let us assume x 2 as some parameter u of t remember this not really a constant quantity this is also a function of time some u of t into x 1 of t and again this is our guess at this point of time. We are just assuming that whether you can really find a solution like that actually. Now, we can substitute x 2 dot we can carry out the algebra u times x 1 dot plus u dot times x 1 that is what you have here. Then x 2 double dot from taking the help of this one we will have u times x 1 double dot plus u dot times x 1 dot and from this one it is u dot times x 1 dot and u double dot times x 1. So, this is simple algebra at this point of time assuming that. Now, we have the term we have x 2 double dot expression we have x 2 dot expression we have x 2 expression as well. So, assuming x 2 as a solution we should be able to go back to the differential equation and try to substitute it back in there.

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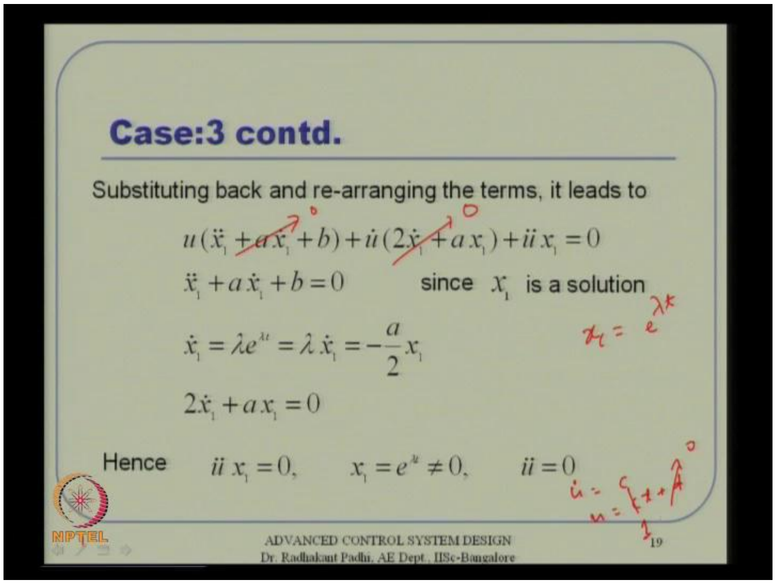

So, we will do that and then the then carry out the algebra and then it turns out that we will be able to write it in this form. There is u multiplied by certain quantities taken together then u dot I am taking common. So, it will turn out to be like that plus u double dot times x 1 equal to 0. Then the beauty part of it is remember, these are all x 1 components actually, within the bracket and x 1 is already a solution. So, if x 1 is already a solution then obviously this part is essentially 0, because that is the solution already. So, that is what you have we have written here actually and that is what it is a solution here. So, this entire term is as if it is not there it is not playing any role here. So, you are left out with only these terms actually.

And let us, see further some analysis sort of thing you have x 1 dot is you have x 1 as e to the power lamda t, that is what x 1 is a solution but just a second, but x 1 we have already know that x 1 is e to the power lamda t remember lamda one t but lamda one is lamda here. So, if you x 1 is lamda t, if x 1 is lamda t then we have this x 1 dot as lamda times e to the power lamda t I mean, that is the simple algebra actually. Then what is happening, if I multiply both sides by kind of lamda I mean lamda times x 1 dot is actually, lamda square times e to the power lamda t and it will again, turn out that it is nothing but minus a by 2 times x 1.

You can I mean, you can substitute it back x 1 dot I mean that equation and all that and then you can get it this expression right here. So, what you are telling x 1 dot actually, a essentially minus a by 2 times x 1 basically, because that algebra tells us right where x 1 dot is lamda times e to the power lamda t and e to the power lamda t I am kind of substituting it back getting into that form sort of thing. Then I what I am having here is two times suppose, I just consider the first one and the last one. So, what I am having here is two times x 1 dot plus a x 1 is equal to 0 that is what I am getting here.

That means, this term is also 0 basically, here I am sorry so, that means what I am getting here is this term is also 0. So, what I am left out is simply that last term and that last term is essentially what I will carry on further. So, the last term is u double dot plus x 1 is equal to 0 that is the solution that I am getting that is the equation that I am getting from here. And then we all know that x 1 e to the power lamda t, which is not equal to 0 x 1 is really not a trivial solution. Then we are left out with u double dot is 0. So, if u double dot is 0 obviously, u dot is a constant quantity I mean, u and then u of if u double dot is 0 then I can really get some solution something like that u is equal to some constant quantity sorry u dot and u is equal to some constant quantity let us say c times d something like that.

Then that is the general form however I am interested in this exercise about a particular solution in other words about a kind of a base line solution about, which I will form linear combination actually. So, this point of time I am just interested in one particular solution, which will give me some x 2, which is independent of x 1 that is all my motivation here in this exercise. So, I will assume d equal to 0 and c equal to 1 that is my choice actually for my simplicity.

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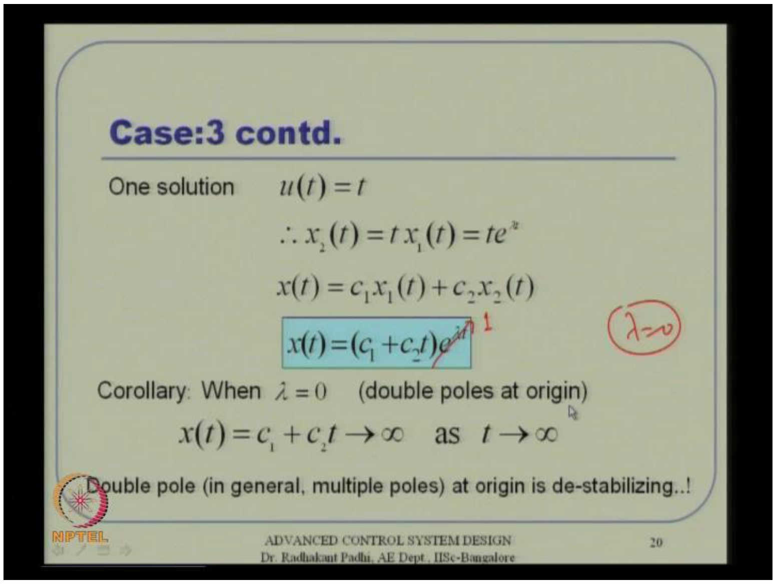

So, I am essentially left out with u t as nothing but t basically, so what I am getting here x 2 is t times x 1 that is how we started with so, I am getting x 2 is t times e to the power lamda t. So, I have got x 1 and x 2 are kind of linearly independent now. And x 2 this particular x 2 will satisfy the differential equation also, I really have two independent solutions to begin with and hence my solution is combination of those two solutions. So, c 1 times x 1 plus c 2 times x 2. So, essentially, if I put it back there then it becomes x 1 x of t is equal to c 1 plus c 2 times t into e to the power lamda t. So, what it essentially means suppose, lamda is 0 that means I have actually, kind of a I mean that pole 0 will see later, but suppose, the solution part of it lamda is equal to really 0 that part of it then this is essentially I mean 1.

So, I am left out with c 1 plus I mean, c 1 plus c 2 t and then that one will the so, essentially what it means is as t goes to infinity. This solution will also go to infinity not necessarily in an exponential way, but at least in a polynomial way, but still it is going to infinity anyway that means, this particular solution is also destabilizing. So, if you have double poles in the origin we will see what is poles and zeros later probably next class. So, if you really have double poles and in origin then it will have kind of instable property basically, the solution is unstable in general. If you really want to generalize that means, third order equation fourth order equation and things like that and lamda is still 0. So, we will essentially we will add one more terms.

So, c 1 plus c 2 t plus c 3 t square by 2 something like that, it will happen in a series way and then it will actually go in a it go to infinity in a polynomial manner. That means, in general, if you have multiple poles at the origin then the solution is certainly destabilizing. So, that will see I mean that is a kind of a consequence of this analysis essentially.

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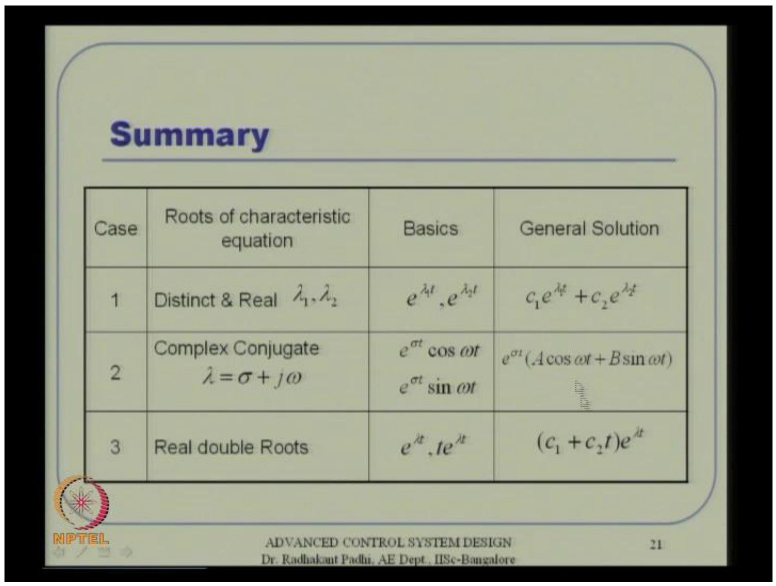

So, in summery there are two cases we have distinct and real case we have complex conjugate we have real double roots. So, the basic solutions are something like this, if you e to the power lamda 1 t e to the power lamda 2 t or this two of that two. And then the general solution turns out to be in this form that is the summary part of it as far as homogeneous equation is concerned.

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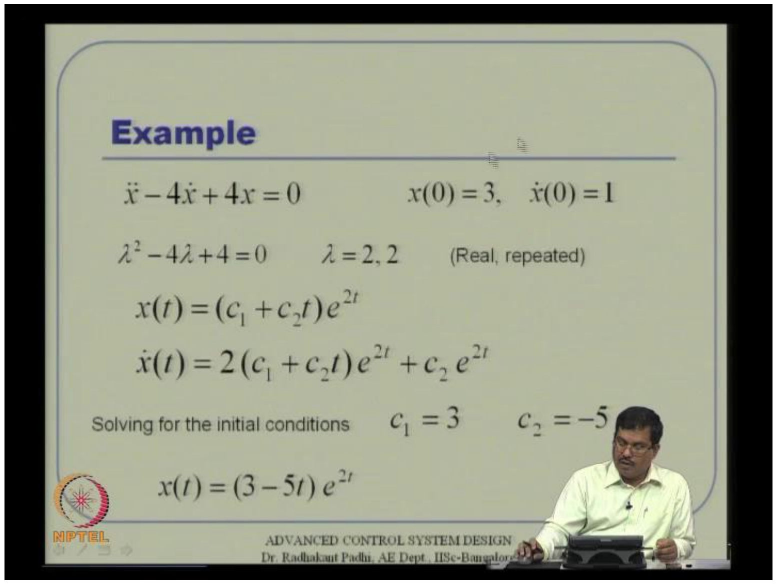

Now, going to the example sort of thing this I can actually a homogeneous system anyway. So, we will start with this equation. So, we substitute it back the characteristic equation turns out to be like that and this is like lamda minus to whole square equal to 0. So, that means lamda is equal to 2 two that means, it is real and repeated also and these are the initial condition that you are dealing with so, this is the general equation general form of the solution that we know. Then we x dot we will we can carry out and then we will substitute these initial conditions what we know then solve for this c 1 and c 2.

So, c 1 turns out to be three and c 2 turns out to be minus 5 you can very easily do that in pen and paper yourself you can do that. Then the entire solution turns out to be 3 minus 5 t into e to the power 2 t that is the solution essentially that is how you get the complete solution actually, accounting for this initial condition as well that is about it.

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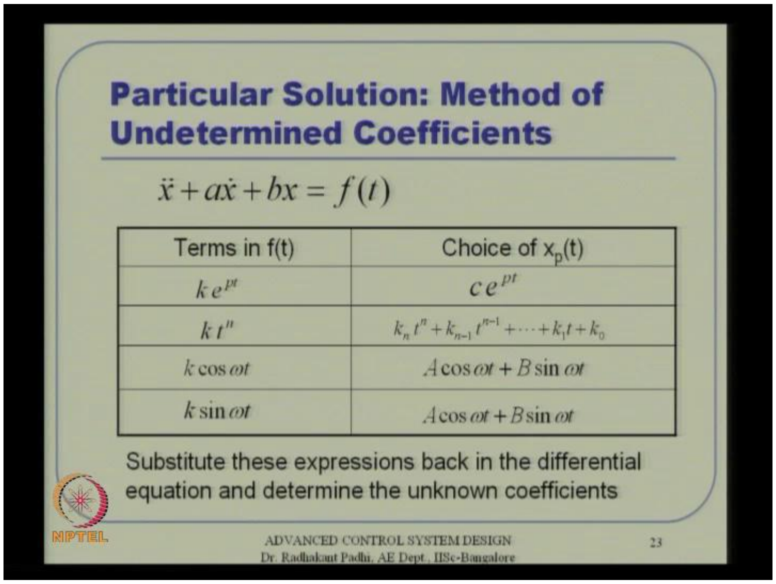

Then we move on to particular solution. So, far you have been taking about second order equations homogeneous part of it. Now, suppose, you have a particular forcing function essentially f of t and remember this kind of equations are very useful in control theory, because later on we are mainly interested in designing this f of t that is the control that is the control problem essentially. So, given an f of t we know how to find a solution, but later we will see how to design this forcing function. So, that this differential equation behaves in a way that you really want it to behave. So, that is I mean that is kind of an inverse problem, but all control problems are typically inverse problem actually, but we will not worry about that part that part of it right.

Now, we will discuss, suppose, given a function f of t in specific forms then what is the solution and essentially this again, there is a method of undetermined coefficient. That there are various methods available for standard functions actually. So, we will just talk about undetermined coefficients. So, this particular suppose, you have a term f of t like this exponential term then you also select an exponential term that way remember e to the power p t also result in e to the power p t it does not the exponential part remains same. Now, if you have a polynomial thing k e to the power k t to the power n. Then you account for the polynomial starting from t to the power n then t to the power n minus 1 n minus 2 all the way up to t to the power 0.

So, if you have just one term here it will lead to entire term there that is the that is the idea there then, if you have either cos or sin or any combination of that then it will essentially have you have to account for both sin and cos together. If it is cos you cannot talk only cos here it has to be cos and sin, if it is sin also it has to be cos and sin together basically. So, all these coefficients that are that are popping up here for example, this a and b here and then k and k n minus 1 all sort of things there all those things becomes undetermined at this point of time.

So, we have to substitute the these solutions back in to the differential equation that will result in some sort of a I mean the coefficient being equal to you have to make it equal to 0 sort of thing. So, you have to get some number of equations will pop up as number of unknowns actually. So, then we will be able to solve it from there.

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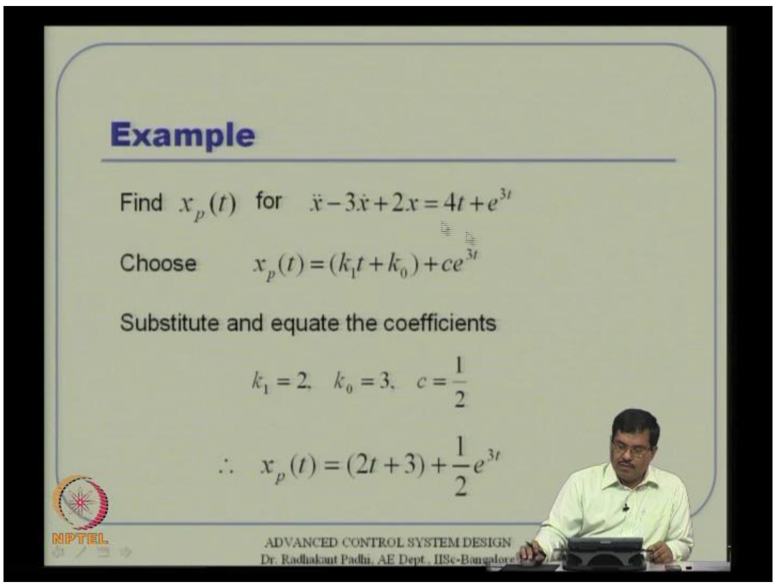

So, we will quickly go through small equation I mean small example here and that example is something like this see this x dot x double dot minus three x t dot plus 2 x equal to something like that, homogeneous part we will assume that we know already. Then the particular part is there is a polynomial term that for that term assuming some sort of a polynomial starting from t k 1 plus k 0 and there is exponential term. So, there is c times e to the power 3 t. So, three remains three here. So, what are unknowns here k one k 0 and c really.

So, while substitute this solution back into the differential equation and then equate the coefficients actually whatever, coefficients will pop up I will equate the coefficients on both sides and then try to solve it. So, I will solve the solution turns out to be k 1 2 k 0 3 and c equal to half sort of thing. So, the entire solution x of x p of t is this part of it two t 2 t plus three plus half of e to the power three t. Remember the solution the complete solution is not yet done that means, you are only getting the particular solution. So, you have the homogeneous solution also there, then put them put them together x homogeneous plus x particular then that will have another two constants that is that c u 1 c 2 sort of thing or a b whatever, it is.

And then you use the initial condition that is given to you for the complete solution. And then solve for those coefficients actually, then as then only you will have a complete solution accounting for initial conditions as well. So, that is the procedure actually, that is all about kind of how do you go ahead and get your solutions and all that for linear differential equation both in first order and second order.

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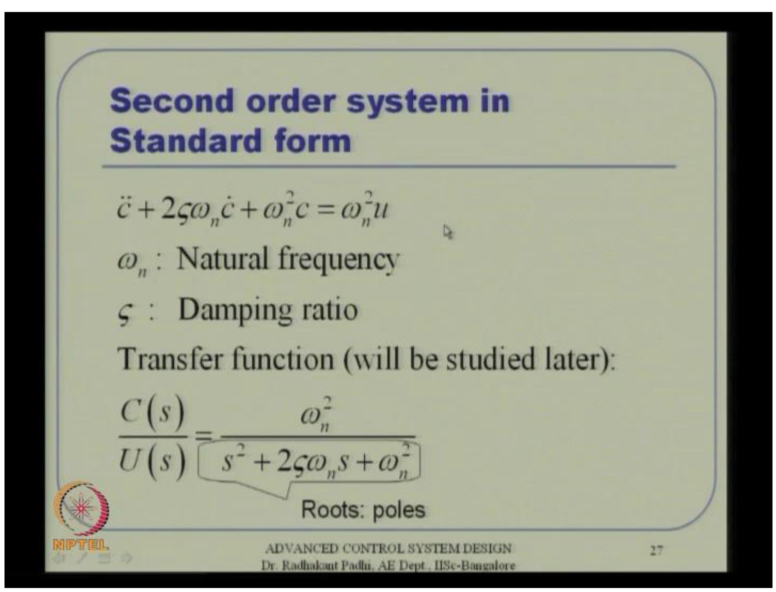

And before, closing this lecture I though we will just revisit this second order differential equation from a different perspective very quickly we will probably do little more in the next class as well. So, if you see classical control books for example, they will typically start with this kind of a differential equation directly there will and this C being the output actually, and u being the input part. So, you have c double dot plus 2 zeta omega and c dot plus omega n square c is equal to omega n square times u that is the standard form.

So, if you really talk about this particular standard form, where omega n is a natural frequency and zeta is the damping ratio. Then we have not discussed the transfer function yet we will just study about that probably next class. But assume that you can apply Laplace transform here that I mean, most of us know that anyway then essentially it leads to some sort of a transfer function C s by U s is in this form. Where the, if I equate this denominator to 0 that is that the solutions of that are essentially the roots I mean, the roots of that equation are essentially the poles of the system essentially.

So, very great amount of study will be done with respect to those poles of the system, because that is essentially which will govern the system stability properties and things like that. Now, let us study let us view that very quickly we will study in detail probably little more detail next class. So, what will happen in the response nature, if I really start with this particular differential equation, then what is my solution close form formulas will be available I mean these are standard text book material.

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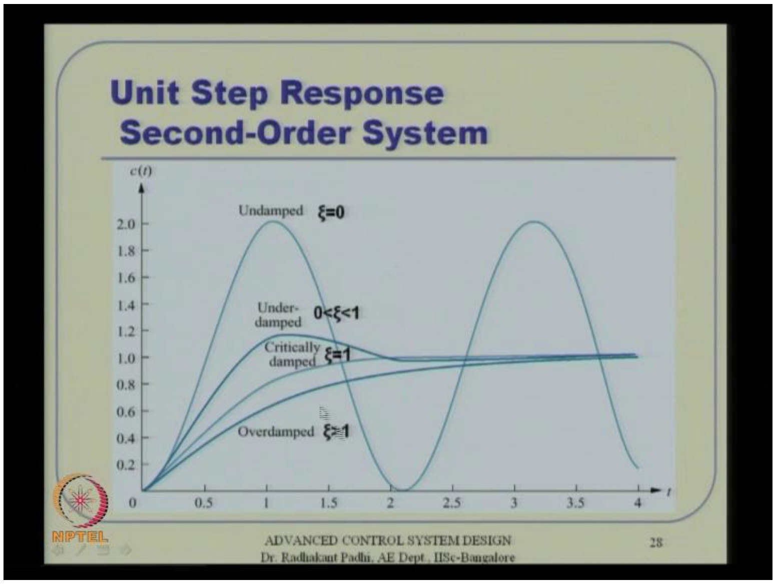

Anyway but very quickly what will happen is c of t if I really want to plot it across time then this is some sort of a graphical idea that I that pops up actually. So, what is happening here, there is a very clear cut distinction between what happens for zeta equal to 0 and zeta is greater than 1 zeta equal to 1 and in between basically. So, if it is zeta is equal to 1 then what happens it is something like called critically dumped that is in between over dumped and under dumped sort of thing but if zeta is equal to 0 that is there is no dumping. So, essentially this zeta plays a heavy role in dumping out the solution decaying the solutions to 0. Because the solution will contain e to the power minus zeta omega n t and things like that in there in the solution part of it if you see that.

So, that clearly pops up here we so, if zeta is 0 that is really undumped. So, that means the solution will never decay it is a keep on oscillating that way. And however, if that is anything other than 0 that means, let us say it is a number but it is somewhere, in between 0 and 1. Then the it will have some oscillation in the beginning, but it will essentially decay out later into stabilize somewhere. And if zeta is greater than 1 there will be oscillate there will not be any oscillation it will almost feel like as though you are operating some sort of a first order system oscillation. And zeta equal to 1 that is a very I mean just took the boundary.

So, it is neither underdamped nor overdamped sort of thing. So, but this is kind of a theoretical thing I mean in practice zeta exactly equal to one it may not happen actually. So, will be either in the this side or that side of the story so, that is what happen actually.

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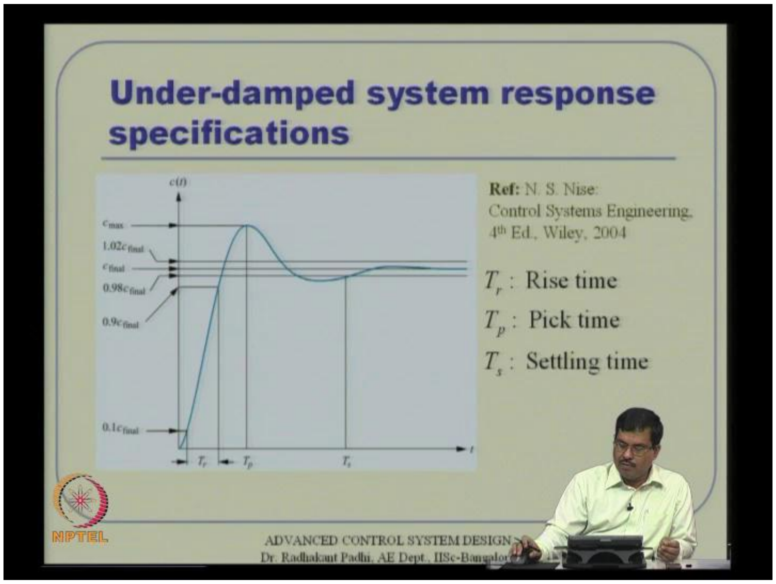

Now, if you really want to analyze the solution little more then, if I really go for this steady I mean this solution response for a step input sort of thing then this solution turns out to be something like this. So, if I have time behavior and things like that typically for underdamped system remember that most of the systems are typically underdamped and even though we want to design a control system we want it to behave like under dumped system also. There are variety of reasons for that, if you really want to make it behave like an overdamped system the response becomes very, very, very sluggish. So, you really, do not want to take that much of time to for the system to stabilize and things like that actually.

So, typically underdamped systems are of our interest. So, let us see that a little more closely actually. So, the solution turns out to be something very similar to this. Then there are various of various concepts first of all there is a rise time, rise time is we start with 10 percent of the solution and let it go to the 90 percent of the solution. So, that part of the time whatever, time it takes to reach that is essentially rise time. And the pick time is whenever; I see the first pick, because subsequent picks will be lesser and all that. Whenever, I see the first pick starting from 0 what about time I is gone there that is kind of a pick time.

And settling time is something very close to the first order system we essentially monitor the solution and tell the solution is within let us say 98 percent of the final solution. However, once it enters once it is 98 percent the subsequent values are within 98 percent only I mean it cannot go beyond that. So, even though the solution enters to this domain here for the first time I am not accounting for that time. I am accounting for that time, because subsequently the solution will not come out of this form that is the whole idea. So, that is essentially the settling time here.

And formulas are available in fact, if you know really this zeta and omega n essentially know this system and of you can write this rise time pick time settling time from the solution. That the solution close form formula and all that available we will discuss, that probably next class. But from the solution you can essentially find close form formulas for these times.

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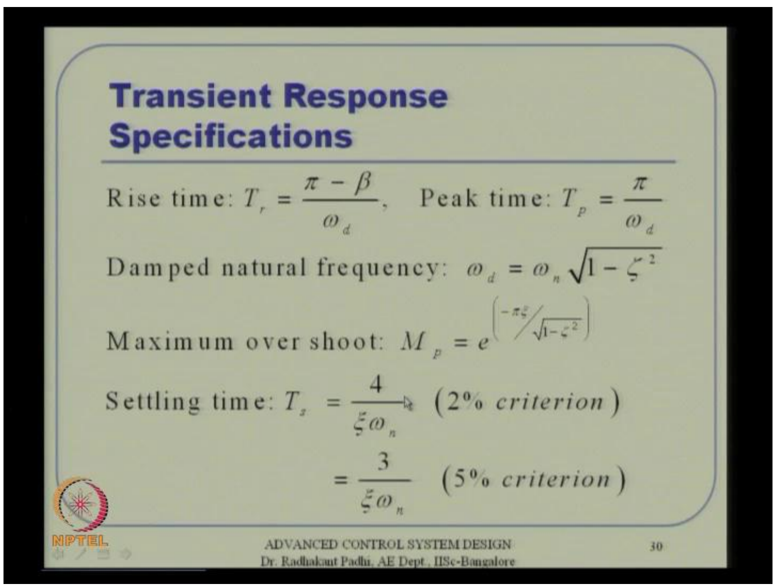

So, rise time turns out to be something like this where beta is an expression I mean that the expression is available to us. Pick time is like pi by omega T, where omega T is damped natural frequency and given by that form percentage over shoot is like that. And settling time typically turns out to be like that as the 2 percent criterion that means, ninety eight percent criterion sort of thing or 5 T 5 percent criterion means 95 percent of the final solution. I mean, if I go back to that picture instead of 98 percent I will talk about 95 percent. So, there are the settling time will be slightly lesser in that sense, but the default value is probably 2 percent criterion. So, 4 by zeta omega n this is not very difficult to remember some of this formula either.

So, settling time is four by zeta omega n and pick time is pi by zeta omega T it is kind of easy to remember. And some of these expressions are actually useful while, we design both linear control system as well as non-linear control system, when you go towards that towards the end of this course. We will see that these expressions are really handy when try attempt to design some control system actually, because that is the time domain specifications that we will start with the control design approach will demand that we assures some sort of a settling time.

And we assures some sort of a pick time we assure some sort of a percent over shoot like that. So, this little formulas that we are studying right now is not really text book material in that sense. We it is actually used heavily in practice both in linear design as well as non-linear design. So, let us just pay a little more attention at this point of time how this these things are so powerful.

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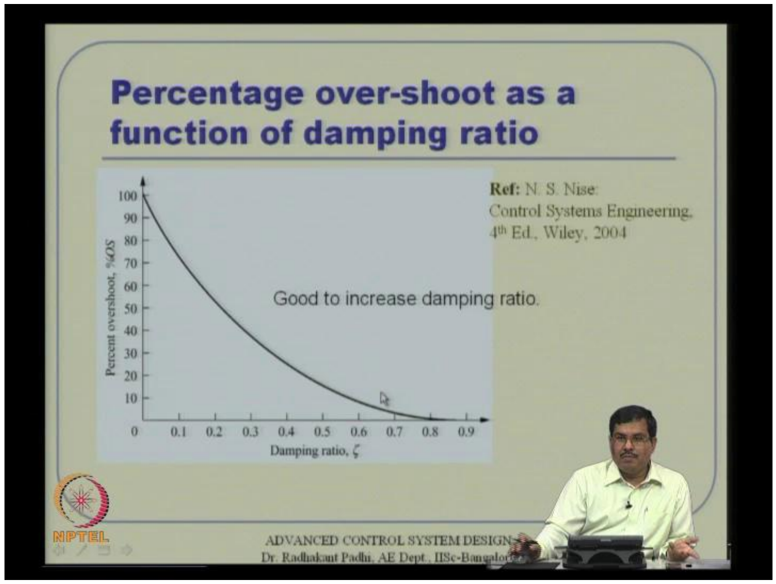

Now, one more comment that comes into picture let us say I will just plot this percentage over shoot I have a formula now. This, what is that there is a there is a percentage over shoot sort of formula available to me. So, I will take this help of this formula and plot it against, damping ratio then it turns out that it decays like this that means, what it tells me the moment I increase damping ratio the percentage over shoot is reducing. So, that means it is good to increase damping ratio that this plot tells me like that. So, when I design some control system do I pick up some sort of a high damping ratio, if I really pick high damping ratio. Then the percentage over shoot is minimum, but engineering say it is all about compromise.

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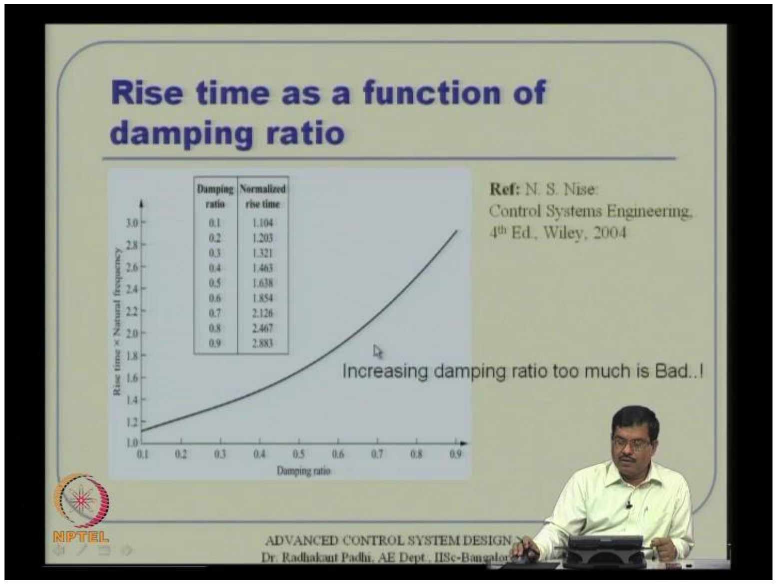

So, see the next plot and next plot it turns about how about plotting some sort of a rise time I mean, it is kind of a scale rise time (( )) rise time into natural frequency sort of thing this is a constant number anyway. So, rise time will turn to increase actually, that means the moment to increase damping ratio the rise time becomes more and more that means the response becomes very sluggish actually. So, on the one hand you want to have some sort of a less percentage over shoot, but then you also you have to live with the fact that it take enormous amount of time to reach there. So, you do not want that either. So, you have to select some number, which is kind of a good combination of good compromise between the two and typical choice turns out to be around 0.7.

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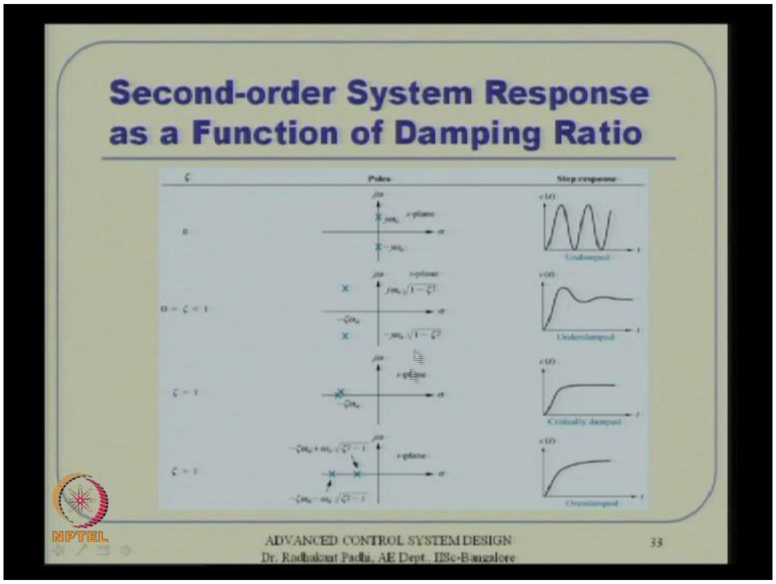

So, that quite kind of now just kind of a summery part of it, if you see this these pole locations and things like that either there can be an imaginary axis then the solution turns out to be like that that is under damped I am sorry undammed. And the if it is like clearly in the left outside it is clearly stabilizing. So, it will have a underdamped behavior it will just oscillate for a little time and then stabilize. If it is a double pole then there is no oscillation it will be some sort of a I mean, if zeta equal to one that means, it is critically damped sort of thing, if the solution will turn out to be double pole basically.

So, anything away from the real line has some damping effect actually, that is the message there otherwise the solution the poles are exactly on the real line then there is no oscillation. So, it is like critically damped. Then, if zeta is really greater than one then the poles will be distinct on the real line. So, that is when we will have this over damp sort of ideas. So, the response when you see something like this you can also guess where these pole locations are roughly that is the message here. So, with this summery probably I will stop here in this part of the lecture we will carry further this kind of ideas in a more in a transfer function approach in the next class.

That will we will start with some sort of a review of Laplace transform and then apply for some differential equation to arrive at a transfer function. And then using the these pole-zero ideas we will introduce and then study the same. This same solution response again, the nature of the solution response and all that in little more detail about what are the variation that you can introduce. And what are the implications that you can have as poles move away in when particular directions and things like that; we will also introduce zeros of the system later. Then tell if there is no 0 that is the that is the response then is there are zeros then what happen.

Now, we have discussed about first and second order systems here, but systems are not really first and second order, we know that they are higher order systems. So, there is idea of suppose, you do classical control design then considering higher order systems are also we are not capable of handling that in a very good way. So, one approach is dominant poles that means, whether there are like two poles that are very close to the imaginary axis, so that I can really neglect some of this poles that are far away from them so, those are the concepts that are available. So, we will discuss that in little more detail in the next class.