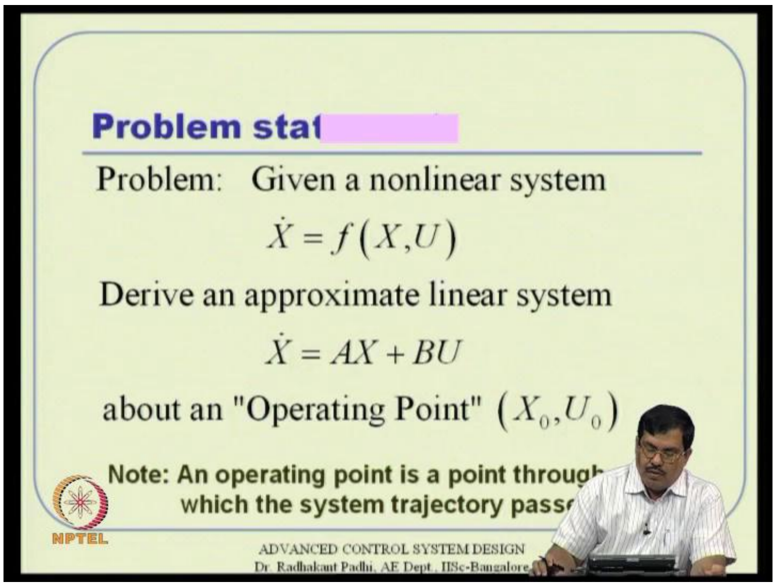
Advanced Control System Design
Prof. Radhakant Padhi
Department of Aerospace Engineering
Indian Institute of Science, Bangalore

Module No. # 07
Lecture No. # 16
Linearization of Nonlinear Systems

Hello, everyone we will continue with our lecture series and previously, we have studied various state space state space representations both for linear and non-linear systems, but if we want to design a linear control system or even want to analyze the stability or controllability things like that for linear systems, first thing to do is linearization of non-linear systems because most of the real systems are non-linear anyway.

(Refer Slide Time: 01:03)



But the tools and techniques that we have are valid for linear systems. So we have to have a some way of a linearizing a non-linear systems so, that is the topic of this lecture actually. So we will see how to start with a non-linear system and what you mean by Linearized system actually, and before even I proceed the point here is a many I mean even if it tells that the systems are linear systems.

I mean most likely it is actually a linearized system. So, that means in reality there is no linear system per se in very rare case you may have it actually, but by implicitly you mean linear systems means linearize about some operating point and things like that actually. So that is why we will study in detail how to get these linear systems actually. So what is the problem, problem statement is something like this given a non-linear system like this X dot equal to f of X U we want to derive an approximate linear system remember this is an approximate linear system that means the system that you are talking about never replaces this system. Well this is only an approximation about an operating point X 0 U 0. So, this particular thing what you see here X dot equal to AX plus BU can have different values of a and b as I mean as you keep on changing this operating point X 0 U 0.

So this non-linear system can remain same, but the linearized system can be I mean will also give an example, linearized system can be different when you change the operating point actually. Now, also remember the my operating Point when we do not necessarily mean that it is a equilibrium point. Equilibrium point just happens to be one of the operating points actually. I mean so what is operating point now. so by definition an operating point is a point through which the system trajectory passes.

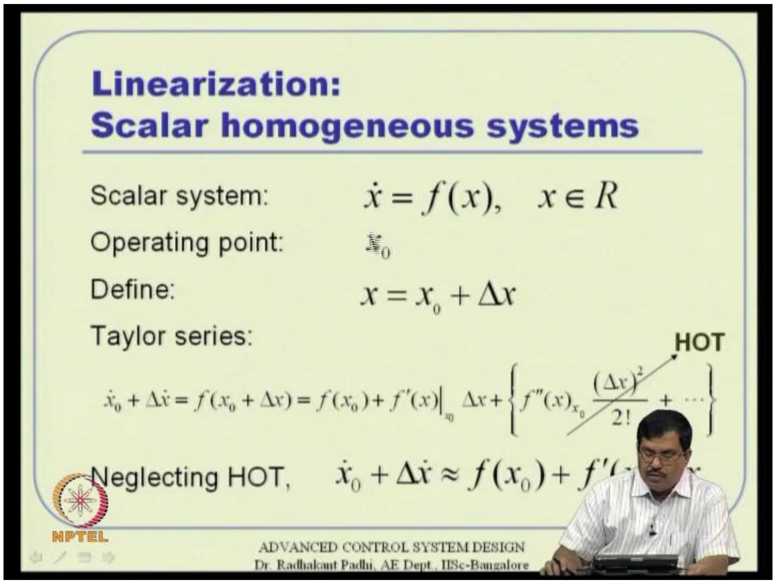
That means if the system trajectory starts with some initial conditions somewhere and then it follows a certain trajectory, associated with that there is a control likes a noise bell and you take any pair anywhere actually and you consider that as X 0 U 0, then about which the system dynamics where A and B matrices can be found out and that is what is called an operating point.

Now equilibrium points are special class of operating points because, if the system is stable ultimately the system goes to the equilibrium point and the by definition equilibrium point will satisfy the differential equation anyway. If you substitute an equilibrium point then this will become 0 and this equation will also become 0 actually, that is that is the definition of equilibrium point anyway.

So equilibrium point happens to be just an operating point, but in general we linearize the system dynamics about operating points actually. And by definition operating point is a point is a point through which the system trajectory passes. That means you can also talk about a nominal trajectory all together x of t of t that is a nominal trajectory about which you can keep on getting a time varying linear system a of t and b of t you will resolve from there.

And then you can talk about a time varying linear system with respect to a nominal trajectory not a nominal point really. So that is also possible actually. So let us see how to do that before you proceed we will start with a very simple idea. Let us say this is there is a scalar system x dot equal to f of x x is just a scalar there is an operating point somewhere. The way to find the operating point is a we will not talk too much on that.

(Refer Slide Time: 04:21)



If we if you can if you do not have anything mean then probably naturally the points to select is equilibrium point and by equilibrium point what you mean what you need to do is just put this equal to 0 X dot because that is the definition of equilibrium point and then associated with that you do some assumptions about this will result in let us say n dimension I mean n equations actually.

But X is n dimensional and U is m dimensional remember that. So, we need to have m further assumptions for that. So, that the number of equations and number of variables will be equal from which you can solve this system actually. This system will consist this system consists of n plus m variables and this equation is only n equations. So we need to have m approximations.

So, that this number of equations and number of variables are same. So, we can solve it from there actually. So, these assumptions can be either in control variable or in state variable for example, if you want this straight and level flight in flight dynamics we consider altitude being a being known to us that mean altitude is a straight variable.

We can also talk about it is a straight and level flight so, I do not have lateral dynamics so delta n delta r that means Aileron and Rodriguez surface reflections are 0. That means these two equations along with h equal to some value. Those becomes kind of input to this equation that means those are known to this and then you can put this equi system equal to 0 and probably solve it.

I hope this is clear. So, now let us continue with that so what you are telling here is just a scalar system. We have an operating point. So we want to find a linearized system about that actually. So, what are you doing what you are trying to do is perturbation analysis really so, any point of, I mean any point of time let us say we consider this x as a perturbation about x 0 remember x 0 is an operating point.

So, that means x is equal to x 0 plus delta x. So, that is what we are interpreting actually and this is not approximately equal to this is equal to this is x equal to x 0 plus delta x that is our definition actually. So, we want to find out a dynamics for delta x using this original dynamics that is what that is what our aim actually. So how do you do that? So, obviously, the this if we substitute x 0 plus delta x there is a term called x 0 plus delta x here f of x 0 plus delta x. So, obviously when we see this term something like this x 0 plus delta x we intend to expand it using Taylor series actually.

So what you what you what I am doing here you are substituting this expression like in there so, the left hand side is nothing, but x 0 dot plus delta x dot that is what it is. And then that is equal to f of x 0 plus delta x and this f of x 0 plus delta x you are expanding using Taylor series. So, what are you getting if you are using Taylor series you expand it about x 0 then, the first term is f of x 0 second term is del f by del x that means in this case it will be d f by d x f dash of x evaluated at x 0 times delta x.

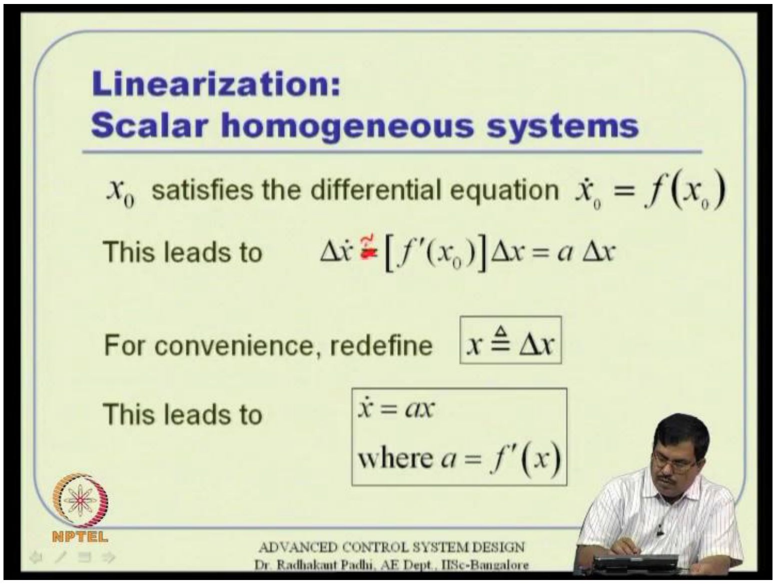
Then, this second derivative evaluated at x 0 times delta s delta x whole square divided by 2 vectorial like that it will continue that is the standard Taylor series anyway. Now,what you are doing here is we are approaching I mean we are assuming that delta x is small, that is why this perturbation theory and all that is valid, because the inherent assumption is delta x is small delta x is small delta x is square and then cube and other things are small and those are also divided by 2 factorial 3 factorial like that actually. So the entire term becomes a small quantity under that assumption that delta x is small.

So, that means these are higher order terms or what we typically call as hot terms that means higher order terms. Normally, we do not want to touch that actually. So that means we neglect this high order terms actually. So what are you left out is something like now this equal to when you when you leave out this one when you neglect this, this equal to is no more valid so, you have an equal I mean approximately equal to term now. So that means neglecting if you once you neglect high order terms the left hand side what you get here is approximately equal to this only the first term plus the first order term this is also called the 0 third order term there is no delta x.

That means delta x to the power 0 basically. So up to first order term you keep then it becomes approximately equal to. Now what is the observation, observation is x 0 is an operating point that means if, I substitute x 0 over here it satisfies the differential equation by definition x zeroes is a point through which system trajectory passes that means x zeroes will certainly satisfy this equation exactly basically.

That is the definition of system trajectory any point you select from the trajectory, it satisfies the differential equation otherwise the system does not does not pass through that actually. That means this x 0 point exactly satisfies this equation. So what is what is the implication this first term and this first term will cancel out that this is exactly equal to that. So that is what you do here and what are you left out with this term being approximately equal to that term.

(Refer Slide Time: 10:12)



So that is what you are doing here, so that the x 0 satisfies the differential equation exactly so that is why this is equal this is valid and this leads to this equation actually. Essentially, well you can I mean theoretically speaking this is no more equal to basically so what you can do I mean what you are telling is approximately equal to.

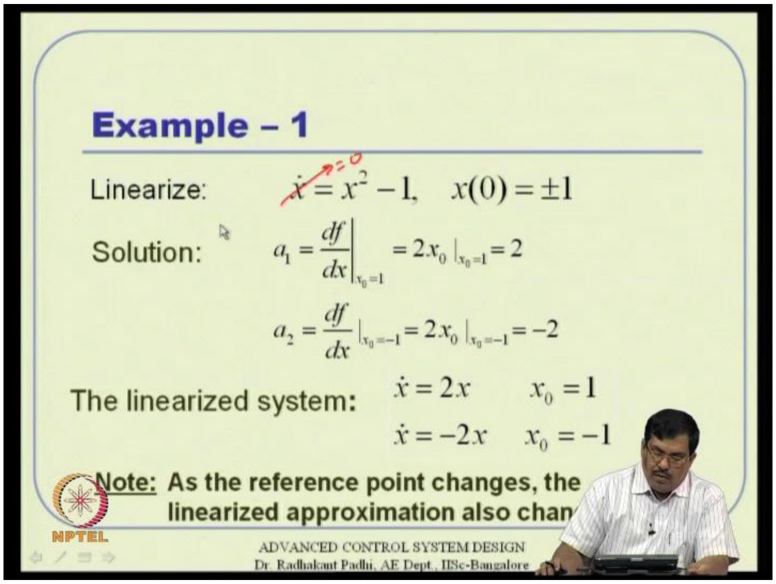
So, delta x dot is the approximately equal to that that term I mean f dash of x 0 times delta x. So, we define this term with f dash of x 0 is some quantity remember this is actually scalar problem so that means this is just a number actually. So, this number we define it as a and then, we are so that means this is our for convenience actually so we just define this quantity as a and then we are left out with a times delta x actually.

And then we would so, that means what, we are getting a linearized system as delta x dot equal to a times delta x that is the linearized system actually approximate system around or about x 0 operating point actually. But we do not want to keep on writing delta x dot delta x delta x dot everywhere actually. So, under the implicit assumption we redefine the variable and we do not want to redefine to a different variable. Let us say we simply little abridged of notation we do with the assumption that mean with the assumption that we know what you are doing here.

That means x is redefined as delta x actually here, so in that sense what you are getting x dot is ax actually where a is nothing but f dash of x, but remember this x dot whatever x you are getting here this x is nothing but this delta x and this delta x is meaningful only when you know ax 0. So this particular delta x what you see is meaningful only with the knowledge of x 0. So, that is sometimes we keep it silent we implicitly assume that is origin that is 0 0 0 and things like that. But always remember that delta x is not equal to x in general and delta x is meaningful only when the knowledge of x 0 is available to us actually.

So, but anyway what you are getting here ultimately is x dot is equal to ax we keep on writing we do not want to write approximately equal to all the time. So, we write as far as linearized system is concerned x dot equal to x where a is nothing but this term well this term evaluated at x 0. So, that means this is actually x 0 this one. So f dash of x that means this is actually f dash you carry out the f dash of expression then, evaluate it about x 0 that is what the term actually that that is what it means. So this is all about a scalar homogeneous systems. So we are not confined only with that. So we will slowly build up towards more general systems and all.

(Refer Slide Time: 13:20)



But before going there let us go through a simple example actually. So we have let us say non-linear system which is x dot is equal to x square minus 1 and also remember x I mean this operating points are given to us as plus or minus 1 which is also happens to be equilibrium point.

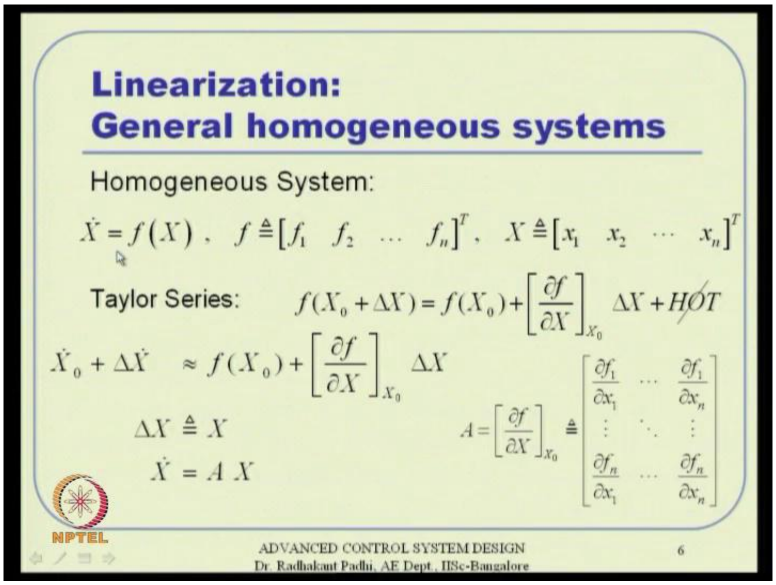
By default, these are taken like that most of the time so what why I get how are you getting that you are putting simply that this is equal to 0 that means x square equal to 1 and hence x equal to plus or minus 1 and if you substitute that it will also satisfy the differential equation obviously, so it happens to be there are 2 equilibrium points for this problem one happens to be plus 1 1 happens to be minus 1. And we know we already we have derived this equation so we do not have to keep it doing again and again so we know tha t x dot equal to a x is a linear system where a can be given like that so we directly evaluate a here. So let us say in one case where you we are taking x 0 equal to 1 that is a 1 and when you take x equal to minus 1 that is a 2. But both of the formulas are del f by del x evaluated at x 0 equal to 1 or del f by del x evaluated at x 0 equal to minus one.

What is del f by del x here 2 x actually. So this is this is evaluated 2 x evaluated at x 0 equal to 1 and this 2 x evaluated at 0 equal to minus 1 actually. So, that means what you are getting here this in this case it is 2 and in this case it is minus 2. So, the linearized system is like even though the non-linear system is whatever you had in the beginning for the same non-linear system when the equilibrium point changes or the operating point changes you have two different linearized systems actually.

So this linearized system is valid about this operating point and this linearized system is valid about that operating point. So the message here is again I again and again I repeat as the reference point changes the linearized approximation also changes actually. So you can also visualize this as a problem for let us say inverted pendulum actually like if you have inverted pendulum you know there are two equilibrium points one is vertical equilibrium when the when the pendulum is down and it is another is vertical equilibrium when the pendulum is up actually.

So, about vertical equilibrium when the pendulum is up then the linear system will be different, when it is down it will be different also and also can you can visualize a problem where it the pendulum is was inverted from the beginning and then it slowly goes to the vertical down actually. So, that means about the entire trajectory you can also talk about a time varying linear system wherever you are the pendulum moves actually about that you can found out the corresponding theta. And then about that you can linearize the system actually. Remember that that be a problem in theta dot and all that where the theta is taken from a reference line actually. Anyway so that is the message that I wanted to give in this particular example actually.

(Refer Slide Time: 16:24)



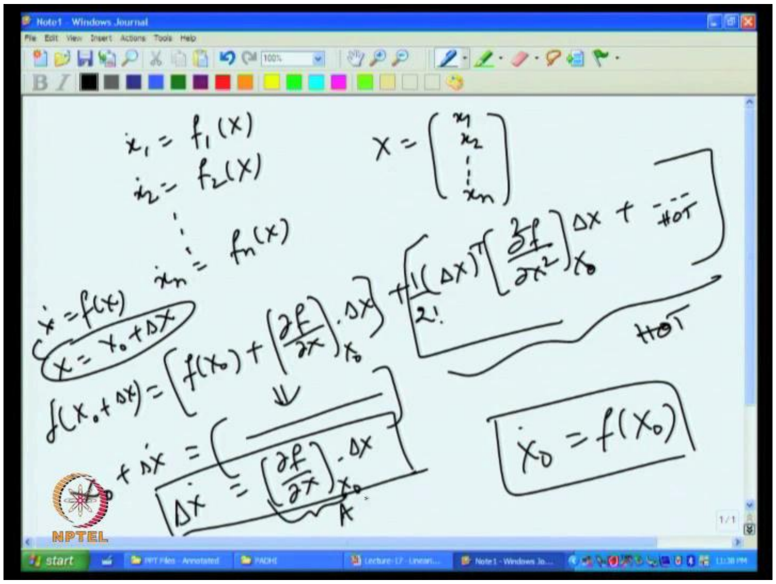
Now, let us continue with the little further development where you are interested in a vector problem now, but still it is a homogeneous system that means there is no control term. With control term that will be our ultimate objective actually. So, what you have here you something like X dot equal to f of X where X is n dimensional and f is also n dimensional remember X dot is n dimensional also basically. So, what you are telling here x 1 dot is something like f 1 of x and x 2 dot is like that actually. So what you are telling here we are telling here that x dot x 1 dot is f 1 of X x 2 dot is f 2 of X like that actually up to x n dot equal to f n of X where x is nothing but x 1 x 2 up to x n actually.

So, that is the problem that you are talking here. Each of this each of the differential equation what you see here, each of the differential equation can be like a function of all the state variables actually. So let us go back to that so that is what you are doing here x dot equal to f of x where f is f 1 to f n compact notation I have written like a a rho vector with a transpose essentially it is a column vector.

X is also a column vector that way by default whenever there is a vector we interpret that as a column vector actually. Now, what you are doing here we also have the very similar analysis basically, what you what we really interpret, want to interpret is again very similar. What if x is equal to X 0 plus delta X where this delta x is a deviation about X 0 X delta X is the deviation of X about X 0 that is the meaning of delta x actually.

And again we will end up with this expression f of X 0 plus delta X. So what you do here again we expand that that is nothing but f of X 0 plus del f by del X evaluated X 0 like that. And then again we will remember these are now this is now a Jacobean matrix actually. And then higher order terms will consist of Gaussian matrix and things like that. It will consists of see for example, the first order term in the higher order series will consist of something like something like this.

(Refer Slide Time: 19:07)



Del f by del X f of X 0 plus delta X is equal to (( )) this is equal to f of X 0 plus del f by del X evaluated at X 0 times delta X plus the second order term if you want to like write actually. This will become delta X transpose del square f by del X square this is hessian matrix now evaluated at X 0, times delta X whole divided by 2 factorial that is that is the term actually. Then, it will continue with the third order thing which you can or which you have to write in a series of matrix and things like that so these are you can talk about higher order term.

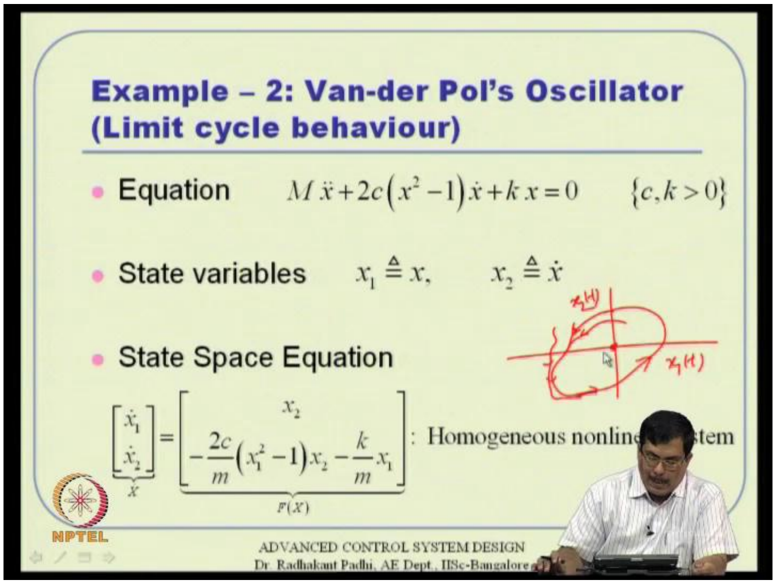
As far as the linearization is concerned we consider everything as higher order terms actually. So what you are left out I mean that means we have started with X dot is f of X and you are considering equal to X 0 plus delta X and then this one is like that. So that means if you substitute this term back in here what you are getting you are getting X 0 plus delta X dot X 0 dot plus delta X dot is equal to all these terms what you have here, entire term actually.

These higher order terms are neglected any way. So, what you are getting here is something like this term will come here actually. Higher order terms you have neglected anyway so what are you having then now you are telling that X 0 dot is equal to f of X 0 because that is an operating point that is there, so if I i can probably I mean I can substitute that and cancel out. So what I have left out is delta left out is delta X dot is equal to del f by del X evaluated at X 0 times delta X.

That is that is the differential equation that I consider as linearized system with that assumption that this is nothing but aero matrix actually. So we are just we will just do that that is what you are doing here actually (Refer Slide Time: 21:32). So you started with X dot equal to f of X X is nothing but X 0 plus delta X and then this f of X 0 plus delta X is that and we neglect higher order term we are left out with that. And then we are telling that this is a operating point X 0 is an operating point so this fellow is equal to that so you cancel out that one.

And again we redefine that actually right, like what we did before we redefine delta X equal to X for our convenience and you are left out with X dot equal to delta X dot now becomes X dot X dot equal to AX where A is this matrix actually this is evaluated that way. So, fairly straight forward so what we did for scalar is also valid for vector the only thing we have to remember del f by del x is no more a scalar quantity it is defined like a Jacobean matrix that way and that that will result in a a matrix actually

(Refer Slide Time: 22:259)



So, let us see an example again, well these are these this example and there are 1 or 2 examples that we will follow we have also discussed that in while you are discussing the original state space equation actually. There is one class you have discussed about state space form of system dynamics and some of those non-linear systems I will discuss here actually and then try to linearize about some operating point for those systems.

So, that is one of the examples that we discussed there is Van-der pol's Oscillator well this is a nice problem in the sense those of you want to know something which is which is like it is satisfies something like a what is called a limit cycle behavior. Limit cycle behavior is something like if you take this is remember this x 1 is x and x 2 is x dot that is what you have already defined it x 1 is x and x 2 is x dot and then with that standard form we can write that x 1 dot x 2 that first companion form whatever. So we defined x 1 is x and x 2 is x dot. So, x 1 dot is x n x n x 1 dot is x dot is x 2.

So that is how it is and x 2 dot is x double dot x double dot you can write everything to the right hand side and tell x 1 is and x 2 is x dot, so I can substitute that and I will get something like this that is how we have done in one of the previous classes. So, this is the non-linear system and the beauty of this system is if I if I just plot these variables this is called space band diagram and all that then x 1 of t in 1 axis.

So, x 1 of t in 1 axis and x 2 of t in other axis then it results in something like you can see that it can result in a trajectory actually in a close trajectory and it will also have like a 0 0. For example, if you substitute x 1 0 x 2 0, then it will also satisfy this differential equation let us x 2 is 0 x 1 dot 0 0 actually that is what you are telling so this is 0 and this is also 0 so 0 0 is certainly an equilibrium point.

But other than that if you try to solve this you still put 0 0 here left hand side and try to solve this equation, then it will also result in a closed loop trajectory sort of thing. And this is also like an equilibrium point actually any point on that trajectory will also satisfies this nice behavior actually that means… So, this system trajectory can keep on evolving like that it will never stop actually. These are some of these departures of non-linear system over linear systems actually.

Linear system we have only I mean 2 options either any trajectory that starts with somewhere can either go to origin or it can go to infinity that that is the option actually. But here the trajectory the system trajectory it may so happen that it can start and still it will go until merge there and once it merges it will just evolve there. And it will start with that and it will merge somewhere and once it merges it will revolve back actually. That means this stable I mean there is some called stability of limit cycle behavior as well so that means if you start with any initial condition it need not necessarily go to origin in fact the origin turns out to be a non stability equilibrium point here.

So, other than the origin itself anywhere it starts it is supposed to converse to the limit cycle only. It will never go to limit cycle. So this is so that is why this is a bench mark control problem here we do not talk about any control here yet. We are not talking about any control input, but if we really want design a control lecture for this unless the control design is good the system trajectory will never go to origin actually. That is why one of the bench mark problems is Van-der pol's oscillator actually.

And there are some practical implications also, me of the springs and all the anyway coming back to this we want to probably design a linearized control system to make the system trajectory go to the 0 not to the limit cycle that is how our problem actually. We should know a linearized system first about the origin.

If you really want to design a non linear control system we should know a linear system about the about the origin which is happens to be equilibrium point here actually so that is our problem about the origin can we derive a linearized system for this particular system dynamics that is what it is we are talking here actually.

So what you are doing here x 1 0 x 2 0 these are 0 0 that means this the origin that you are talking here. X 1 is 0 x 2 is 0 also, we have the we have the I mean this a matrix formula which is like Jacobean matrix remember this is f 1 f 1 of x is x 2 and f 2 of x is whole lot of that so that is the that is the formula that I will use in this actually. So this will this will consist of del of 1 by del x 1 and then del f 1 by del x 2 the second term the second 2 by 2 1 element will contain 2 by del x 1 and the 2 by 2 will be f 2 by del x 2 that is how you have to do actually. So this is our f 1, this is our f 2 here so we have to just substitute that actually there.

So if you do that then obviously del f 1 by del x 1 is 0 this is x 2 del f 1 by del x 2 is 1 obviously, so that is so that is what you get 0 1 here now del f 2 by del x del x 2 let us say that is that is how we do the second one this is easy actually. So del f 2 by del x 2 this is only del x 2 term here so this all this will be 0 you are left out with that actually so that is that is the term what you get here and then del f 2 by del x 1 if you see one terms will come from here another term will come from here which is nothing but 2 c x 2 by m that is the coefficient into 2 x 1 actually.

So that is what you get here 2 c x 2 into 2 x 1 divided by m and then one more term coming from this 1 minus 1 k by m actually that is what you have here. So what you are having interestingly, it turns out that what you are having is the first if once you evaluate at 0 0 you have to evaluate remember that actually. So this 1 is all 0 this is also 0 this term happens to be just 0 so this is left out with 2 c actually.

So you are having a a matrix which is in this form so the linearized space equation talks about x 1 dot x 2 dot taken together is nothing but that actually so what you see here so there is a little bit observation also here if, I just write it in equation form x 1 dot is something like x 2 this linear system as well and x 2 dot for the linear system is nothing but 2 c by m into x

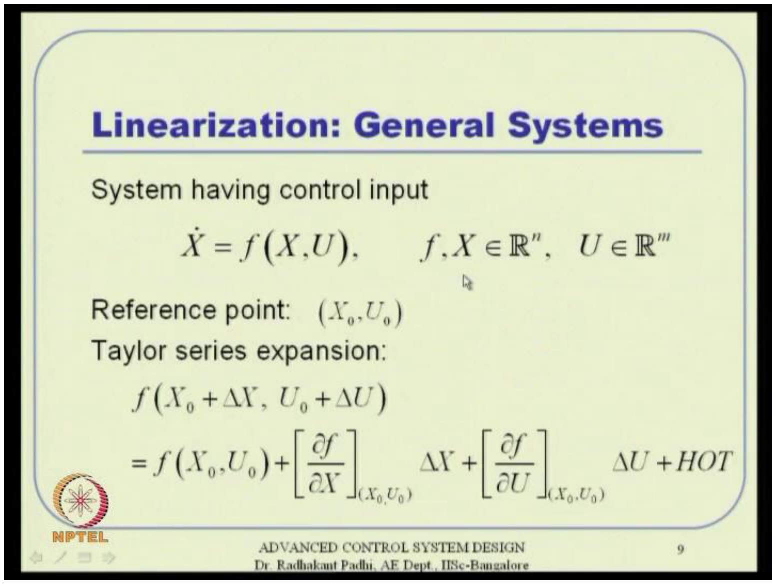
2 again 2 c by 2 c by m x 2 basically, so essentially what you mean this entire row becomes kind of 0 0 actually.

So in other words there are jump properties will get suppressed actually by doing this Euler systems are nice, but you also suppress various properties by doing this actually. Suddenly this if you see the non-linear system it is tightly coupled with x 1 remember that x 2 dot x 2 dot if you see this equation here it is really tightly coupled with x 1 x 1 appears here x 1 appears here all that actually.

What the what non-linear system I mean even though the non-linear system depends on that x 2 dot depends on x minus 1 only the linearized system x 2 dot depends only on x 2 which is actually I mean not a good thing to see. So linearization is not a very good thing for all practical applications that is the message actually in fact if you carry out further analysis and all it may also lose control ability things like that because the 1 root column is also 0. So I am not very sure that depends on the way the control will appear here we do not know which way it will appear, but if it if it may so happen that it may lose the system may not be controllable also using linear system theory, but it is a very nicely controllable using non-linear system theory we will see that later.

So this is all about vector thing now we will continue further discussions on what you do that in General Systems. That is our ultimate objective remember we started with that objective really so we have this system dynamics we have an operating point which is X 0 U 0 combination not just X 0, X 0 U 0 combination operating point and then about this operating point can we get that and that is what our aim was actually.

(Refer Slide Time: 31:31)



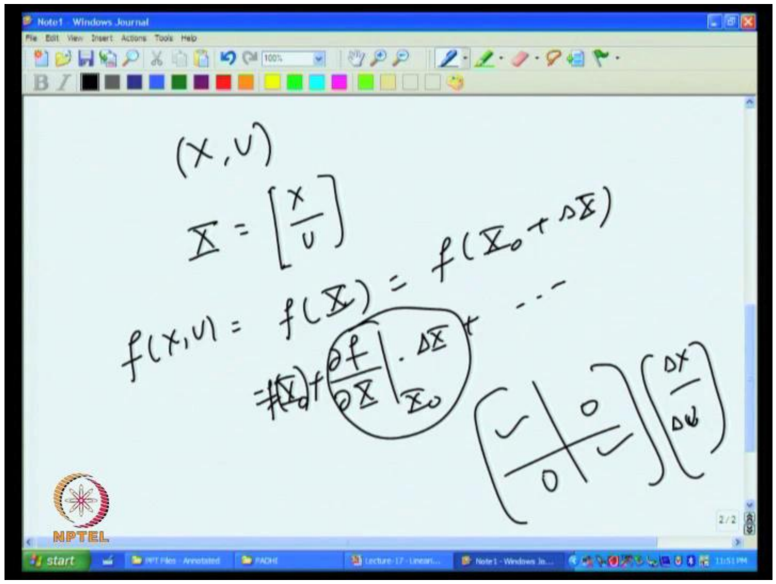
So now we will talk about the real problem the full problem actually. So obviously we are talking about the non-linear system when an operating point or a reference point is X 0 u 0 combination X is of m dimension and u is of m dimension. We know that the analysis is fairly straight forward again whatever you are done before is also relevant here. So what is happening here this right hand term will again result in this term with the with the same assumption that X equal to X 0 plus delta X and U equal to U 0 plus delta U.

So, we are interpreting this x as a perturbation about x 0 and U as a perturbation about U 0 delta x x equal to x 0 plus delta x and U equal U 0 equal to delta U so if you substitute this x and U this x and U combination here that is what you will result f of X 0 plus delta X comma U 0 plus delta U. And this particular term I can analyze I mean I can expand it using Taylor series that way so that means X 0 U 0 is the first combination then del f by del x evaluated at X 0 U 0 times delta x and then will be one more linear term del f by del U evaluated at x 0 U 0 delta U plus higher order terms.

Now, how do you all get this I mean that is also I mean you do not have to keep on doing. This in the sense that once you once you have this we can also derive that fairly easily actually. How do you do that? One way to look at the problem is something like that I have X I have U combination. So, I can also visualize another big vector something which is like

I will put them together X first and U next actually. Then, what I am telling here f of X U what I have is nothing but f of this big vector actually you know.

(Refer Slide Time: 33:41)



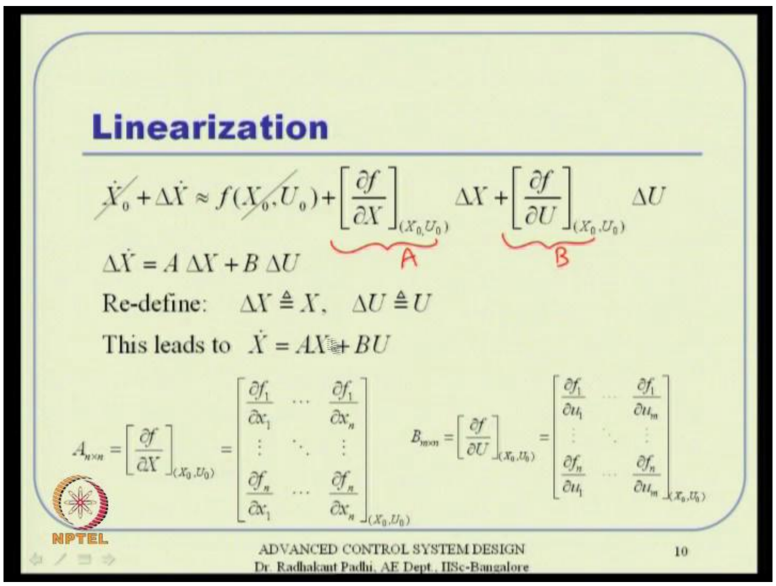
So I can evaluate this del f by del X rather obviously X 0 dot I mean if you want to evaluate expand that X 0 plus delta X big delta is like that actually. So, this is obviously f of X 0 plus this one evaluated at X 0 times delta X things like that. And then this term what you what it results in actually will be a big vector again this will nothing this is nothing, but delta X first and delta U next actually. And X and U are decoupled terms and all and if you see this partition matrix something is here something is here nothing will be there actually.

So, this particular term will get coupled with delta X and this second term will get coupled with delta e actually partition sense. So, that is how you will get the other one actually so we get of some term like this. So, this entire thing f of this X 0 plus delta X comma U 0 plus delta U you will expand it that way again the same trick we will neglect the higher order terms. And again remember this pair is an operating point, so that means this pair will exactly satisfy the differential equation.

So, that means X 0 dot is equal to f of X 0 U 0 anyway that is what you will do here, so we will neglect the higher order term and then make this equal to as approximately equal to, and after that this is this will exactly satisfy that, so that that is there we will cancel out and then we will be left we will define this as a matrix, and then we will define it that as b this is nothing but a matrix this is nothing, but a b matrix remember these are numbers because these are evaluated, some formulas evaluated at operating point X 0 U 0 pair actually. So these are always numbers actually.

There can be time varying numbers or time invariant numbers depending on various condition whether you have time varying parameters already or whether you are interpreting this X 0 U 0 as time varying trajectory sort of a thing nominal trajectory sort of thing either way. Either way it will resolve in time varying quantization of that. Anyway the same thing I mean after you neglect this you redefine this delta X as X and delta U as U. So, this linearize system we will conveniently write as that x dot equal to AX plus BU with the assumption always that x that this x is nothing but delta x and this U is nothing but delta U actually.

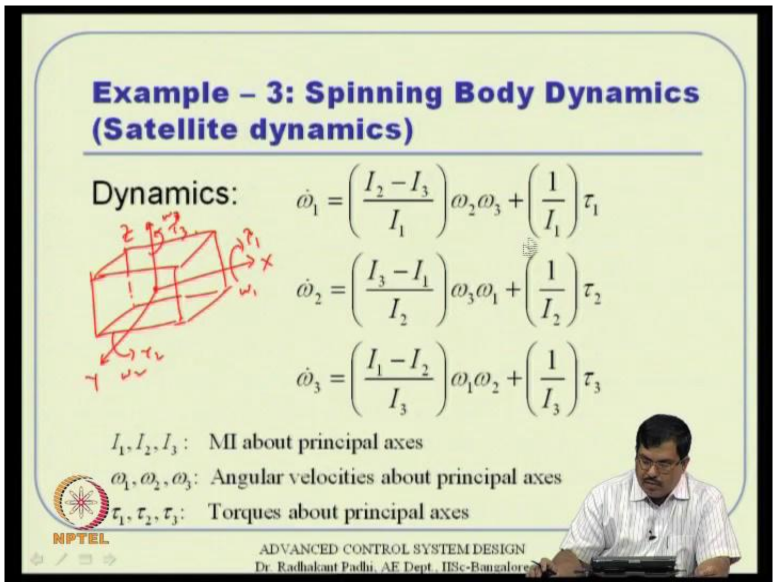
(Refer Slide Time: 36:00)



And a matrix is as evaluated before, but this will also be evaluated at x 0 u 0 pair now it is no more a function of only X 0 it is X 0 U 0 pair a matrix and similarly, b matrix will be X 0 U 0 pair actually. Now, also remember that a matrix is always a square matrix where b matrix need not be a square matrix this is the system dynamics which I will suppose you see this dynamics for example, X is n by 1 vector, so X is also n by 1 vector X dot is also n by 1 vector.

So the obviously, there is n by n actually. Whereas, U is an m dimensional vector, so the corresponding B matrix is actually n by m and most of the time b matrix happens to be non square actually.

(Refer Slide Time: 37:16)



I will continue with a new example this happens to be a this again an example we discussed in a in a while you are discussing this state space form of equi system dynamics actually.

So this happens to be a rigid body dynamics essentially, you can visualize this as something like this. You have a box sort of thing let us say some satellite something like that visualize something or it can be any arbitrary safe thing. So, you have a axis frame one going towards that, one going towards that and one coming outwards that let us say this is X this is Y this is Z sort of thing 1 2 3 sort of thing.

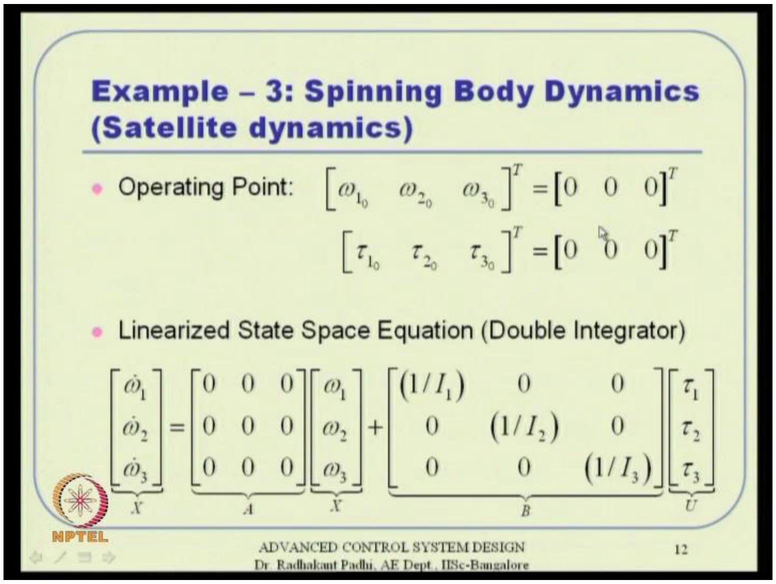
And if you simply apply this torque tau 1 here and then let us say tau 2 here and tau 3 here then you want to see what all system dynamics results and these are remember these are velocity level equations. And if you connect to the flight dynamics what you discuss before, these equations are nothing but like the velocity level components essentially in the in the aircraft dynamics we have interpreted that as p dot q dot r dot kind of a thing actually.

So similar thing you can interpret as omega 1 dot, omega 2 dot, omega 3 dot and this the resulting equation will be like this actually. And this I 1, I 2 and I 3 are nothing but moment of moment of inertias about principal axes and omega 1 omega 2 omega 3 are nothing but angular velocity angular velocities about the principal axes that means this is torque actually.

So you can also have this velocities that is what the dynamics you are talking about so there is a velocity omega 1 here omega 2 here and omega 3 here. To control that these quantities omega 1 omega 2 omega 3 we are applying tau 1, tau 2 and tau 3 which are nothing but torques actually. So, the resulting system dynamic happens to be like that obviously this is a non-linear system where cross there are multiple terms here actually. That means omega 1 dot is not a function of omega one really, but it is a function of omega 2, omega 3 and things like that actually. So we want to linearize this system dynamics. Let us say one of the objective is for control design is arresting the tumbling actually. That means if it keeps on rotating in a different direction one direction it rotate that way and the other direction and then the other direction it will all get coupled actually. So it is something called tumbling effect actually. So, if you want to control that that tumbling effect using linearize system dynamics we really need to have a linearize system to begin with actually.

So let us try to do that means this arrest arresting this rotation motion actually. So anything that that happens to be I mean we are not particularly interested in a specific direction or orientation let us simply arrest that tumbling actually so, stabilize somewhere exact orientation where it will stabilize we are not bothered so much this basically that is the problem actually. But we are not talking about a controlled design we are we are in this particular lecture we are talking about derived deriving a linearized system that is what you are doing actually. So, these are the definitions and all.

(Refer Slide Time: 40:43)



So we now let us discuss that operating point is nothing but 0 0 0 that is what we want and the controlled values around that should happens to be 0 0 0 obviously. I mean if it is a good controlled design then after the tumbling is arrested I still do not need any torque actually it should happen that way actually so that is a good way to work with actually.

So this is this is your x 0 this is your u 0. Now, let us try to apply the formula whatever formula is there and that the A matrix is given like something like this way and the B matrix is that way. So this is del f 1 by del x 1 like the del f 1 del f 1 by del x 2 like that and similarly, del f 1 by del u 1 del f 1 by del u 2 like that actually. So, if you start applying that then it all happens by let us say for B matrix first (( )) comeback to that b matrix is fairly straight forward because we have only tau 1 here tau 2 here tau 2 3.

So if you take f 1 del f 1 by del u 1 it is 1 by I 1 or del f 1 by del u 2 there is no u 2 term here, so that is all 0 basically and similarly, u 3 is also 0. So that is why you get this only 1 term here the other terms are 0 first of all you have 3 states and 3 control so that means b matrix is also a square matrix remember that this particular case. So similarly, if you go back to this second equation you tell del f 2 by del tau 2 is there, but other things are not there that means my this term is 0 this is the only term what I have here and the last term is 0 again.

And similarly, if I have the third row of the b matrix so this matrix is fairly straight forward now what happens see here. So, this is a del f 1 by del x actually evaluated at omega 1 omega 2 omega 3 0. So first of all there is no term for omega 1 here that means this term is again 0 and this there is no omega 2 term here remember that so that is the second 2 by 2 element is also 0.

There is no omega 3 term here 3 by 3 is also 0 it need not be evaluated anywhere the terms are simply missing so this diagonal things are all 0 0. Now, what about off diagonal term let us now consider this one this is this term partial derivative with respect to omega 2. So what is left out is omega 3 this coefficient multiplied by omega 3 that is that is the term which is like partial derivative of this term with respect to omega 2.

But that is evaluated getting evaluated at omega 3 also 0 omega 3 0 is nothing but 0 here. So that is also is evaluated at 0, so that term becomes 0. Similar thing if you just talk about omega 3 term that means this is the term that you left out after I mean the partial derivative with respect to omega t where is omega 2 term, but omega 2 0 is also 0 that means this term this term happens to be 0 actually. So this diagonal terms are 0 this term all the term happens to be 0 actually.

So, whatever inter coupling that you have here is all gone actually by linearizing. So what are having now otherwise you are just having a b times u matrix x dot equal to BU simply here, and b matrix happens to be diagonal that means you have a need to decoupling of the system dynamics actually. And this is omega by definition omega 1 by definition is also theta 1 dot actually you can visualize that way.

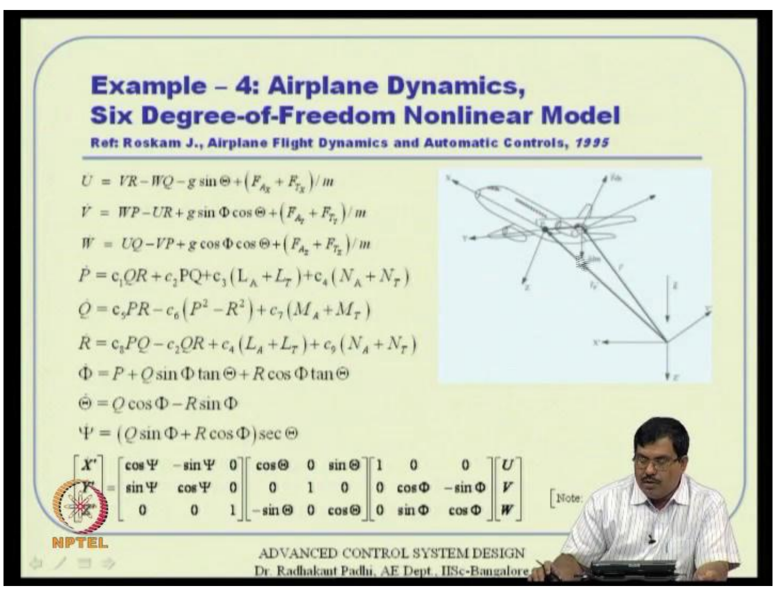
So omega 1 dot is nothing but theta one double dot. If you have some reference line or something there you can omega is nothing but theta 1 double dot is equal to 1 by I 1 times tau 1. So, this particular thing is called like a double integrator actually. Theta 1 double dot is a function of only tau 1. So that is a nice suppose you take a transfer function analysis and all that this is a nice linear system you can take a transfer function there all that it will have a double poles at the origin. And that happens for every channel does not happen for one channel omega 2 dot is also like that omega 3 dot is also like that actually.

So this double integrator problem that you might have studied before or we are going to see the I mean I do not know whether we are going to see that or not in standard control literature actually is very popular, because of several reasons. One of the reasons is like that you want to apply linearized control theory for this applied control problem let us say then this double integrator is relevant to that actually whatever.

So, this is this essentially leads to a double integrator problem this, but remember that this is really not nice because again you are neglected all the coupling effects for omega 2, omega 3 on omega 1 and omega 3 omega 3 omega 3 omega 1 on omega 2 like that actually. So,, this coupling effects are simply gone actually so that is why this linearized control system using the whole integrator does not work very well for arresting large tumbling actually is a small oscillation around which it is.

For normally you remember satellites are in 3 d space any small disturbance can really lead to a large tumbling and things like that and this where this is where linear control theory is typically not sufficient actually.

(Refer Slide Time: 46:14)



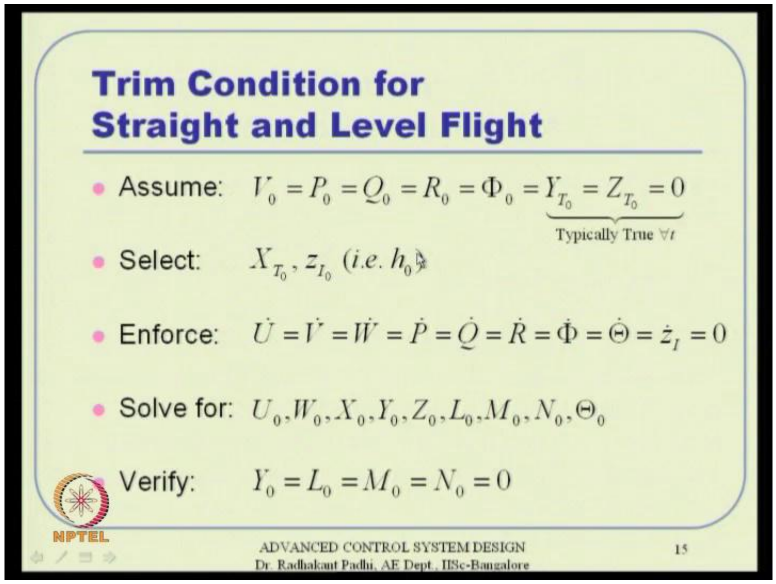
Now, the last example we have also discussed this flight dynamics in detail in the previous lectures and all. We all know these are the things system dynamics given in twelve inter coupling state variables and typically it will have 3 control 3 or 4 control surface depend on whether you are talking thrust as a control or not actually.

If you thrust is also a control vector you have 4 controls other 3 control deflections primarily. So you have 12 state variables and 3 control variables we have also discussed before, that the first 6 are dynamic level equations that the next 6 are kinematic level equations and things like that. So, here it is all getting coupled here and it is too much to deal with so we have observed it in the flight dynamics lecture as well that the three equations are decoupled from rest of the things.

And these are psi X and Y so if you take out these three then you are left out with 9 equations really. And out of those 9 equations you can visualize some something like a perturbation around a let us say state and level flights. So height is known to us probably, so that means height is also gone. So, you are left out with 8 equations and out of those 8 equations you can linearize and probably that linearization also leads to decoupling of the system dynamics longitudinal lateral.

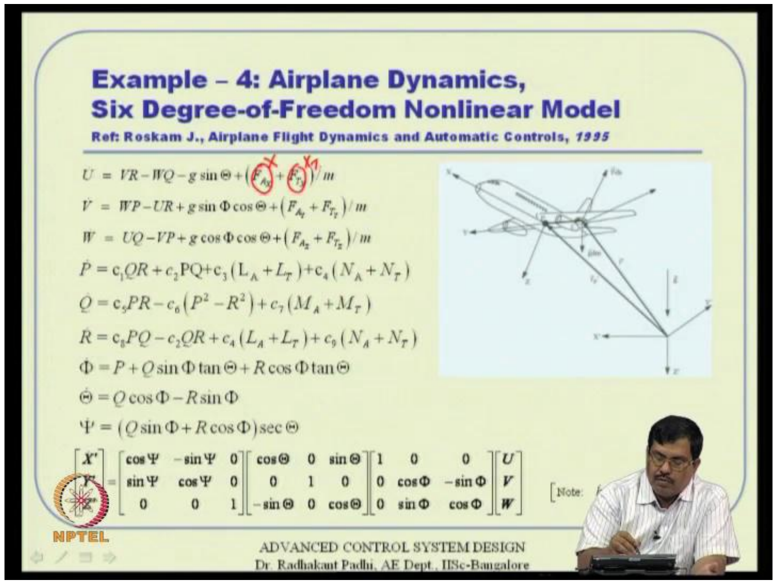
So let us quickly review that I mean this system states and what you are doing here is perturbing these variables all over. Now these variables with a suffix 0 are nothing but operating variables actually. That is what either you have to select or you have to solve for either way. So, what we have discussed before is like one way to do that is to assume straight and level flight and assume all these conditions are 0 then, we select these variables and we enforce these equations actually.

(Refer Slide Time: 48:00)



Then, you will be left out with these variables for to solve for that means you are talking about 1, 2, 3, 4, 5, 6, 7, 8, 9 9 equations there I mean 9 variables 1, 2, 3, 4, 5, 6, 7, 8, 9 9 free variables and 9 equations actually. So you can solve for which will be valid for this particular thing under these assumptions and this will result in this X 0 U 0 (( )) combination actually. So first thing to find out in linearization process is what is my operating point actually. If that is not there linearized linearized system dynamic does not make any sense actually. So this process this process what you call as stream condition and all that is one of the ways to find out X 0 U 0 pair combination actually.

(Refer Slide Time: 49:05)

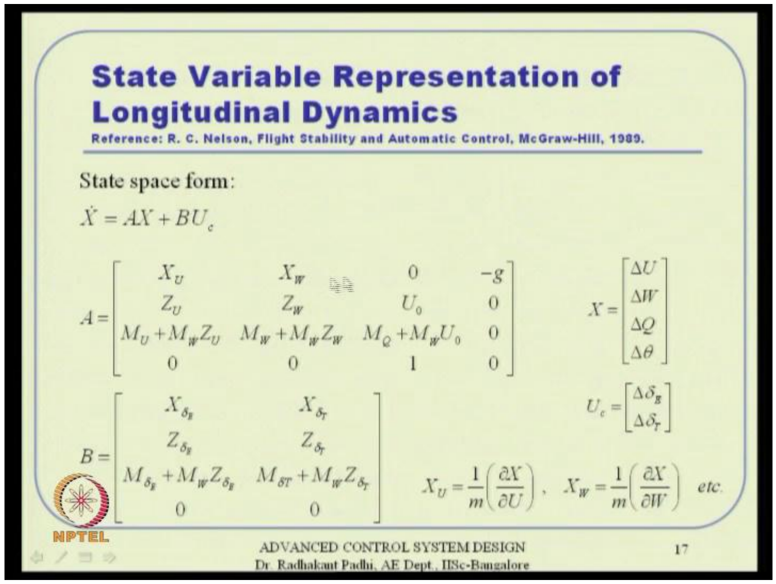


And then once you have that you also have this aerodynamic forces and moments and things like that what about this system variable will system dynamics will consist of this X Y Z L M N.

Actually by the way there is a small printing mistake in this particular slide and correct that this f of x actually we talk that as x and this f t x is something like x t actually that is what you derived in applied dynamic lectures. So this is like y and this is like y t sort of thing so like that actually. And then this l a is actually l and this l t and things like that actually. So that is what you are doing here and this delta X delta Y delta Z further expand we expand using this component build off ideas and all that.

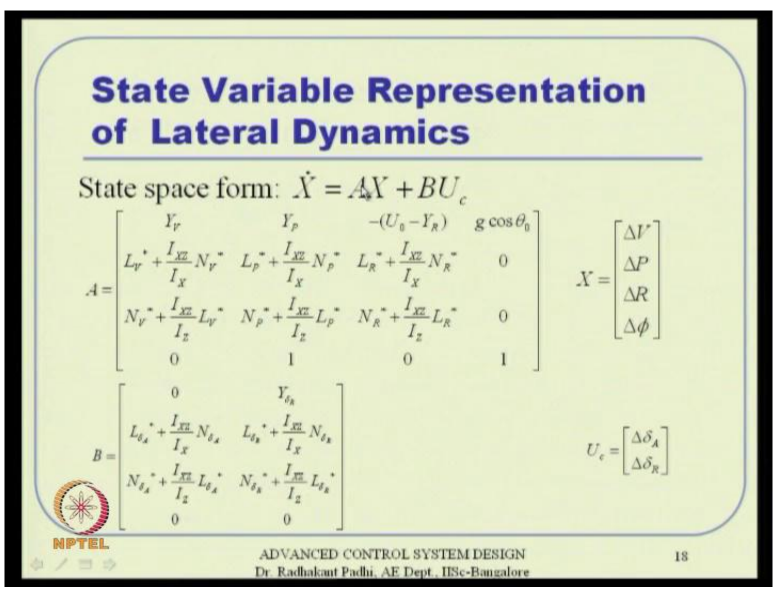
So this delta X is a function of delta U and delta W these are like primary dependence actually. And this delta Y is like that we expand that way delta Z we expand that way when we keep primary function dependence and all that actually. So similarly, these moments also we these are primarily these aerodynamic forces and moments and this is how we build up this components actually.

(Refer Slide Time: 50:32)



And then we write this way that ultimately if we apply this standard equation I mean this theory what we did just learnt del f by del is nothing is evaluated at X 0 U 0 is a matrix and del f by del u evaluate at X 0 U 0 is d matrix and we consider only these 4 variables in one group and the other 4 variables in other group.

(Refer Slide Time: 51:36)



So these 4 variables result in that the other 4 variables result in other one. And this I mean this also assumes that we both have aerodynamic and thrust control actually.

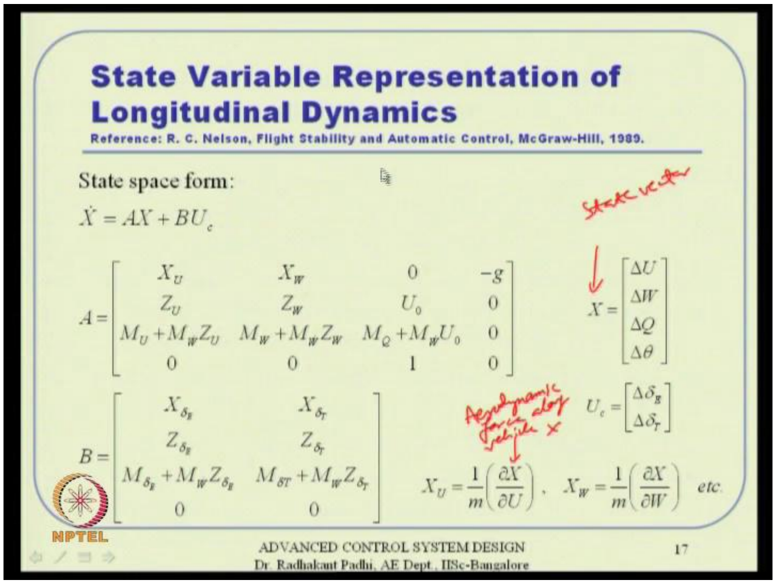
Elevator deflection and as well as percentage of maximum thrust actually that is how we did that. So, this will result in that A and B matrix further we analyse you can decouple this 4 into 2 2 by 2 and things like that we discussed about figurative short period also that thing actually. We will not discuss so much that here, and now with this is something called longitudinal dynamics because it all happens in pitch plane actually x z plane.

Now similarly, you can also talk about this group of variables along with these two control surfaces both are both are aerodynamic control surfaces here and then we talk about this is my A matrix and B matrix and that is my linearized system dynamics. So, but also remember that this X what you see here and this U what you see here and typically we are we denote that U C not U because u is forward velocity. I mean by flight dynamics notation we do not want to confuse that actually U C's transfer control.

So this is what you have here and then this x and u c what you see here are very different from what you see in this system dynamics actually. So, this x means the non-linear x as it is whatever it is and this particular thing this x means this perturbation variables actually. And these do not do not get confused with this X with that X. I mean these are all book keeping sort of thing this X is aerodynamic force along the vehicle X axis or that X means this is like a state variable photo state variables grouped together.

I think this is all very self explanatory these are not nothing to get confused with. But if you consider X U whatever X U you see here this is del X by del U this X is aerodynamic forces actually. So let me also probably write it here this X is is aeroforce aerodynamics force along vehicle X, vehicle X direction, this X is in a straight vector probably, you could write you could define a different variable not to get confused with that actually.

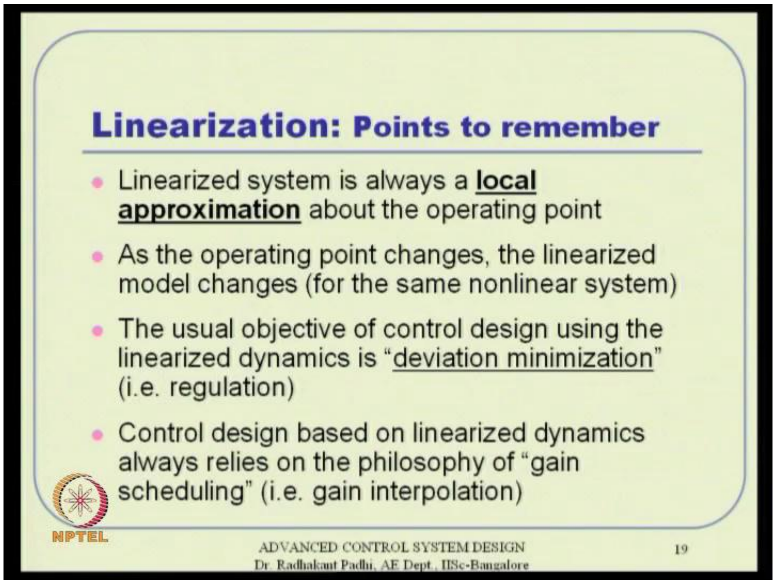
(Refer Slide Time: 53:33)



So this is how it happens there and similarly, also remember this X 1 you talk about lateral variable lateral dynamics and that X what you talk about longitudinal dynamics are also different. So these are kind of implicit implicitly we know what you are doing you keep on you do not have to keep on defining new variables actually, if you talk about a longitudinal dynamics that is the trade vector.

If you talk about a lateral dynamics that is the state vector and similarly, the control vectors are all also different actually. So with that so with that we have linearized longitudinal dynamics and the linearized lateral dynamics, so we will probably we are ready to reply linear systems theory analyze the system further as well as to design control system also actually. So, what is the what are the points to remember here the entire linearization process, they have been keep I have been repeating this and the linearized system are always local approximations about the corresponding operating point.

(Refer Slide Time: 54:09)



One linear linearized system is always a local approximation about some operating point so without operating point linear systems do not make any sense actually. Now, as the operating point changes obviously, the linearized model also changes so that that means this linearized model keep on changing for the same non-linear system, non-linear system does not change was a linear was a linearized system it can keep on changing actually. So that is one way of getting the time let us say time varying linear system if you want to do that.

And time varying resulting linear systems result from various conditions one condition is this operation point keep on changing and the other condition is the system dynamics it itself can have time varying parameters. For example, if you talk rocket dynamics or something the as the rocket flies you have large amount of mass coming out actually. So thrust as a thrust is also a function of time as well as the mass of the vehicle and hence the moment of inertia because the mass is gone the c g somewhere there the c g also keeps on troubling actually c g is about c g is a point about which the net moment is 0 actually.

So the c g variation mass variation and all these things are embedded into the system dynamics and that is and that will also be there as part of the a and b matrices actually. So if you have time varying parameters in the model to begin with or you have this system operating points keep on changing any way. So, then you have time varying linear systems actually otherwise you have time invariant linear system.

And then, the third point is and once you have a linearized control linearized system dynamics obviously the objective in control design using linearizing dynamics is 99.99 whatever percent is deviation minimization that means you have a deviation dynamics now and using those deviation dynamics you just have to suppress the deviation actually. So, that is something called a regulation problem and that is also like a stabilization problem any deviation around that, around the operating point I want to suppress actually.

So that is the usual objective that is put in linear control system design you even if nobody tells us explicitly that is by default actually. So, the linearized system the objective of the control design is to suppress the deviation or delta x goes to 0 that is why you see x going to 0 x going to 0 always is there in linear system design kind of an assumption it is linear system control design the usual objective is to take x to 0. X means delta x anyway because that is a linear system which means delta x and delta x goes to 0 that means you have a deviation minimization problem actually. Deviation suppression or regulation problem actually. And the fourth point is because this linear control systems are based on linearized system dynamics which keep on changing with time anywhere you use a linearized system dynamics to design a control system you must also follow the philosophy of gain scheduling.

Because whatever linear system you get for one particular operating point is different from different operating point. So, if you want to have a control system which is valid all over actually then also you have to bring in the concept of gain interpolation or what is called as a gain scheduling and then interpolation variables are (( )) themselves what variable select for interpolation and thing like that and flight control design typically we follow max number and dynamic pressure as the free variables for which we want to interpolate the gain.

That means different different gains what you select are considered as functions of max number and dynamic pressure. And then using these 2 free variables as free I mean as interpolation variables and all we keep on interpolating the gains are different operating points actually. We will talk more on that as we go to the gain scheduling lecture actually, but that is usually the way to explore this linear system dynamics actually that is why you are interested in linear linearization and things like that to begin with actually.

I hope this is clear we know how to do linearization and what are the implications for linearization how do we make use of that and things like that I i think I will stop here for this lecture actually. Thank you.