Advanced Control System Design
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Lecture No. # 14
Review of Matrix Theory – III

Let us continue with our matrix theory review for this particular class as well and this is going to our last in this series. So, we will see some of the concept that is quickly kind of skipped in the previous two lectures as well as little more examples and I mean one or two little more concepts as well as if you couple of examples to make our ideas clear actually.

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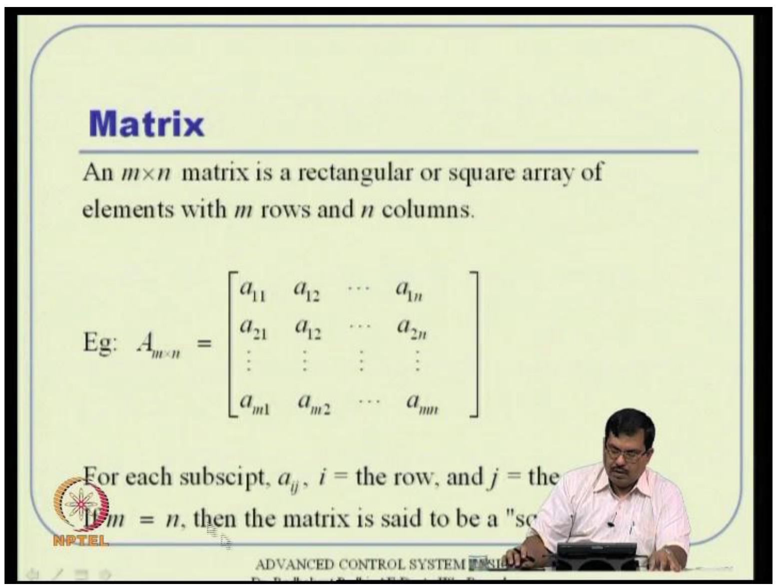

So we will start with the very basic definition and that matrix was nothing but a rectangular matrix in general is of m by n it is a square of elements with m rows and n columns and obviously, for each subscript i j i means the row and j means the column and if m is equal to n then it suppose to be I mean is said to be a square matrix.

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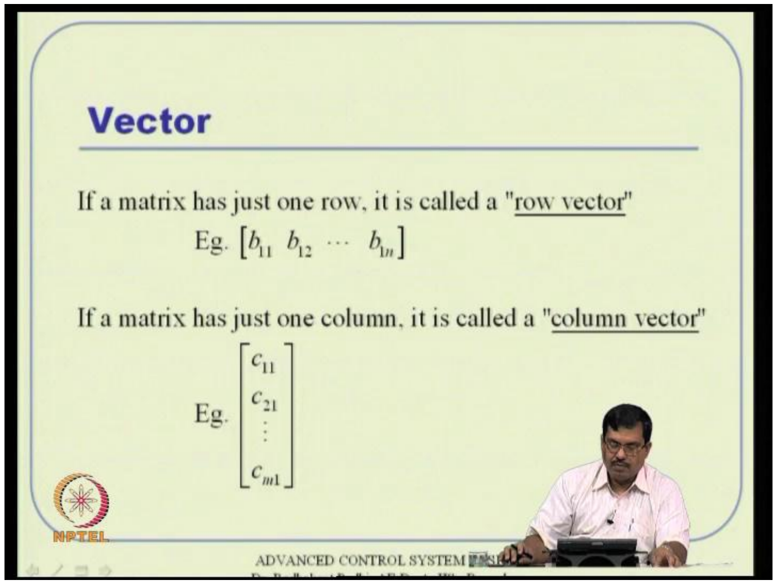

Actually, are very basic definitions and subsequently there is a related concept call vector and if a matrix has just one row it is called a row vector and is a 1 column it is called a column vector examples can be somewhat like this.

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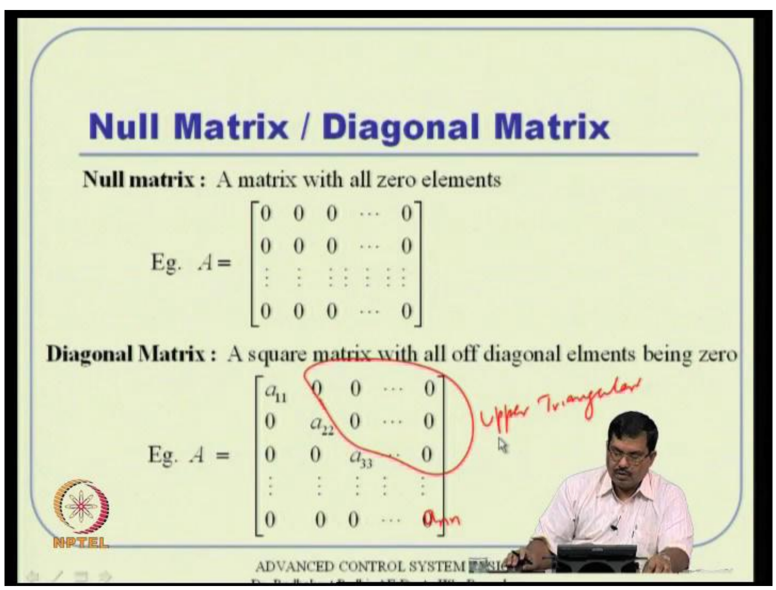

Then related concept is null matrix where the a matrix with all 0 elements. The matrix contains all 0 elements all the elements are 0, then it suppose to I mean it is called an null matrix actually. And diagonal matrix is a square matrix where all of diagonal elements are 0 I mean you have some entries only here, I think this is a mistake here, this is supposed to be this is supposed to be a n m this is not 0 this is a n m, but anything other than that all the elements are 0 basically.

And related concept is triangular matrix where you have upper triangular matrix means all these elements are also non 0 some of them that is if you have that kind of a situation this is upper triangular. And if you have something like the other side that means the lower half is 0 then I mean, lower half is non 0 these elements where the upper half is 0 then it is called lower triangular matrix. So diagonal element is a special case of a triangular matrix of course.

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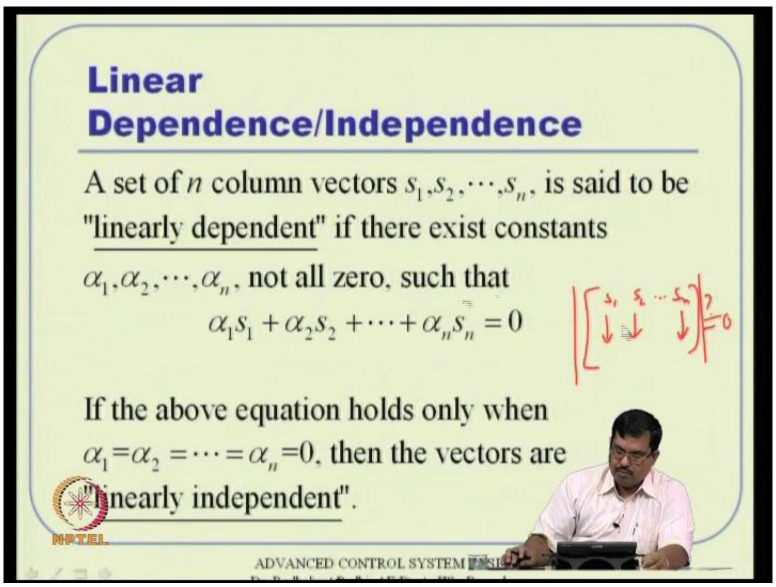

Then there is a concept called linear dependence and linear independence which place a very heavy role in many analysis including control theory. So let us see that the formal definition is something like this, a set of n column vectors and you can generalize it to row vectors as well there is nothing specific to column vectors as such actually. A set of a set of n column vectors s 1 to s n is said to be linearly dependent. If there exist a some constants alpha 1 to alpha n and this is critical not all 0, that means every every one of them cannot be 0.

Such that this equation is satisfied that means we are we want a nontrivial solution for alpha 1 alpha 2 up to alpha n that mean trivial solution being all of them are 0, nontrivial solution if you formulate this equation and try to find out a solution for alpha 1 alpha 2 up to alpha n, all of them should not be 0 actually. Then it is called linearly I mean, the then it is called linearly dependent and if the equation holds good for all these things then the vectors are linearly independent actually.

What does it mean suppose I mean, those of you know this what is called linear combination and all out of all this vectors s 1 s 2 to s n 1 or more vectors are linear combination of the other vectors actually. If that happens then those set of vectors are linearly dependent on each other if they all of them are independent then obviously, nobody depends on any more I mean a combination of others cannot substitute any one of them and hence this kind of equation would be satisfied alpha 1 s 1 alpha 2 s 2 up to alpha n s n equal to 0.

And if you formulate this equation, then you find a non trivial set I mean solution set that means alpha 1 alpha 2 or up to alpha n all of them should not be 0 actually. And if they if they happen to be 0 then they are linearly independent how do you find out quickly that these are all 0 or not? Now if you know this is actually a vector equation that means this 0 is actually 0 0 0 n zero’s there actually.

That means if such a system of equation is satisfied for getting a nontrivial solution, if I formulate this vector, I mean this matrix if I formulate this matrix where s, is s 1 s 2 I just put them together up to s n. This particular matrix must be non-singular. So what I need to see that if I formulate the determinant equation whether this is 0 or not actually the question is that. If they are if they are 0 then there exist a non-trivial set of solution obviously.

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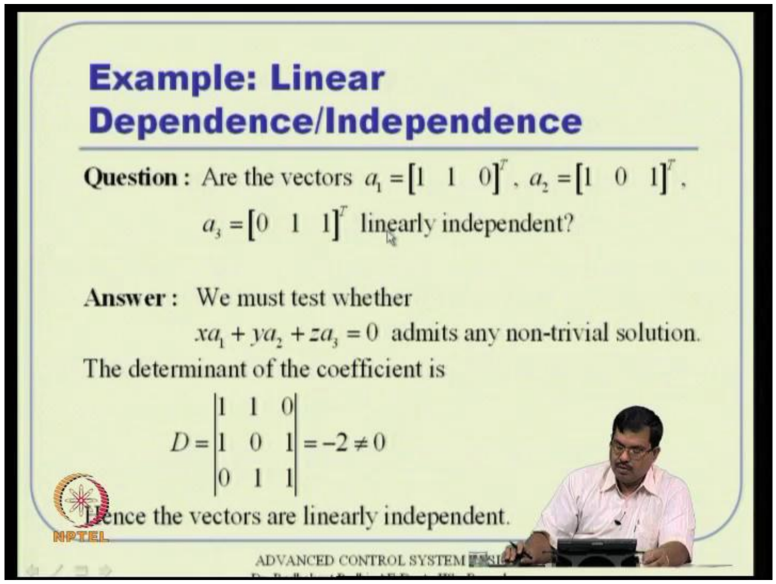

And if they happen to be non 0 then obviously, this is a a non-trivial solution and hence the vectors are linearly dependent actually. So let us see an example now we want to see the whether this a 1 a 2 a 3 are linearly independent or not. And going back to the definition it is slightly difficult that means if you use keep on asking for the question whether whether our a 1 is linearly dependent a 2 or a 3 or a 2 is linearly dependent on a 1 a 2 a 3 then it is not that easy to find linear combination set, because they are infinite sets and how do you know whether there that condition exist or not.

So we will go back to that that idea that we will formulate this equation and ask this determinant whether this determinant is 0 or not actually. So, what is the answer to that is we must test whether this equation x times a 1 y times a 2 z times a a 3 is equal to 0 whether, if I formulate this equation. Whether it admits any nontrivial solution or not actually, that means if you put this a 1 first 1 1 0 then a 2 1 0 1 and then a 3 0 1 1 together and then take the determinant of that and if this case happens to be minus 2 which is not equal to 0 that is that is more important actually.

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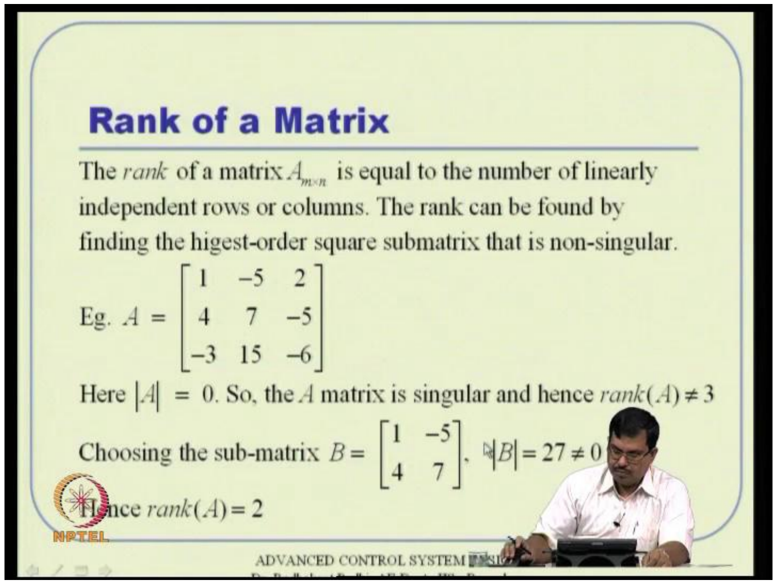

And hence because it is not equal to 0, the vectors are linearly independent actually. Now the rank of a matrix is equal to the that next concept is rank of a matrix and this is also very critical observation in place you have lot of role in in control theory as well as numerical methods things like that. So we will see that rank of a matrix and it is in general matrix need not be square here it can be m by n that means it is a rectangular matrix in general.

So the rank of a triangular matrix in general is equal to number of linearly independent rows or columns. And the rank can be found by finding the highest order square submatrix that is non-singular that means, if I take a matrix like this and first thing I have to see that the largest order highest orders square submatrix that is non-singular the height order submatrix here is the matrix itself. Then if I start checking from there, then this determinant happens to be 0. So the matrix is singular and hence rank cannot be 3 it is a 3 by 3 matrix, but the largest order submatrix is a matrix itself for which the determinant is 0 and hence the rank cannot be 3.

So we have to select all sort of 2 by 2 matrix out of this for example, I can select this particular set, I can select that one, I can select that one whatever actually. Whatever possible submatrixs i will select for 2 by 2 next actually. So I will select B matrix out of that 1 minus 5 4 7 and then determinant of B happens to be 27. You can see clearly that 7 plus 20

27. So not equal to 0 that means we got a submatrix of dimension 2 by 2 for which the determinant is not zero.

So then the rank is 2 actually so that way you can find the rank of a matrix and there are efficient algorithms also to find rank of a matrix and all. Because in general if you start using this definition stating from the highest order and things like that not that easy to I mean it can be lot exercise before you find the the answer actually. So there are efficient algorithms to find rank actually

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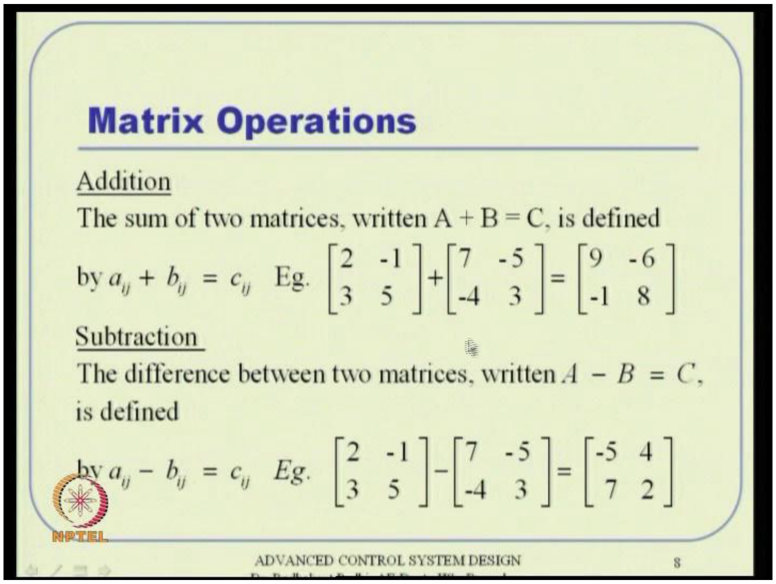

Now basic operations I think all of you know anyway. So this I mean some of this concepts we we skip it in previous lectures so let us quickly review that.

So we have a matrix A matrix B if you just want to add them together first of them first of all they have to be of same dimension. If a is m by n B must be m by n so that you can add them together. Otherwise this is not possible and if that is the case then it is just element by element addition actually. So if you take this matrix and that matrix just add them up element by element one one and one one from B matrix 1 one from a matrix and 1 one from B matrix if you add them together it will formulate 1 one from C matrix actually.

So 2 plus 7 is 9 minus 1 and minus 5 minus 6 like that actually and subtraction is similar you all that you have to do is you subtract element by element actually. So again if you take the same example

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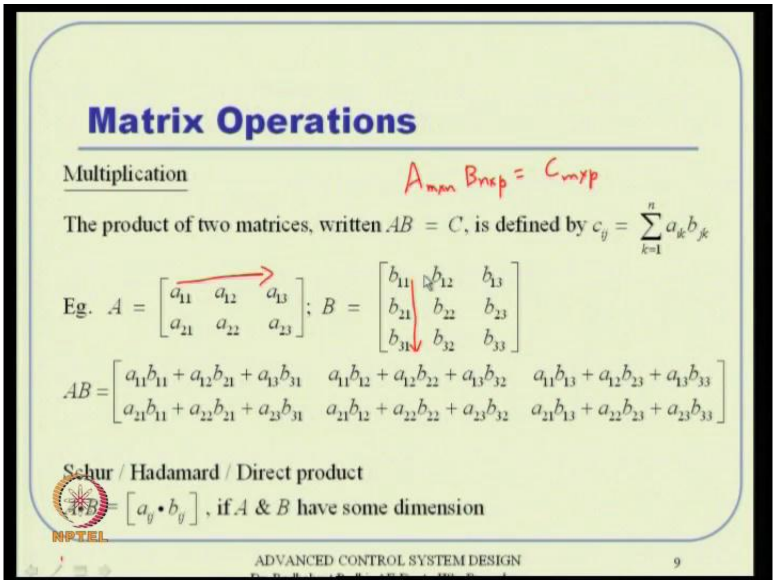

This is the matrix basically. Multiplication however, is a slightly tricky affair you can if you want to take direct multiplication, this is a direct multiplication that means if A and B are of similar dimension same dimension rather. Then you can just do element by element multiplication and that is called Schur product or there is a particular name Hadamard product more popularly known as Direct product because you have a simply directly multiply actually.

And these are some there are some applications I have, but most applications are related to this kind of an idea what is going on here, it comes from vector space theory and more details you can find in a linear algebra book or things like that. So the product of matrices defined something like this and the first of all a if a is m by n then B has to be n by P so that C has to be, if a is m by n then B has to be n by p. So that C can be m by p that means these 2 elements what you see here this n and this n they have to be same actually otherwise you cannot do multiplication.

And after you multiply this the product matrix you will have the dimension m by p basically so one example is something if you a if a is something like 2 by 3 matrix like this here and B is a 3 by 3. So the product has to be 2 by 3 into 3 by 3 that means 2 by 3 so this is how you the the product is obtained all this elements how do you get that it you take this row and multiply with this column element by element and then add them up actually. So you take this way you take this way and then take that way.

And then do element by element multiplication actually that means, if you multiply with a 1 1, then you do b 1 1, then plus a 1 2 into b 2 1 plus a 1 3 plus b 1 3 b 3 1 like that, we will come in this particular term will contain the AB 1 one term actually and similar things are therefore, the second column third column like that. So you take first row elements and first column elements that will formulate the 1 1 element. Otherwise first column first row element and third column element will formulate 1 3 element this is what will pop up actually.

So all this things have nice geometrical mean I mean this algebraic meaning in vector space approach and because of that these are different that way and the most of you I think known how to do this operation also. And once again, this this direct product does not have so much of application specially in control theory we will not worry so much about that, but this one will be of actually.

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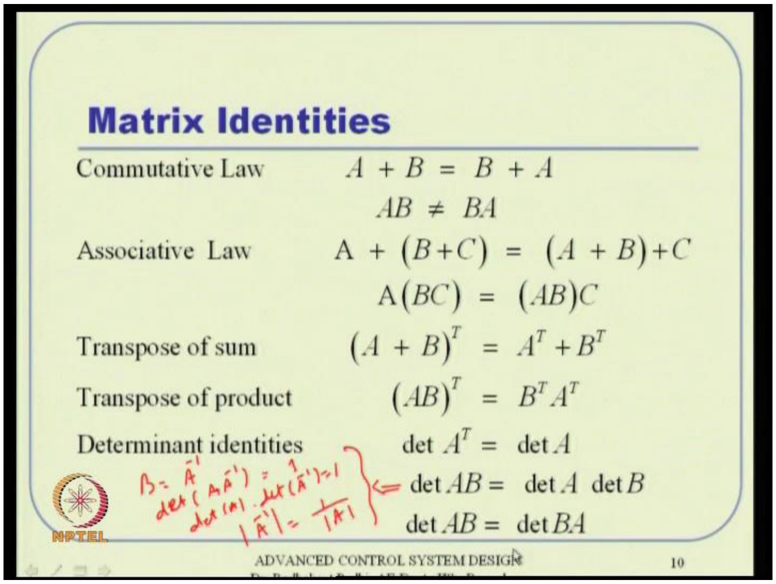

Now certain identities that we can very clearly see and sometimes it is not that apparent first of all is commutative law. Now say I mean if you add A plus B does not matter whether you had elements of A to B or vise versa A plus B equal to B plus a simply because of the definition whatever definition you see here is element by element addition anyway. So it does not matter which order you had to add. However AB is not equal to BA in general, that is a critical observation. So we cannot we do not have the luxury of writing AB equal to BA or interchanging the multiplication order is not allowed actually and that is also very apparent suppose I take reverse way that means this matrix becomes m by n this becomes n by p i mean, p by n then obviously, m is not equal to p. Hence I cannot take the cannot even take the multiplication forget about visualizing whether it is equal to or not multiplication will not be defined in general.

One side multiplication will be defined, but other side may not be defined. And even if they are defined, if you see this these elements and all the way it will pop up this element the the product elements and all. In general it may not satisfy this characteristic AB is not equal to BA which we clearly have to remember. And if AB is equal to BA then that the definition is like a and B are suppose to I mean reside that A and B commute with each other that means if a matrix A and commute that means if you see some statements somewhere like this then that is what it means AB equal to BA. So very special case it will happen that way.

However, associative law sense it is there for both addition and substraction and multiplication that means whether you add A I mean, B plus C first and then add to A or you add A plus B first and then to C all both are same actually and similar thing happens from multiplication as well. So when you see associative law it is happens to both hessian subtraction whereas, commutative law is only valid for addition actually. Well also remember there associative law you do not have a luxury of changing the sequence specially in the multiplicative term.

In addition term probably it is because you can exchange whether you whether you write A plus B plus C or a plus C plus B probably. But in associative law you do not have that luxury the order has to be same actually. Anyway then the next thing is transpose of the sum if you have A plus B transpose it turns out is a transpose plus plus B transpose also apparent from the addition law actually. If you because it is all its term by term or element by element addition it just happens that this is very apparent actually. A plus B transpose if you take nothing but element we put first take the transpose of a then B and then then transpose of B then add it up actually so both are same.

And interestingly transpose of a product on scenery reverse sequence which is very interesting actually. See if you a AB whole transpose is not a transpose B transpose is rather B transpose a transpose again in a longhand algebra you can show this actually. So these are some critical observation specially with respect to the product law actually summation law will be fairly straight forward I mean, many of this but, when you see a commutative thing I mean this product thing which is first of all not necessarily commutative I mean this one is, but transpose of a product is like B transpose a transpose reverse sequence actually.

Now certain determinant identity also exist. then first thing is if you see determinant of a transpose is, nothing but determinant of a because suppose you evaluate determinant of a like row expansion sense, then I can evaluate determinant of a transpose using a column expansion sense it is one of the same thing actually and this one is most of the time we will use it also determinant of A B is equal to determinant of a into determinant of a is a very useful property rather. So this is this is rather useful actually, so it will using this you can also show some interesting things for example, if you really want to show some interesting thing let us say B is equal to i take A inverse actually.

If I take that then determinant of I mean determinant of A B which is a A inverse obviously determinant of I actually. So this is 1 and that means determinant of so this is determinant of a into determinant of A inverse equal to 1 that means, determinant of A inverse is nothing but, 1 by determinant of A. So this is actually a very useful property in in that sense many times you will make use of this this property actually.

And also you know the determinant of A B happens to be determinant of B A as well. So this is I mean provided this A B and B A are both defined both are defined and then you can do that actually

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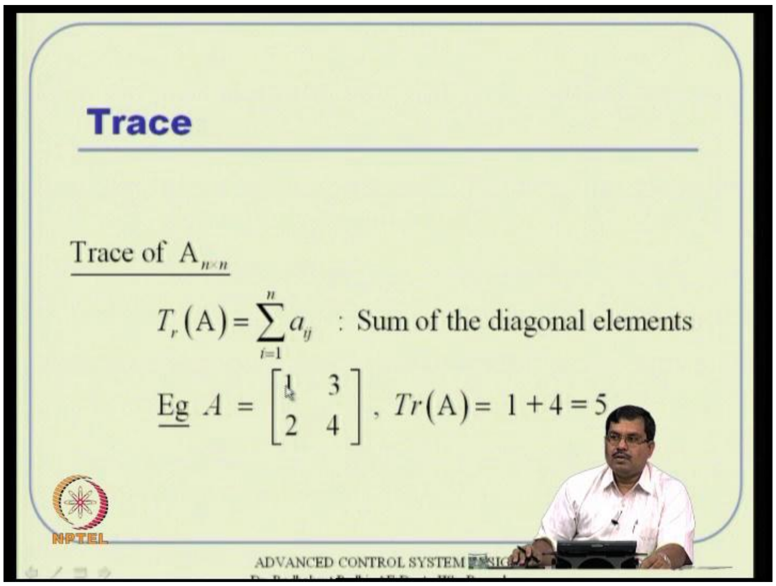

There is a another concept called trace operator trace is a very simple operator, but very handy sometimes because there are something related to some matrix now and things like that later. So trace trace of an like n matrix and specially it is difined with a square matrix is, nothing but some of the diagonal elements actually that is all you have to do you see the sum the diagonal element just add them up no absolute values by the way this is simply algebraic addition actually.

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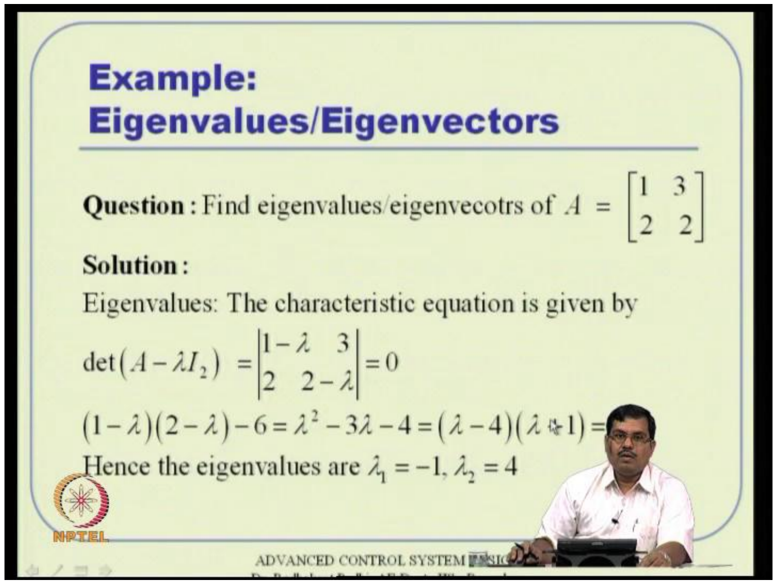

So we have in this particular example 1 3 2 4 then you simply add 1 plus 4 becomes 5 actually that is all. Now we will see is last one or last class I think we did not we just had the concept of Eigenvalue Eigenvector, but we did not give an example. So let us quickly see an example how to find out this Eigenvalues and all Eigenvalues and Eigenvectors of this particular 2 by 2 matrix. So it is very clear that first we have to start with a characteristic equation so that is determinant of either you take determinant of lambda I minus A and make it equal to 0 or take determinant of A minus lambda I equal to 0 I mean either way because.

Eventually, because whether you have A minus 1 to the power n it will pop up now it is determinant of minus A is something like minus to the minus 1 to the power n into determinant of a something like that, and then if because it is equal to 0 that minus 1 to the power n does not play a role I mean either way you can cancel that actually. So you take A minus lambda I 2 then expand that up whatever equation pops up and then I mean this particular case you can just find out that this these Eigenvalues are like this minus 1 and 4.

So this also has a small comment here, that not that this Eigenvalues are actually real, but the vector, but the matrix is not necessarily symmetric. However if you have if you start with a symmetric matrix you are guaranteed to get real Eigenvalues, that one of the theorems we we saw last class actually .Here we get Eigenvalues as real Eigenvalues, but the matrix not does not necessarily have this one, this I mean the symmetric. So if you remember one of the theorems, so what we really need is the I mean the guaranty will be if the Eigenvectors will be orthogonal actually. The if it Eigenvectors are orthogonal then it is guaranteed to have a symmetric matrix otherwise not.

So that is just a comment, so now coming back to finding Eigenvalues and Eigenvectors. So this lambda 1 is minus 1 and lambda 2 is 4 that is what we found by the way there is a small print mistake here this is Eigenvectors by the way the spelling error there actually so that is all right.

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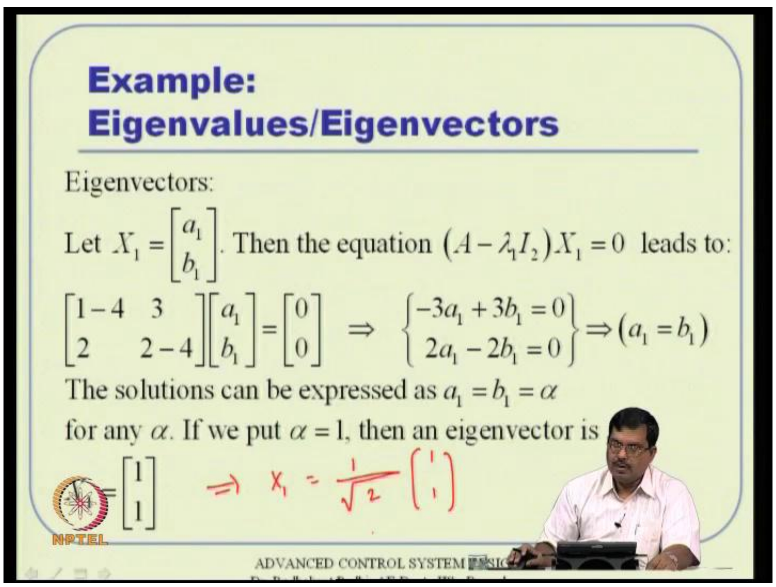

So then, now next thing is to find out those Eigenvectors so how do you do that? So first let us take X 1 a something like a 1 b 1 and then we will go back to this equation where we where we all started A minus lambda i 2 into X 1 has to be 0. So I put lambda 1 and then X 1 corresponding vector X 1 that is to satisfy so I put it a 1 b 1 then make it equal to 0 and interestingly both of the equation will lead to the same consequence a 1 equal to b 1

So here there are infinite solutions, but remember Eigenvectors do not have magnitude they are only directions, so I can just put any value I want if I select a 1 equal to b 1 equal to alpha, then for any alpha that is are also an Eigenvector, but for particularly if I put alpha equal to 1 then Eigenvector is given as 1 1 actually

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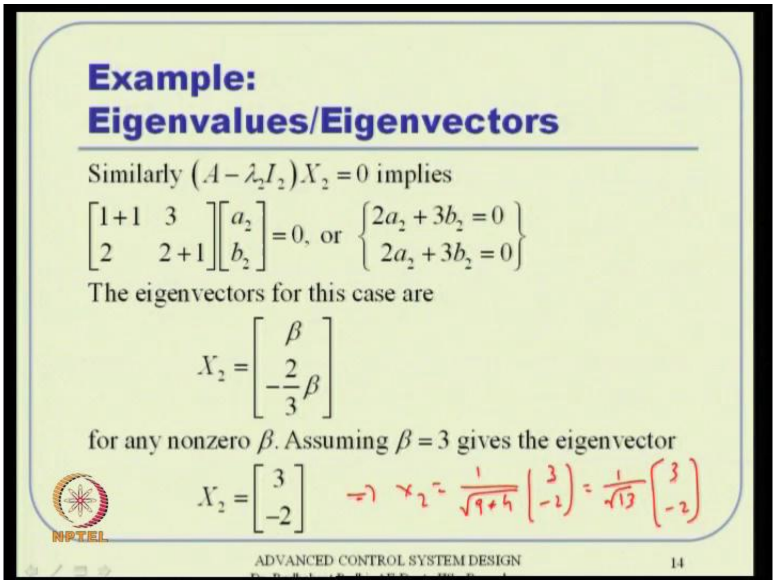

And similarly, this I go to the next one, if I go to the next one i will put lambda 2 and X 2 and hence, I start with a 2 b 2 and similar algebra i will carry out and here i will end up with something like that.

And just to cancel out this denominator 3 I will just put beta equal to 3 here, then I will be left out with something like 3 and minus 2. So that is another Eigenvector actually. Now suppose if you want to normalize it you can also normalize this one I mean, sometimes people prefer that actually and the by the way the previous one is also not normalized I mean if you want to normalize this normalize vector will be something like X 1 is 1 by square root of 1 square plus 1 square, so that is 2 and then 1 1 that is a normalize vector normalize Eigenvector.

And similarly, if you want to normalize the other one you have to do this something like normal normalized vector. So equal to 1 by square root of 9 plus 3 square is 9 plus minus 2 square is 4 and then it is 3 minus 2. So that is if I this 1 root square root of 13 will come and then 3 minus 2, that is what I get actually.So depending on the application for example, if you are interested in in reducing this matrix for example, similarity transformation if you want to do then you will probably would like to start with is normalize Eigenvector actually. There will be like some nice properties there, if you start with orthonormal columns for that p matrix what you discuss before then p inverse you do not have to take p transpose is p inverse actually that way.

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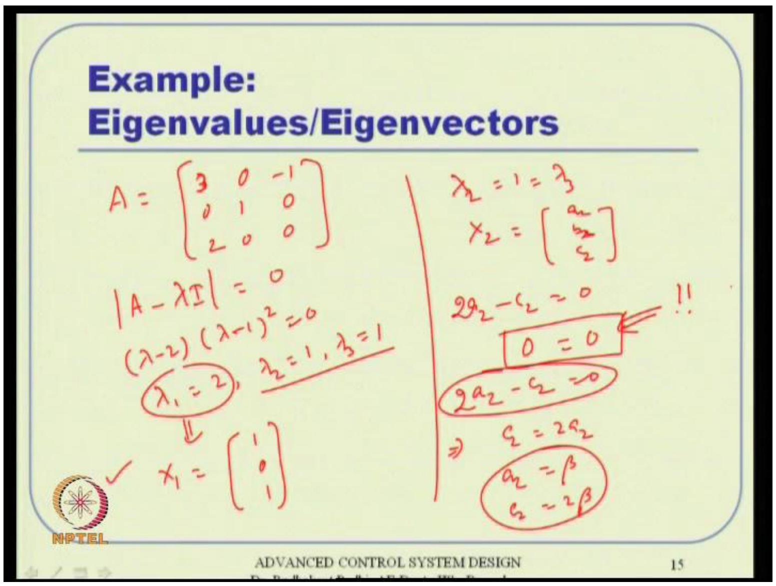

Anyway, so coming back to this let us start with another interesting example which i will try to demonstrate here, which is something like this A is let us say 3 this is 3 0 minus 1 and then this is 0 1 0 and this 2 0 0. If you start with this this particular matrix you start with again the same thing you start with A minus lambda I 3 basically, is equal to 0 and then you will end up with something like lambda minus 2 into lambda minus 1 whole square equal to 0 if you if you do this exercise it will find out.

So what does it give us lambda 1 2 3 is equal to something like lambda 1 2 lambda 2 1 lambda 3 also 1 remember that. That means we have this this repeated Eigenvalues here. So if you go back to our observation before then tell there may be a possibility of finding independent Eigenvectors still a possibility just because you have a repeated Eigenvalue does not mean you do not have the possibility of finding out linearly independent Eigenvector that is what I want to demonstrate here in this example and see that in this particular case it will be possible to find out linearly independent Eigenvectors still even though Eigenvalues are repeated.

So how do you do that none of our lambda 1 equal to 2 this particular thing is fairly straight forward you do the same same exercise. So you find out something like it will turn out to be like X 1 if you find out it will be 1 0 1, so that is fairly straight forward you do not have to see too much of that actually, but if I put lambda 2 equal to 1 which is also lambda 3, then I will be left out with some equations like that actually and then probably X 2 i will take it I mean this.

So something like a 2 b 2 and c 2 I start with that, and then i will end up with some equations like that so 2 a 2 minus c 2 is equal to 0 and then there interestingly there will be an identity actually, there will not be an equation. If you see that go back and put it there in this equation if you and then the last one will be also 2 a 2 minus c 2 equal to 0 so this 3 equations I will end up with what does it give us actually.

So if I take a 2 equal to what are that give me c 2 equal to 2 a 2 that is for sure. So if I take a 2 equal to some some value beta then c 2 equal to 2 beta that is. Because one of this equations giving that actually. Now this particular equation what you see here is an identity really it is not an equation, it is very apparent from what you put there actually. Because remember this is lambda I and all that actually. So it will get a lambda is 1 here lambda that means this 1 minus 1 is 0 if you say this this particular.

Why this is coming because there is an element one here and then you have a lambda value also 1. So this particular 1 minus 1 become 0 actually here, so this entire row becomes like 0 0 0. So 0 into a by 2 plus 0 into b 2 plus 0 into c 2 that is a 0 basically. So that is why it will get 0 equal to 0 here and if you have something like this that is an indicator that there is still that there is an Eigenvalue vector which is independent actually.

So that is a if you have if it happens to be like an identity then you have an independent Eigenvector why I will see the I will show you that actually and if you have 2 identities like that so obviously you will have 2 independent Eigenvectors like that actually. So let us see why it is so, because see because what you got what you got here is a 2 is beta and c 2 is 2 beta that is all it shows me actually. So let us carry on with that.

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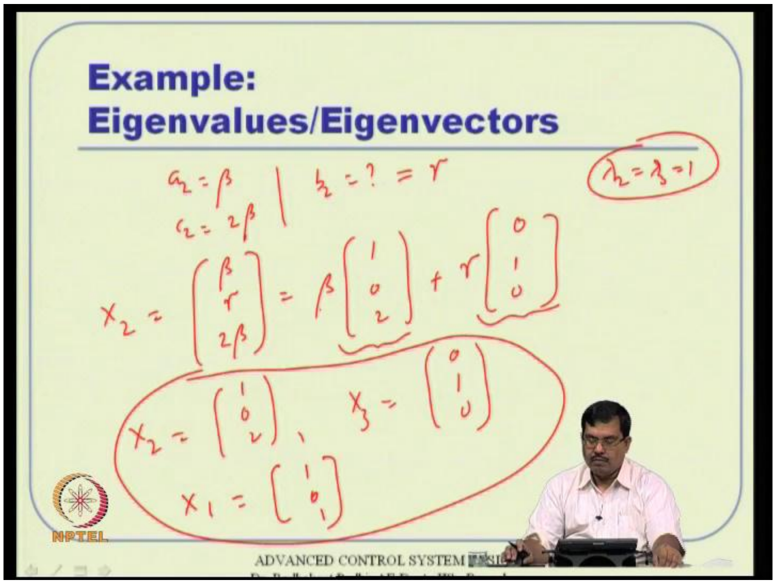

So what I get I get a 2 sorry I get a 2 equal to beta and c 2 equal to 2 beta I got that actually. Now what about b 2 what about b 2 and b 2 can be anything actually really. I mean, I because that is not related to a 2 and c 2. So I can select some some gamma value like that actually, so if that is the case then my X 2 Eigenvector is something like what I am getting a 2 which is beta b 2 is gamma and c 2 is 2 beta. So that one I can simply write something like beta into 1 0 2 plus something like gamma into 0 1 0 and remember all these are with respect to lambda 2 lambda 3 equal to 1 that the case that you are analyzing lambda 1 equal to 2 got an independent Eigenvector already.

So that is what I am telling this one we already got actually. So that is not a problem for us we are just exploring the other case for which we can serves now if you select some I mean for any choice of beta and gamma we will have a Eigenvector with us actually. So I can comfortably tell that this is like an Eigenvector and that is another Eigenvector actually. So in in principle what i will have I have X 2 equal to something like 1 0 2 and I can select X 3 to be 0 1 0 and as you I mean seen before X 1 equal to 1 0 .1 So X 1 equal to 1 0 1 and these are linearly independent with each other.

So that is why you get a set of Eigenvectors which are linearly independent of each other even when you have a repeated Eigenvalue actually that is the message that I wanted to demonstrate in one example actually. So the key observation here is whether you have something like an identity on the way or not actually

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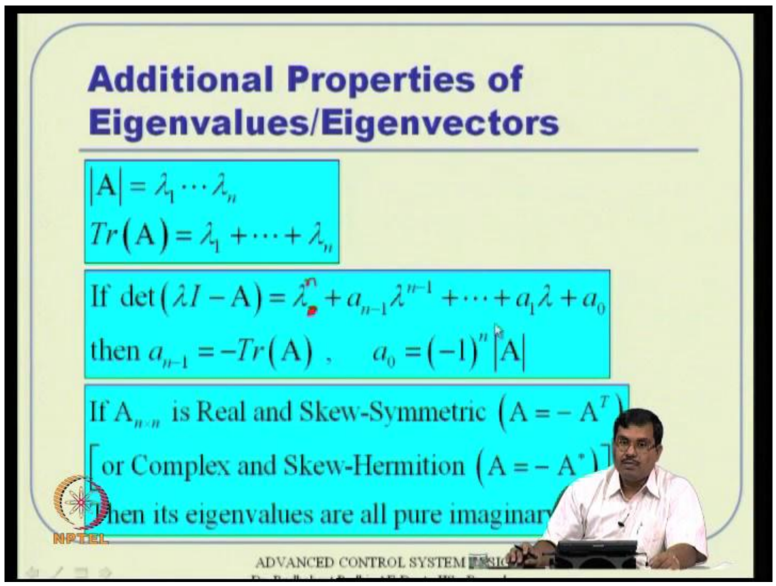

Now let us continue with our I mean discussion some additional properties of Eigenvelues and Eigenvectors we will see here we saw a lot of properties last last time and there are interesting additional properties available here, which we will see there actually.

One thing is when you talk about determinant of a matrix you can just multiply the Eigenvalues and get the determinant. In other words if somebody has given us a set of Eigenvalues then we we do not have do not need the entries of the element of A can simply multiply the Eigenvalues and say I do not need to know all the entries individually about the a matrix, but I can still know the determinant value and similarly, I can still know the trace of this matrix actually. So that is a summation of Eigenvalues summation of Eigenvalues are, nothing but trace of the matrix A product of the Eigenvalues are determinant of A. So these are sometimes very handy, but what does it tell.

There is another consequence that if one of the Eigenvalue is 0 then the determinant is guaranteed to be 0. It is a multiplication term anyway. So that is another consequence which turns out from these interesting results also actually. Now this characteristic equation determinant of lambda e minus A even though we saw in the this example these are all A minus lambda I, but typically we will interpret as lambda I minus A.

So if we if you expand this determinant of lambda I minus A and assuming that A is an m by n matrix we will land up with this n-th order polynomial I think this is a I mean this is a small error out here this is not lambda n this is lambda to the power a. So we will we will land up with an n-eth order polynomial and then what interestingly turns out that this A to the power this this coefficient what you have here n minus 1 is negative of trace of A and then this a 0 is, nothing but that way.

So this particular thing this particular property, I have not seen too much of usage in our way of I mean, whatever we are going to discuss later. So that is I mean that is a observation that we can probably just remember and now coming back to the the example that I mean the property that we discussed before remember

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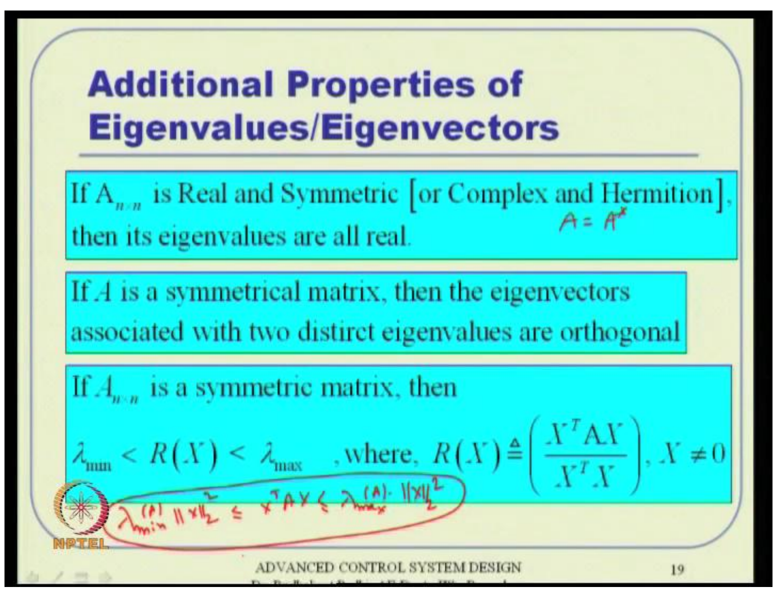

That this is the thing we discuss before that if a is real and symmetric or equivalently if it is complex and hermition remember real and symmetric means A equal to A transpose and complex and hermition means this particular thing means, A equal to A a star rather star is conjugate transpose actually. Then these are all complex I mean if you if your a matrix is complex entries then we will worry about those things actually.

Anyway, so if it is real and symmetry then that means A equal to A A transpose then we know for sure that Eigenvalues are real. We discussed about that and probably I am going to show that in this class also. Now as a related thing we ask the other question what if the a matrix is real, but skew-symmetric that means A equal to minus of A transpose or similarly, skew-hermition that mean A equal to minus A star.

Then it turns out the Eiegnvalues are all purely imaginary there is no real component. So the E igenvalues will sit on the j omega axis actually that means it is a pure imaginary only. So that is an interesting thing you can see that actually. Then there is another associated property that this all talks about Eigenvalues being real and we also know that the the symmetric matrix A, if a is symmetric then Eigenvectors associated with 2 distinct Eigenvalues are also orthogonal, if lambda 1 lambda 2 are distinct they are not same not repeated and they are real of course, but if you take lets say 1 2 2 3 like that.

So, if you compare the Eigenvector with one I mean corresponding to lambda equal to 1 and lambda equal to 3 lambda equal to 1 lambda equal to 2 like that actually. Then these Eigenvectors are guaranteed to be orthogonal actually. We will see that. Then there is important property here, this particular what you see here is called Ralley Quotient actually this X transpose A X divided by X transpose X remember X transpose X is, nothing but second norm square actually.

So, if you not alternatively what some people write that this is lambda minimum of lambda minimum of a into norm X second norm square is less than equal to X transpose A X is less than equal to lambda max of A into norm of norm 2 of X. So this is one and the same thing whatever is written actually and this particular way of writing probably, we will see many non-linear control analysis and all. We want bounded information actually whether the value remains error remains bounded and thing like that.

And if you have a symmetric matrix that goes along with this these Eigenvalues then this a nice result that you can tell X transpose A X is guaranteed to be bounded from both open both opened below. So this is a nice property that comes handy actually. So in a in a alternate way you can define a Ralley Quotient something like this, and then tell this ralley quotient is bounded between this minimum and maximum Eigenvalues actually remember these Eigenvalues are guaranteed to be real numbers. So you do not have any problem of comparing this this.

Now, let us see some of the proofs here that what is the proof that these 2 specially if the if a matrix is real and symmetric, symmetric matrix then the Eigenvalues are real that is what we want to show and we also want to show, that if a is matrix matrix is symmetric and Eigenvectors associated with 2 distinct Eigenvalues,there is kind a small print mistake this is n actually 2 distinct Eigenvalues are orthogonal we want to show that quickly.

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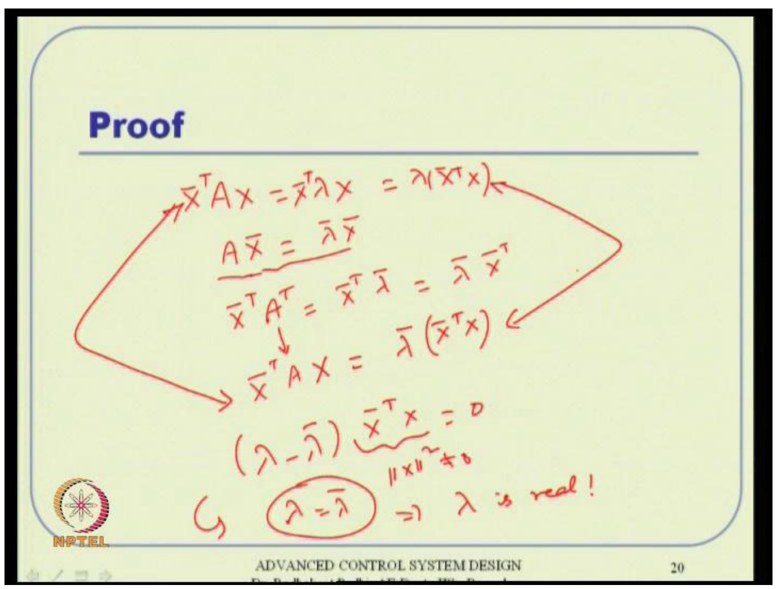

So how do you see that now let us say you have a vector you have a matrix A for which you are talking about a lambda and X pair actually. So you have A X equal to lambda X. That is for sure, what you are going to show is something like this. If for a real symmetric matrix Eigenvalues are I mean, for a symmetric matrix Eigenvalues are real that is what we want to show here actually. So we start with this definition and interestingly turns out that, if I take conjugate I mean complex conjugate of lambda the corresponding Eigenvector is also a complex conjugate of X actually.

So with that, I can also write a into X bar is, nothing but lambda bar into X bar. So that I can that that way actually. Then I what I do i will multiply this side with an X bar transpose and this is also like way X bar transpose that means, I will get something like lambda into lambda is a scalar remember that X bar transpose into X bar I mean, into X. Similarly, if I take this particular thing I will not do directly as such i will take something like a transpose basically.

So what I get X bar transpose remember A B transpose is B transpose a transpose so I will get a transpose equal is to lambda bar into X bar transpose actually, because X bar transpose into lambda lambda bar is a scalar quantity and because it is a scalar quantity I can write that way actually I mean, if I really want to actually. Now what I do here I want to have a similar expression like that, so I will do multiply this this expression A bar transpose remember A bar I mean this is also equal to A basically. So I can take out this transpose actually here, because a transpose is equal to A.

Then I will do a multiplication of X then what I get lambda bar X bar transpose into X. Now if you see this equation this is same as this and hence these 2 have to be same actually. So if you if that is the case what I am getting here lambda minus lambda bar multiplied by X bar transpose X is equal to 0. Now if I take X bar into X this this is, nothing but like a norm actually. This particular quantity is, nothing but like norm X square sort of thing. So that is cannot be equal to 0 you are not looking at a trivial Eigenvector here. So that gives me so this is absolute I mean, this is not equal to 0.

And hence, that gives me lambda equal to lambda bar and lambda equal to lambda bar that means, lambda is real actually. So we cannot have I mean for a symmetric matrix where the critical observation comes critical comes here to here actually. This a transpose is equal to a if it is not there then you do not have to return this actually. So if it is symmetric matrix then lambda has to be real actually. Now let us show the other one also quickly that if you have Eigenvectors associated with 2 distinct Eigenvalues are suppose to be orthogonal. How do you show that ? It is also fairly similar similar proof sort of thing.

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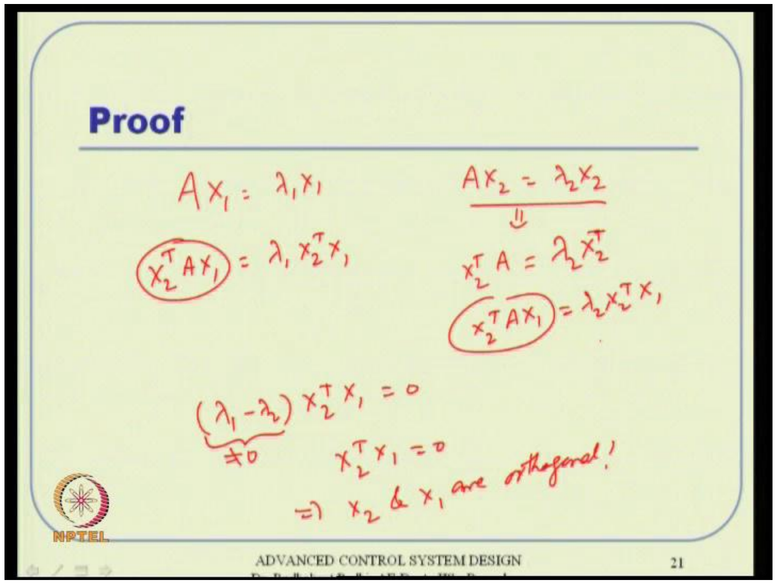

So let us say if you are having the lambda 1 X 1 pair and lambda 2 X 2 pair, so what is this? A X 1 so by definition, I will talk about A X 1 is equal to lambda 1 X 1 and similarly, if I talk different pare A X 2 equal to lambda 2 X 2. Now if I take again this this transpose of this what do I get X 2 transpose and a transpose, and a transpose is a remember that we are talking about a symmetric matrix again and then is equal to this is equal to lambda 2, lambda 2 into X 2 transpose because lambda 2 is a scalar anyway, so I can I can do that actually.

So what do you getting here I mean if I do this X 2 transpose from this equation. If I multiply with let us say X 2 transpose AX 1 then I get lambda 1 into X 2 transpose X 1 and if I do this post multiplication here, by X 1 then I get X 2 transpose AX 1 is nothing but, lambda 2 X 2 transpose X 1. And similarly now you say this term is same as this term, and hence the other terms have to be there. So that means lambda 1 minus lambda 2 into X 2 transpose X 1 is equal to 0, but lambda 2 lambda 1 and lambda 2 are distinct. So they are not this this is not equal to 0.

And hence, X 2 transpose X 1 has to be 0 that means this 2 are orthogonal actually. X 2 and X 1 are orthogonal. This inner product between X 2 and X 1. So this is like a proof where it can very clearly show that if these are the 2 properties that you can show actually

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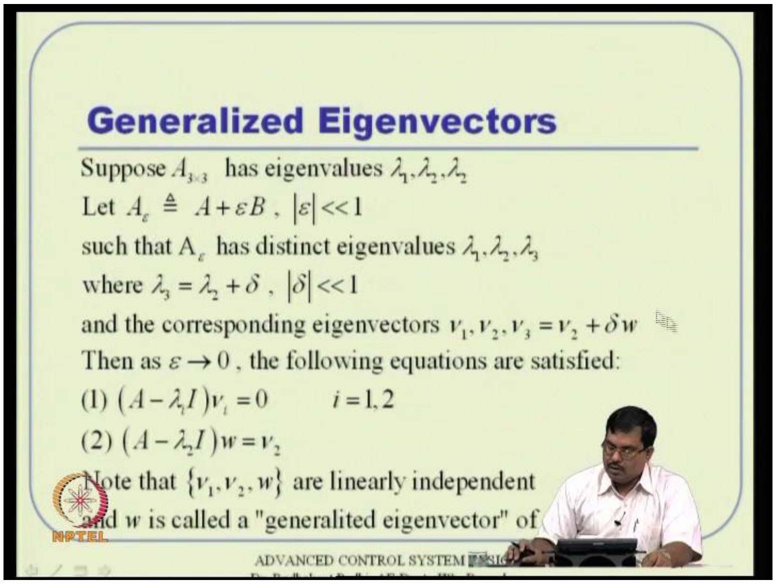

Now coming to the generalize Eigenvectors we saw last class our then the definition part I did not cover. So let us see like what does it imply what does it mean actually. Now let us take a 3 by 3 matrix for our simplicity.

And then tell this 3 by 3 matrix is lambda 1 and lambda 2 lambda 2 repeated twice actually, then what you have is lets because lambda 2 and lambda 2 repeated I have I may not be have I mean, I may not be able to find a linearly independent Eigenvector and that is the case I am talking about in that case what will I do actually. Let me perturb the same matrix by a small quantity epsilon where B is an arbitrary matrix actually of same dimension epsilon is a very small number and i will take arbitrary B matrix of same dimension and formulate this epsilon matrix actually.

Then I will perturb it in such a way that a epsilon will have distinct Eigenvalues now. This is not lambda 2 anymore it is lambda 3 actually. However because this matrix is perturbed by very small quantity lambda 3 I can write it as lambda 2 plus delta basically. So delta is also a very small quantity. Now it turns out interestingly that this v 3 can be represented as v 2 plus delta times w delta is a small number and w is what is called as generalize Eigenvector, this v 3 is very close to v 2 in that is sense and then this equation when epsilon tends to 0 that means extremely small quantity very small quantity then this 2 equations will be satisfied.

That means this first 2 equations are regular Eigenvectors and the last one this equation will get satisfied actually w equal to v 2 that is why these equations pop up actually. So what does it mean, actually the this w which is generalize Eigenvector is very close to the Eigenvector v 2 but, it is perturbed away by a small quantity and then I mean the w I mean this w what you see here, will satisfy this kind of I mean these identities and things like that actually. So what you are doing here I mean you are telling that I will I do not have I do not have to work with the same A matrix that is given to me, but I am i will perturb the same matrix by a small quantity arbitrary selecting v and putting this summation actually, and then find out the Eigenvalues of this A epsilon.

And then when epsilon tends to 0 then this 2 equations get satisfied actually and this w also goes close to like v 3, but w is different from v 3 of course. So these are all implications more on that if I as I told in the last class you can show in a thomas book actually.

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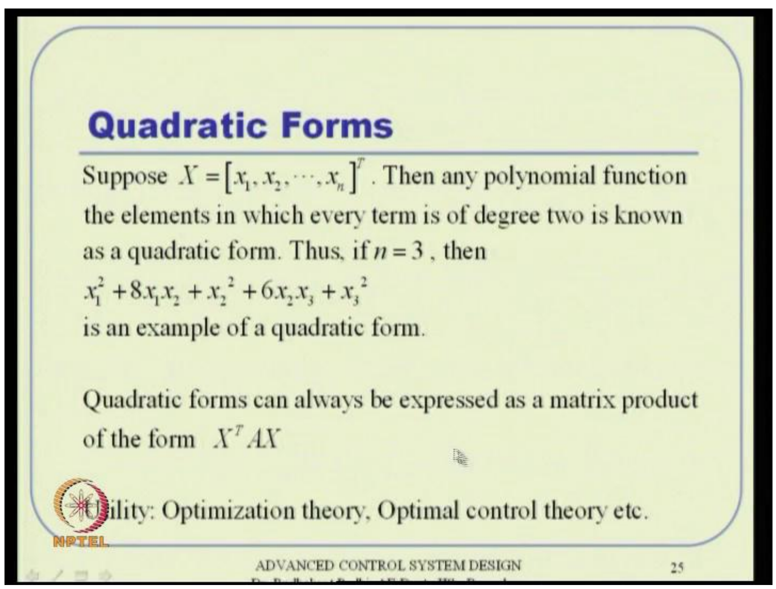

There is another concept in matrix theory which is also which is a important optimal control theory specially. And most of the time we will use in optimization theory optimal control theory like that and what is this actually. Now if you have a vector x 1 to x n. Then you take any polynomial function for which every term is of degree 2, that is critical actually every term will of will be of degree 2 actually.

Then this particular polynomial is called quadratic form actually, and interestingly if it is a quadratic form, then you can always express this in terms of X transpose AX and given a choice we will select a quadratic as a symmetric matrix. And I think by now we have sufficient knowledge to see why we want symmetric matrix many nice properties for a symmetric matrix actually .So you can for example, if you see this this expression each of the term has degree 2

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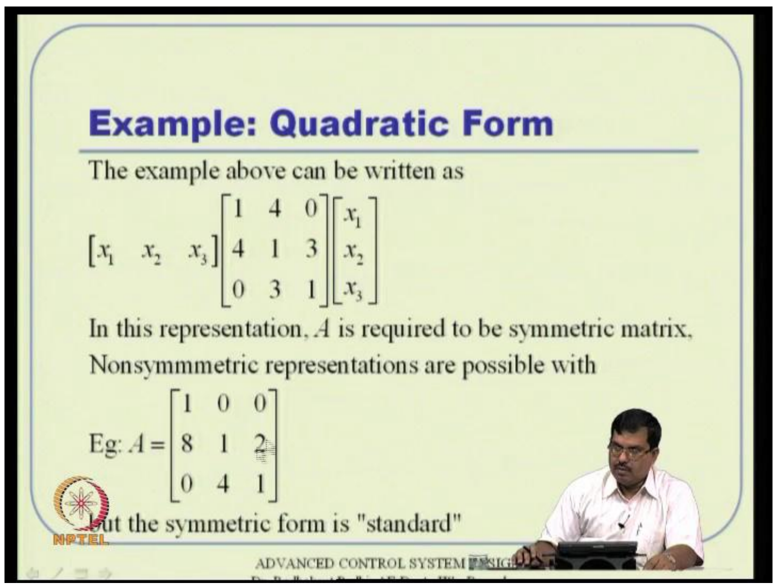

And if you want to put it in this form, then you can you can do that either this way or that way. If you select A matrix like that you can do that or A matrix this way also you can do that, but given this 2 choices obviously, you like to see this way this matrix is a symmetric matrix this is not actually.

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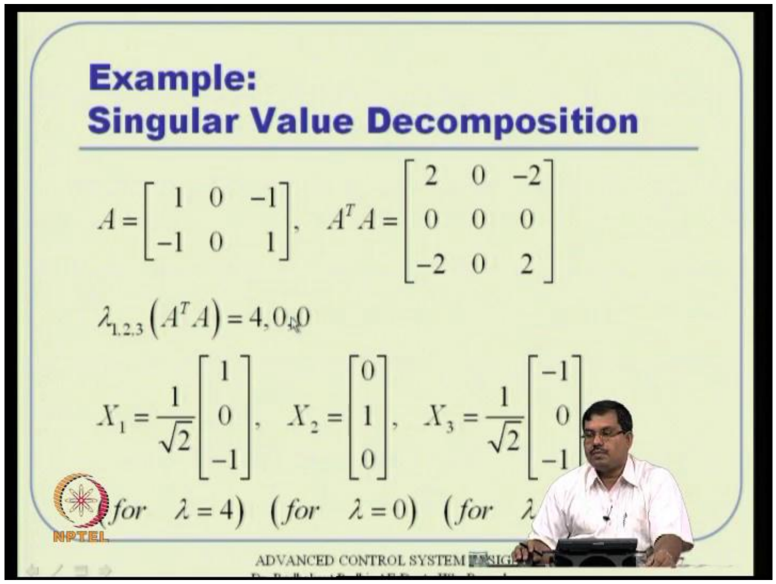

Now we will quickly review the singular value decomposition same example, what you discuss last class how do you do that actually. We started with an with an A matrix something like this. Then first thing is to find out singular values and singular values as positive square root of Eigenvalues of a transpose A and A transpose a is something like this and then A transpose A is guaranteed to be positive semidefinite matrix both A transpose A as well as A A transpose both are positive semidefinite, we seen that before and then a transpose a Eigenvalues are 4 0 0. This particular Eigenvalue if I find out it happens to be 4 0 0.

Then Eigenvectors associated with that is all given to us actually and in this case also see that the Eigenvalues are repeated, but Eigenvectors you can still find out linearly independent Eigenvectors. So do not need to find out generalize Eigenvector what I mean actually.

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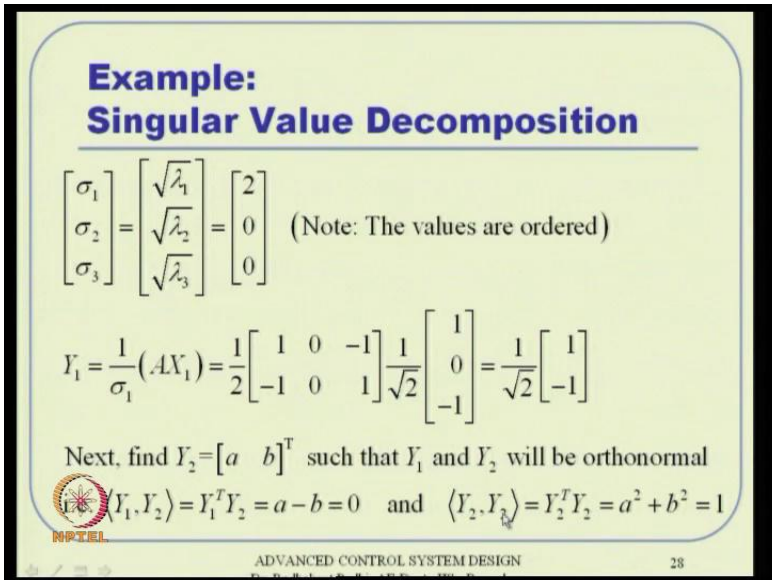

Then singular values are given as that way and the values are already ordered, if it is not ordered then you have to order that first actually. The zeros will be lost and the non 0 values will be in the top actually. Then Y 1 is given something like 1 by X sigma 1 into A X 1 which is all algebra is given.

Then Y 2 you have to find out in such a way that it is orthonormal to Y 1. So Y y 2 if I select that then Y 1 and Y 2 are orthonormal provided Y 1 and Y 2 inner product which is in this case is like this equal to 0, and Y 2 and Y 2 inner product with respect to itself it is to be like norm one basically this is norm square, so norm has to be one actually.

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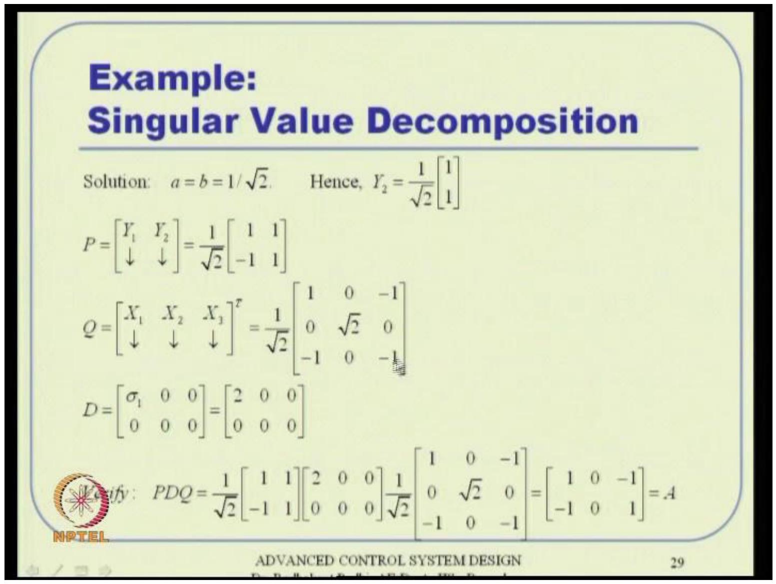

So if you put these 2 identities I mean this 2 equations then you will find out what is Y 2 actually. So you had Y 1 and you will have Y 2 now. So P matrix is Y 1 and Y 2 that is given to you and Q matrix is X 1 X 2 X 3 you put them together and take a transpose actually so that is given to you like this actually.

So the D matrix is given like this only one non 0 value. So that is 1 sigma 1 here, and then your all these other elements are 0 actually. So what you have now this singular value decomposition which tells you that a equal to PDQ remember that, so that is a equal to PDQ that will and if you do that PDQ it turns out to be a actually. So that is that is what it it tells you actually that is a singular value decomposition

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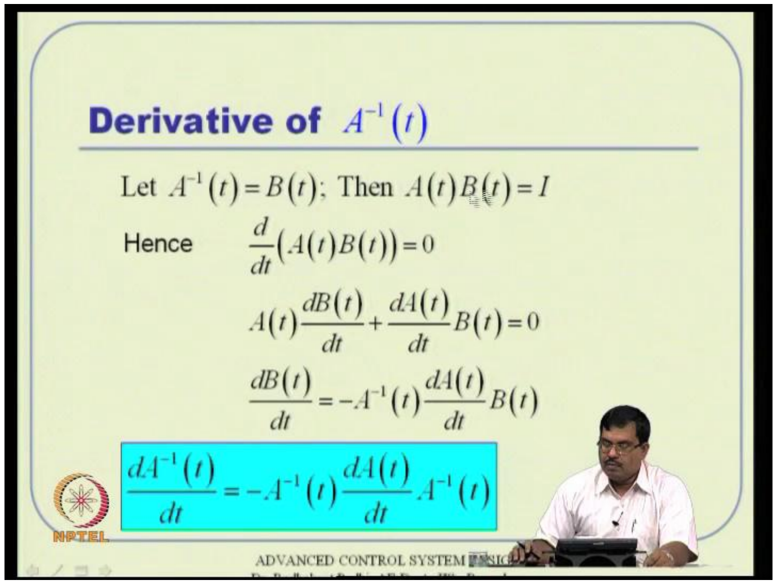

And one last observation here is that I told that derivative of A inverse of t is something like this how do you get that that is also easy to see that, if you if you take A B matrix which is inverse of that A inverse of t is equal to B t you just define that way.

Then because B is inverse A times B equal to identity, and you take derivative both sides what do you what do you get you take derivative both sides this is identity matrix. So derivative of that is 0 and then you tell if I take ,if I continue with this multiplication identity then this is a multiplication identity remember order cannot be changed here order has to be same. And then d B by d t is, nothing but that I take this one to right hand side this is that with a negative sign, and then A inverse I pre-multiply so I get that way.

And d B by d t B by definition is A inverse actually. so d A inverse by d t is, nothing but like this. This is very close to what we know in scalar operations actually. So I think all of this has given us some idea of matrix theory we can do so many I mean you can make use of so many properties you can do algebra with respect to matrix theory you can do calculus with respect to matrix theory, you can apply chain rules and you can do several several things with respect to matrix theory all of which will be very handy for our modern control I mean tack ticks and techniques which we will discuss further actually.

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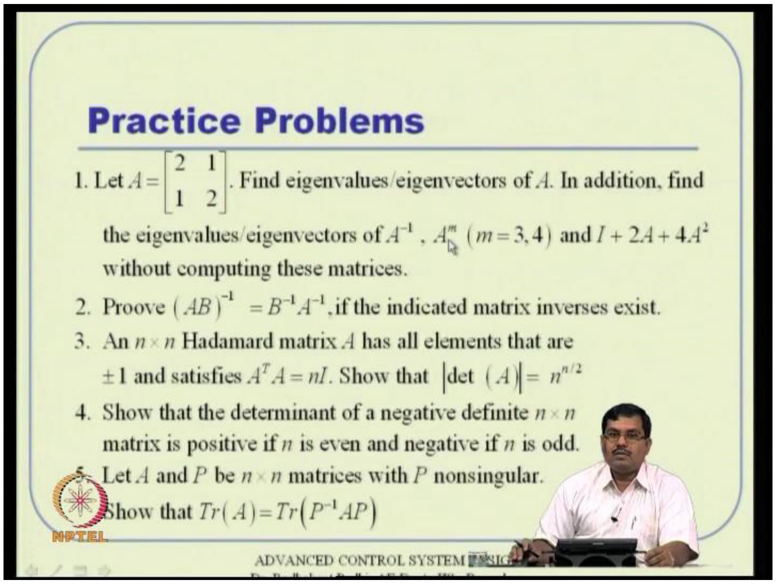

Now just for you to get comfortable with this matrix theory notations I have given some practice problems, you can probably see some of this thing very quickly we will see what what do I mean in this actually. So this problem 1 a matrix is given. I suggest that you solve this yourself to get comfortable with what all you learn from this couple of lectures actually. So if A matrix is like this then you have to find out the Eigenvalues and Eigenvectors for this A matrix. And in addition you have to find out Eigenbvalues and Eigenvector for all this actually like A inverse at the power 3 at the power 4 and as well as this polynomial, but remember you do not have to compute this matrix and find out. You have to use some of this properties that you studied and then you can quickly find out the Eigenvalues and Eigenvectors actually.

Now you have to show that this particular identity remember, we discuss about a transpose A B transpose is B transpose a transpose it is also valid for for inverse sense actually. A B whole inverse is B inverse A inverse provided the matrix inverses exist. So if you assume that the matrix inverses do exist then you have to show that this identity is also valid actually. Now problem number 3 ask you so do that this it is a something called Hadamard matrix is defined as n by n matrix for all these values are either plus 1 or minus 1 there are some nice properties for that and on the and it also satisfy this identity a transpose a equal to n into I. You cannot just keep on putting A minus 1 plus 1 arbitrary you have to put it these elements in such a way that a transpose a this equal to n terms identity that has to be satisfied.

Now under those conditions you have to show that determinant of this particular matrix is n in n to the power and n by 2 for n is the dimension of this matrix remember a is n by n matrix actually. Now problem number 4. So that the determinant of a negative definite martrix n by n matrix is positive, if n is even and negative if n is odd. So this very clear statement. So i will not elaborate on that and then problem number 5 it tells you that if A let A and P be n by n matrices both are square matrices and P is nonsingular of course.

Then you have to show that trace of a is equal to trace of this remember this is also like similarity transformation and if you I mean probably one small hint is you can make use of some of these Eigenvalue properties for trace and all that, but you have to also show that why these Eigenvalues of a similarity transformation, I mean if you take a similarity transformation Eigenvalues are preserved actually. You have to show that first and then you can relate those properties and all that actually.

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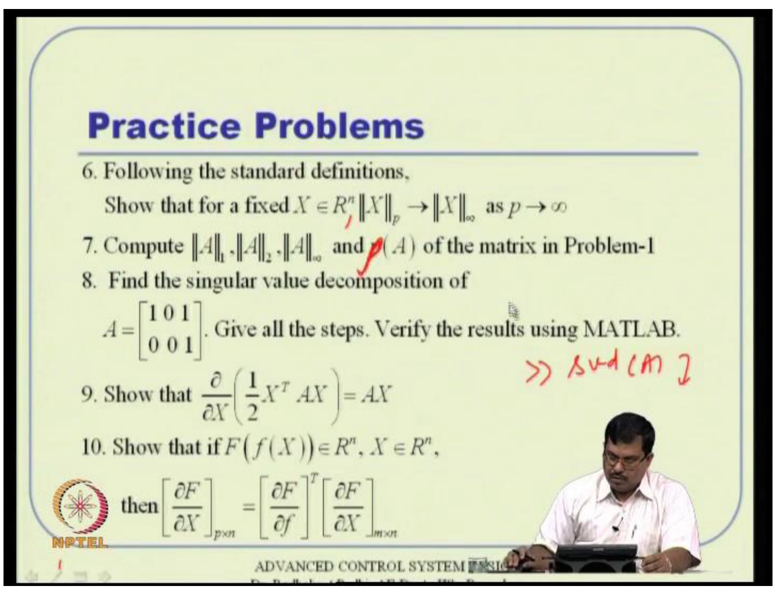

So that is number 5 and then number 6 is something like this. You start with the definition of norm p. And probably there is a comma missed out here. So let me put that here, so X is in

R n space that means X is n by n basically. And in that case you have to show that as p goes to infinity p tends infinity the p-eth norm is, nothing but infinity norm. You know infinity norm has a separate formula and compare to p norm actually. So you have to when p goes to infinity you show that both of this are compatible actually.

And then you have to compute these matrix norms or induce norms what you discussed as well as this I mean this is not p of A, this is rather this is rather rho of A spectral radius this is rho of A of matrix for problem 1, problem 1 is like that. So you can so that actually. And then you have to find out the singular value decomposition for this 2 by 3 matrix, you have to give all the steps here, and you can also verify your results quickly either by longhand calculation or quickly by MATLAB is also fine or PDQ what you find out then you have to tell P time D times Q if you multiply using MATLAB you will see that this is, nothing but A.

And if that does not turn out to be a obviously, you have done some mistakes somewhere and then you have to go back and correct your steps actually. In fact you can if you use s v d command of MATLAB matlab use that I told you before s v d command and then s v d of a will probably give you all this P Q all sort of matrix actually. PDQ all the 3 will give you actually, but I do not want that I want you to give all the steps with longhand calculation and probably you can verify your results with that actually.

And problem number 9 and 10 some of this identities that we discussed with respect to vector vector matrix calculus are we will not take it for for granted. We will try to show that actually, so that means if you take this particular quadratic form you have to see that it is there del by del X equal that and then you this n ruled formula what we discussed also need to show that actually.

So this couple of problems like some 10 problems I have listed out. If you solve it yourself then I think you will get lot of confidence to understand this as well as proceed further actually.As I told this 3 lectures as given is a quick over view of a variety of concepts from matrix theory which will help us proceeding further in our control theory concepts later with that message I will stop here thank you.