**Introduction to Computational Fluid Dynamics**

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**Lecture - 05**
**Hat functions, Quadratic & Cubic Representations**

So are there any questions? Any questions regarding box functions, hat functions what we have done so far, no okay that is fine.

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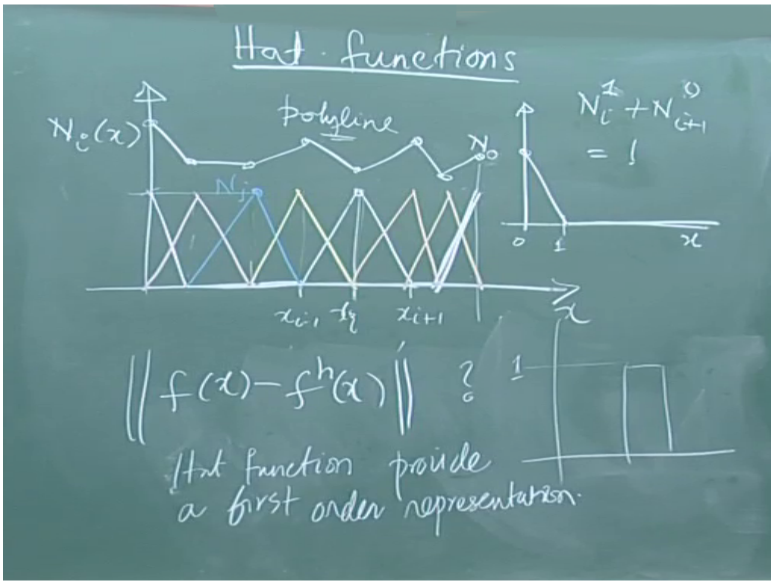


So what we have done so far in the last class is we have defined hat functions, and the hat function was defined as follows. The hat function centered about the point i right, so there is some interval a, b, we have created N+1 points which results in N intervals, the hat function centered about the point i for now for just for the sake of this discussion we will assume that all the intervals are equal in size, but you see the way I have defined the hat function they need not be of equal intervals okay.

So the points involved are xi, xi+1 and xi-1. So the hat function that is centered about xi takes the value 1 at xi, and drops in a linear fashion to 0 at xi-1 and xi+1, and from there over the rest of the interval in our problem it is 0. So the support of the function it is the support of the hat function spans 2 intervals as supposed to the box function which was over only one interval fine. So the Ni's are not quite orthogonal to each other.

So if you take Ni dot Nj, Ni and we take a dot product with Nj they are not quite orthogonal to each other, so what does this turn out to be? What is Ni dot Nj? Well, if Ni and Nj were 2 functions I and j are 2 functions that have absolutely no overlap, then we know what the answer is, so they have absolutely no overlap right, so this is Nj then Ni dot Nj is 0. On the other hand, if they do have an overlap, the overlap can come in different forms.

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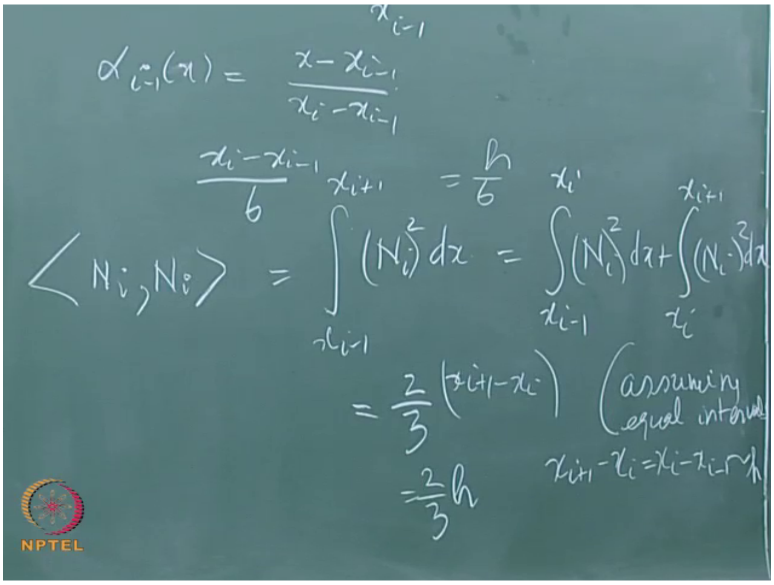

One is that the overlap can be over the interval xi-1, xi, it could be over the interval xi, xi+1, or it could be over the whole xi-1 to xi+1 okay, so we have 3 cases. So what do they work out? What is Ni-1 dotted with Ni? Right, Ni-1 dotted with Ni is a hat function that is centered at i-1 and is 0 everywhere, see the overlap clearly is over the interval xi-1, xi okay. So this in fact turns out to be integral xi-1 to xi, what is this xi -1 to xi obviously right alpha i x and 1-alpha i alpha of i or i-1 call that i or i-1 dx.

Where alpha i is a function of x is xi-xi-1/xi-xi-1, I have a feeling I defined it as alpha i-1 yesterday, it does not matter, I have feeling I may have defined this alpha i-1 it does not matter. What does this integral turn out to be xi-xi-1/6, you can check this out check that out okay. Now what about the other possibility which is Ni dot Ni that i=j right, this was j=i-1, this is j=i, just give me the integral xi-1 to xi+1 Ni squared dx okay, and what will this turn out to be?

You have to split this into 2 this is just integration so I am going to leave you to do it xi Ni squared dx+ xi to xi+1 Ni squared dx, each one of them is basically each of these elements here is basically linear, linear square gives me a quadratic integrate it gives me a cubic right, there are 2 of them. So this in fact turns out to you can check this out 2/3 xi+1-xi, and we are assuming equal intervals you have to a little more careful here.

Because this interval may not be the same that is xi+1-xi is the same as xi-xi-1 is like some h okay, since I have introduced the idea of an h here this is like h/6 and this is like 2 h/3 fine okay.

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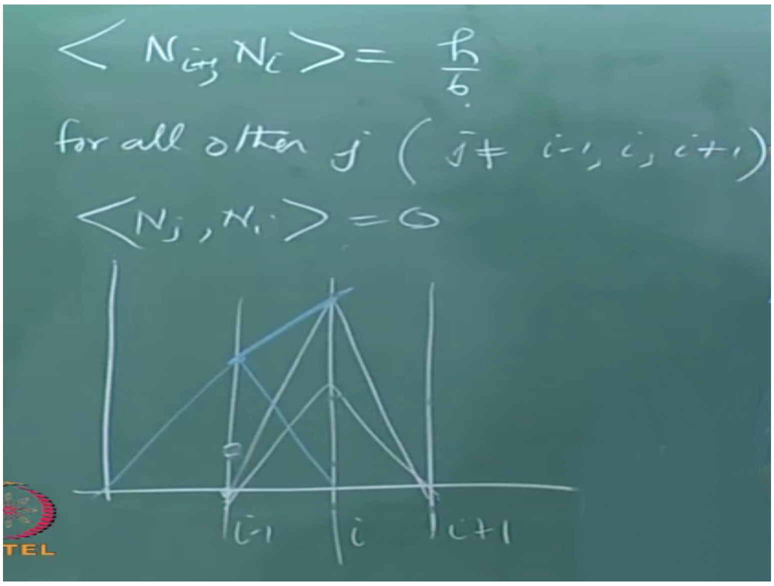

And what is the other one? The other one is going to be the same now, Ni+1 dotted with Ni is going to be basically h/6, so Ni dot Nj otherwise so for all other j for all other j meaning j !=i-1, i, i+1 for all other j, what do we have? Ni, Nj, Ni or Ni Nj=0 okay, not quite orthogonal. So it is not as good as the box functions right, so we have an overlap that consists of 3 of these functions, so we will have to worry about that every time we take a dot product or something of that sort, we will have to worry about that.

But what did you get in exchange for that? What we got in exchange for that is that we can represent on any given interval, we can represent a linear function. So if you take i-1, i, i+1 using hat function of course using one hat function, you have only one hat function, you can only change that one value right using one hat function you can change only one value. But the minute you introduce another hat function you can actually represent a linear interpolant between the 2, is that fine.

So what we have got? What we have is that we have an overlap of the hat functions, and therefore, we do not have pure orthogonality but we have locality. We have locality in the sense that if I were to change if I were to move change this one value, all it could do is influence the function in these 2 intervals, it is not 1 interval like in the box function, it is only 2 intervals. It is not as though the whole interval if I have a polynomial and I change the value, then the function changes on the whole interval okay, on the whole length of our problem domain.

Whereas here we have broken it up into sub intervals right, and in the length i- we can restrict any change that I make here, we can restrict it to those 2 intervals, is that fine okay. So we do not quite have orthogonality, but we have some element of orthogonality between the various functions okay right. So both in the case of the box functions and in the case of hat functions, I am not going to bother to make these orthonormal okay.

I am not going to try to make them normal, which means I am not going to try to make their magnitude 1, and in the case of the box function also I did not bother to make the magnitude 1. Why would I do that? What do I gain from it? Why would I do that, look at this function or look at this function why would I do that, why would I leave the value to which the function grows the hat function grows, why would I leave it at 1?

Clearly, Ni which is the magnitude is 2/3 right 2/3 h, so instead of going up to 1, if I go up to square root of 3/2 h, then the magnitude Ni dot Ni would be 1. why do not I bother to do it? **"Professor - Student conversation starts"** (()) (11:48) so constructing and finding the coefficients will be easier, that is the key, finding the coefficients will be easier I will tell you what I mean by that. **"Professor - Student conversation ends."** So let me leave that because I will need that a little later, I will tell you what I mean by this.

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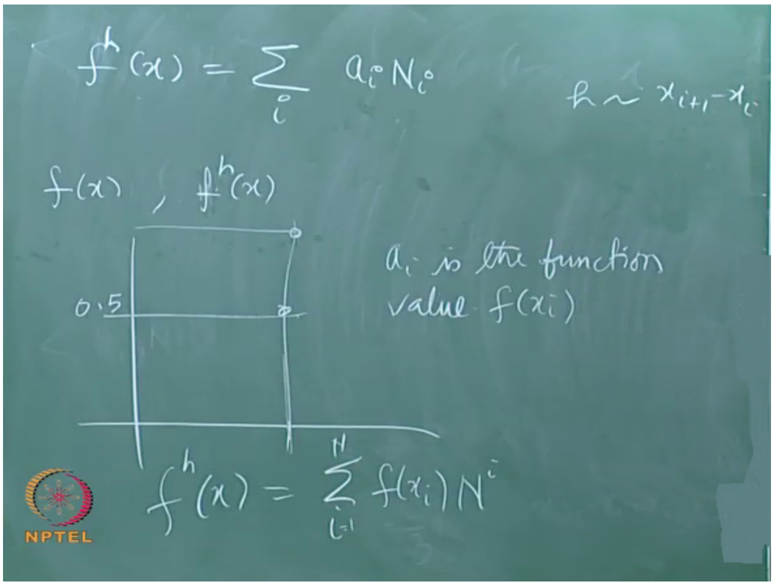


So if you want to represent, if we were to say that f(x) I am going to represent as summation over i I am not going to locate the interval right upper limit and lower limit we look at that, that is why that figure is left there, ai Ni and since I have already introduced the idea that h is like xi+1-xi for any i right, I change it from each equals which is like okay. Because we know that they need not be actually equivalent rules.

So I will introduce this notation I may not consistently follow this notation, but this is just so that you are aware of the notation okay, where it is clear I will drop the h, but very often to distinguish between the function that we want to represent and its representation to distinguish between them we stick an h on top right, I may use different f's but it is stick an h on top right. To distinguish between a representation that is on some grid right.

I am using this word grid for the first time some grid which is typically of size h, I will call it f of h and f is the original function that we are trying to represent, I do this because we are aware now that these 2 need not be the same that there is an error in representation. So the distinguish between the 2 I give it a notation that is like this, so f of h is ai Ni, if we followed the usual process, what we would have to do is we would have to take dot product to find the coefficient.

But if I let Ni go only up to 1, then I have the advantage that if the function value at the ith grid point is say 0.5, and Ni itself takes a magnitude only 1, then ai turns out to be 0.5 that is the idea okay, so finding the coefficients is easy. If you have tabulated data, you perform an experiment you have tabulated data you want to find the hat function representation to it, you have the coefficients in the tabulated data, you already have it that is the idea right, that is the advantage.

That is why I do not make it normal, I do not make it a unit vector, if I make it a unit vector then I lose that I will have to go through and write the process the data more, so it helps me in a situation here it helps me not to have to process the data right. So the ai is become automatically the values the function values at those nodal points is that clear fine everyone okay. So ai is the function value or the value of the function f(xi).

So this in fact becomes f of h x is summation over i f(xi) which I will very often just call i f i Ni, and they should be from this would really be the summation should go from 1 through N but we need to have we have to say something about what happens at the end points okay. So let us go back here and see what happens at the end points? So what do I have here. so apparently I have 1 interval here, so I have that.

And finally there is nothing on the left hand side my problem starts here, so I only get half a hat function. So my very first function, if I were to draw it separately my very first function would be this N1 or N0 depending on I have to actually you do have to be a bit careful with my, because I have count starts from 0, so I have that is 0, that is 1, and it goes on that is i, i+1, then we go to the other extreme let us go to the other extreme.

What do we have at the other extreme? We have one more hat function here, one more hat function here, and finally I have one last hat function here, and that will turn out to be the last hat function will turn out to be this function, there are N intervals, and this is going to be, the count starts at 0 that is going to be N, so the upper limit is going to be N is that fine okay, has any questions.

Using these functions, it is actually possible now that we can represent any function that we want as a piecewise linear straight line, polyline. So if you had some functional variation, you have the functional values the f set xi it is actually possible for you right to represent it as a polyline right, this is piecewise straight lines, and it is continuous. **"Professor - Student conversation starts"** are there any questions? (()) (18:25) unlike the box function integral f(x) and integral f h of x is the same, well we actually have to find out this is not conserved.

So we need to do a little analysis as what we have right now okay. **"Professor - Student conversation ends."** So one of the things that we have to do is maybe something I will do something I will leave for you to try it out, so first question that you have is what is f(x)-f h of x the norm of this okay, so this is one question, what is the norm of this, am I making sense. So in a sense that is connected to what you are saying right now.

So what is this now? What does this turn out to be? Why did we work there? What is the in the case of the box function right. Here, it is very clear here I said I left it as one, so that the function value comes out immediately, in the case of the box function what did we get in exchange? So in the case of the box function do you understand the question, I said that I have a box function and I said that I left this at 1 right, I left this magnitude the magnitude of the box function I left it at 1.

I could have made it a unit vector by making its magnitude square root of N, I could have made the magnitudes square out of N, and it would have been a unit vector the dot product with itself would have been 1, whereas I did not do it. So I left it at 1, there must be some advantage that I got because of that okay, so this will turn out that this is basically the average value, the area under this is what is the value here will be the average value of the function over this interval, you can just check and see whether that as a consequence with right okay.

So on the other hand here, we need to know what this is, just like we did for the box function, you need to try it this out and make sure that right you get an expression for this just try it out. I would suggest that you try functions of various kinds, let us look at I will just name a few and we will see what happens. So if I have a constant function, hat functions I can represent it exactly right, this norm is 0, meaning I can represent it exactly.

If I have a linear again 0, so the representation is definitely of order 1, if it is quadratic then I am going to have a problem right, so you can use a quadratic, try to represent the quadratic and see from there what is this error, see if you can get an expression for the error okay. If you can get an expression for the error, so that will tell you something about the nature of the error the error term, please do this error term will tell you the nature of the error.

And it will also tell you what happens as we shrink as we allow h to go to 0, what is the rate at which that error goes to 0? So it is important that you derive that expression fine. The other thing that we want is I want to make sure that we get this that hat functions provide a first order representation, right, the box function was a 0th order representation that means a polynomial of constant was represented exactly.

In the case of the hat function it provides a first order representation, we can up to a linear can be represented exactly, is that fine okay. Let us see what else we can now do, is it possible for us to go for higher order representations is there a way that we can get higher order representation okay, so we would obviously want to use polynomials of higher order, you want polynomials of higher order okay.

But I suspect just like going from a constant to a linear, when we went from constant to a linear we went from 1 interval to 2 intervals, I suspect if we go from linear to a quadratic that possibly will have to go to 3 intervals okay.

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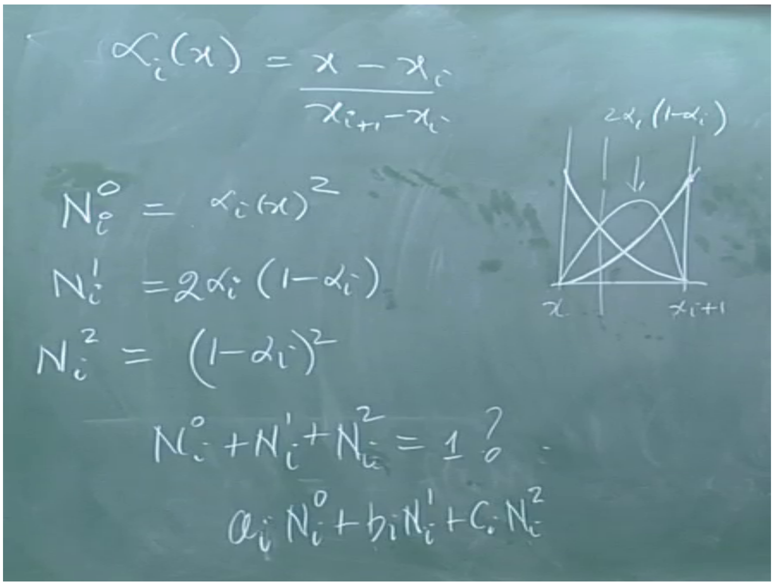

So what does the nature of those functions that we can use? So we saw that alpha of x was like x-alpha i of x-xi/xi+1-xi, what is something that we could start of with linear, it turns out we can actually use this to construct polynomials of higher order okay. So if I define 3 functions now on a given interval, Ni 0 is alpha i of x squared it is clearly a quadratic, Ni 1 is alpha i*1-alpha i also clearly a quadratic, Ni 2 have 3 such functions will be 1-alpha i squared, it will be a 2 there.

And ask question how do you know there is a 2 there, how do I know there is a 2 there, all of this Ni 0+Ni 1+Ni 2 at 1 is that work, why should they add to 1, no, no but why you see this is over the same interval, please remember this is over the same interval, this is over the interval xi, xi+1, this is over the interval xi, xi+1. What do these functions look like? One is going to be a quadratic that is like this, one is going to be a quadratic the slope is not 0 that is like this and the third is going to be a quadratic that is like this okay.

This would in fact be this one okay, so question that we have is why should this add up to 1, in fact if you look at the hat functions you will see that they are 2 even there they add up to 1, why do they add up to 1? You look at the hat functions here Ni 0+Ni 1, Ni 0 my notation is changed a little, I called it Ni+1 0 and Ni 1 add up to 1, why do they add up to 1? If they did not add up to 1 what is the big deal? You cannot represent a constant, you will not be able to represent a constant right.

And there will be some sort of funny interpolation going on now over and above right, if this did not add up to 1 that means that it could be it is varying in some fashion, if it did not add up to 1 it is varying in some fashion, if it did not add up to a constant, if it did not one is convenient if it did not add up to a constant that means it is varying in some fashion, am I making sense right okay.

And at any given point in the interval we want the function value to be there to be it is a linear combination of these 3 convex combinations of these 3 is that fine okay. So we do not want any other variation, we want the variation that you have picked here that should be picked up by these 3 functions. And therefore, they have to add up to a constant, and one is a very convenient constant okay.

So in this case just like we did ai, bi there, we could actually say ai Ni 0+bi Ni 1+ci Ni 2, you would have the sum of these 3 qualities, is that okay. As we go along you will see that I leave more and more of the details of working things out to you, is that fine. This is a quadratic, so what it allows me to do again is you have a function that is 0 at this end, 1 at that end right, starts at 0 at this end, goes to 1 at that end, okay fine.

So it is possible for me to independently vary what happens to the function on either sides of the interval right, and also there seems to be the same symmetric about this, this figure does not but it should be symmetric about that right, and there is the slope that is fixed at both sides. I am not I do not have that much of a great performance for quadratic, I just do it for completion so to speak. I will go on to something that is more important which is cubics.

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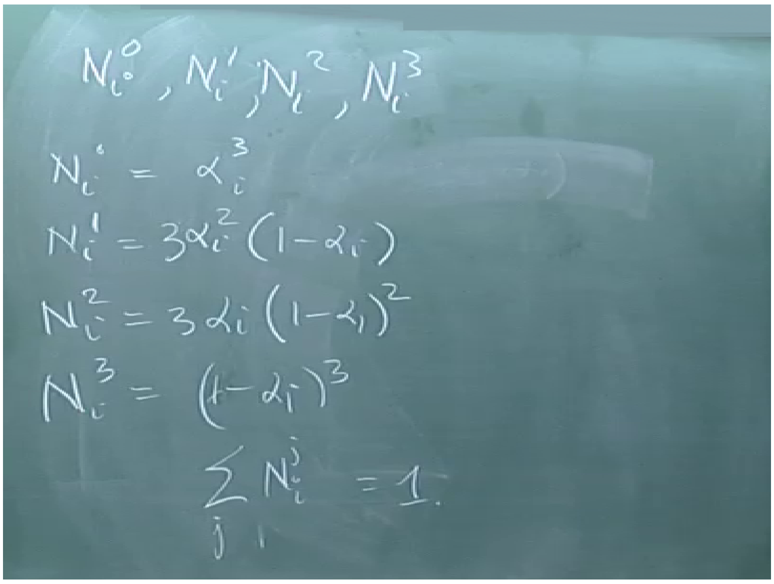

So in the case of cubics we would define how many? 4 functions, we define 4 functions which are Ni 0, Ni 1, Ni 2, Ni 3, and this should be relatively easy for you to figure out now. So Ni 0 would be alpha i cube it is function of x I am not going to keep writing that, Ni 1 would be alpha i squared 3 of them*1-alpha i right, so in your mind you should be thinking a+b whole cube. Ni 2 would be 3 of 3 times alpha i 1-alpha i squared, and Ni the last one would be 1-alpha i cube, is that fine.

So this is just a simple algebraic process that I am going through, and clearly the sum of these sum over j have Ni j is again 1, because the sum of the 4 of these will give me alpha i+1-alpha i whole cube right which is 1, which is where these constants came from, and now it is actually possible for yourself to construct right any order that you want. Now that you have seen this process, it should be obvious that you can construct any order that you want, is that fine.

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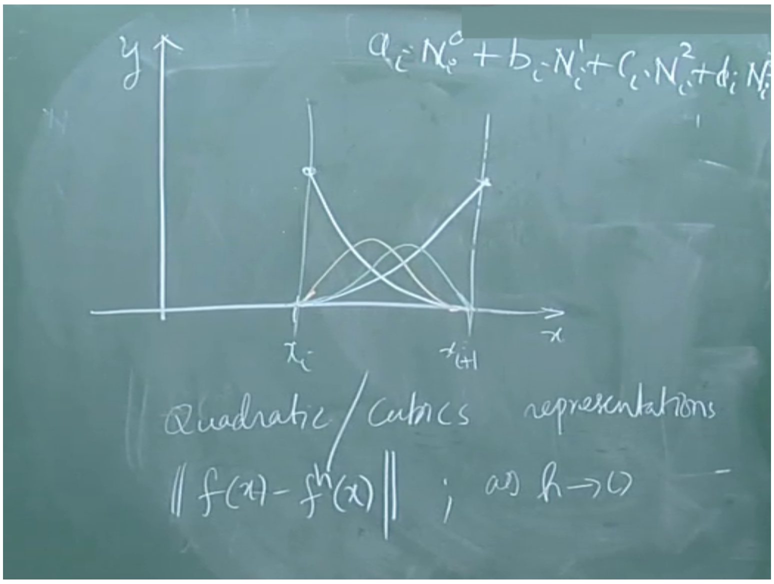


So why did I skip quickly over the quadratic and go to the cubic? Well, the cubic is an interesting function, that the cubic is an interesting function, because it gives me control over the function values at the end points, and so this is well I just call it y right now, x, xi, xi+1 the cubic allows me to do this. The slope here is 0, the slope there is 0 right, so it is going to go the 0 slope to 1 there right, the other function 1-alpha i cube is going to start at the 0 slope and 0 value here that is important 0 slope and 0 value, and goes up to 1 there is that fine okay.

The next 2 functions do something that is interesting, let me choose the next 2 functions do something interesting, so they start with a non-0 slope on the one side and have a 0 slope on the other side okay, they start with a non-0 slope on one side and have a 0 slope on the other side okay. So by on this interval by using these 4 functions, and using ai let me ai Ni 0+bi Ni 1+ci Ni 2+di Ni 3.

By using these 4 functions and these 4 coefficients, it is possible for me using the first one ai to move this value up and down, you understand what I am saying, just to change the single value alone. It is possible for me to use di and to change move this value up and down independent of the others. It is possible for me to use di and ci to actually manipulate the slope, you understand what I am saying.

So in the sense in the representing the function value, I have now got what I would call as the ultimate and locality and manage now right it is not quite the ultimate obviously you can go to higher order, but we like the direction in which we are going that we are able to manipulate the function value, and you are able to simultaneously manipulate the slopes at this point okay. And what is the cost that we paid? What is the price that we paid in exchange for this?

We have a higher order polynomial of course, you have more coefficients to handle, we have more intervals involved right. There will be more intervals involved, so we will end up having overlap who are multiple intervals, so what I want you to do is I want you to try the following for both quadratic and cubic representations right, I just say basically say quadratic and cubic representation.

If you know that the quadratic will represent anything up to a second degree polynomial right, and a cubic is going to represent any polynomial accurately up to a third degree polynomial, and beyond that of course then you are going to have errors. So for both of these you need to find please make sure that you are able to find f(x)-f h of x the norm, what else do we want? Which means that for a quadratic obviously it will represent quadratic properly it is likely that the cubic will not work.

Try a cubic, try a quartic even with the quartic try a higher order polynomial to see what is it that you get. And for cubic it is the same thing right, up to a cubic it works if you have a doubt try it out, and try quartic, try a quintic to see whether, what does the error term turn out to be? How does the error term work out? Okay, how does it work out? 1 what is the error term? And how does it converge, so one is find this and look at it to see as h goes to 0, how does it converge?

What is the order? What is the rate at which it goes to 0? Is it like h or is it like h squared, is it like cube right, what is the rate at which it goes to 0? Is that fine okay right? So what we have now is a way by which we are able to represent functions of various orders on any interval, we have a mechanism by which we are able to find the, what should I say the error in our representation.

We are hopefully once you have done the assignment you will see that for all of them you are in a way you have a mechanism by which you are able to find, the rate at which you get convergence okay, we will say a few words now about what I mean by that, the rate at which it gets convergence, the magnitude of the error and you have a mechanism by which you know what is the order of the representation right. The idea what is the order of representation is something that is clear okay.

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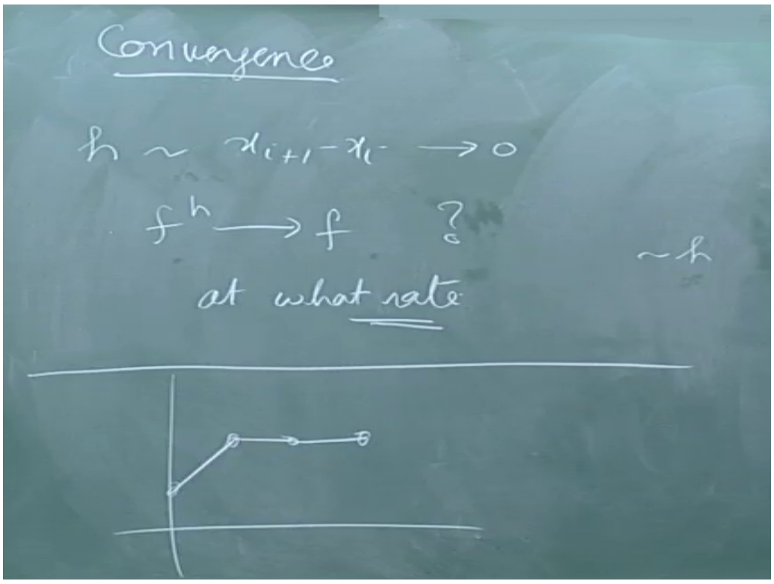


So what do I mean by convergence, in this context what do I mean by convergence? You take all these terms that we have used, what do I mean by convergence? So as h goes to 0, h which is like xi+1-xi for all i's, so they are all of the same order of magnitude as this goes to 0, which means that N goes to infinity the intervals get smaller and smaller, and the number of intervals becomes larger and larger.

We are asking the question does f of h go to f, first question does f of h go to f right it means that it convergence. And the second question is at what rate, do you have convergence and at what rate? Okay right. So this is the first definition that you are seeing of convergence we will see different types of convergence, I want to make sure you have to be depends on the context I want you to keep it in mind.

So this is as h goes to 0 we have a representation, the question that we have is what is the rate at which the representation that we have goes to the actual function, what is the rate at which the representation error goes to 0 okay. So and as I said there is always this question does it indeed go to 0 and if it goes to 0, what is the rate at which it goes to 0? Is that okay fine. So what we will do is maybe we will consider now how useful this is in what we have set out to do which is solve differential equations.

If I figure out the way to find out what is the order that is if you give me the data, that is why the example that I gave you earlier was if you perform an experiment you have tabulated data, from that tabulated data it is very easy for you to fit, you understand very especially for hat functions it is very straight forward for you to just extract the data and use it right, directly plot all you are doing is linear interpolation, if you think about it.

The only thing that I am telling you now is every time you use linear interpolation earlier you are actually using hat functions, it is that awareness that I want, every time in an experiment or in any work that you have done earlier, every time you had data and you are connected it by straight lines. You are actually using hat functions that is what you have to be aware, you understand what I am saying, every time you connected it by piecewise straight lines, you are actually using hat functions okay, that is the key.

So this is a situation where you have the data points ahead of time, this is a situation where you have what happens if you do not have the data points ahead of them? This is what we encountered typically when we are trying to solve differential equations, so if I had a dynamical equation a dynamical system a system that is varying in time, I want to predict its behaviour in the future.

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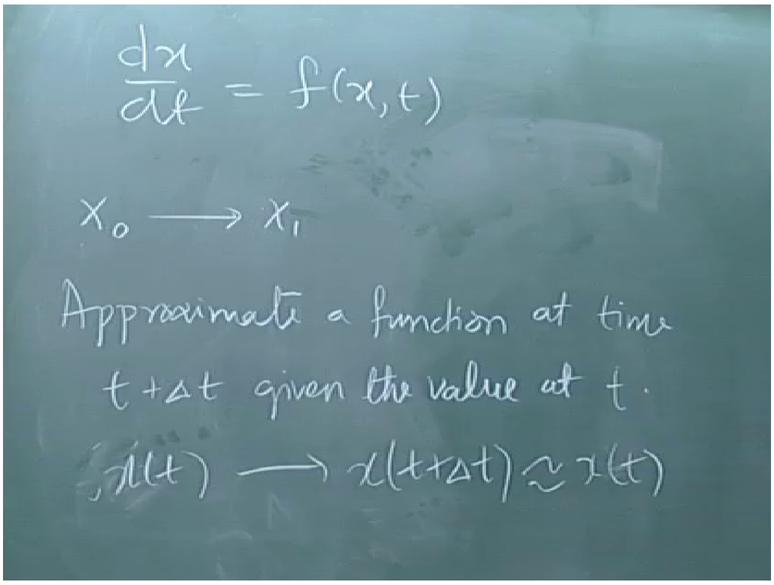


So I could in a sense have some dx/dt which is f(x,t). And this is something that is varying in time, and I do not have x this could be a particle that is travelling for instance right, propagation of a particle x could be its position, so I have dx/dt related in this fashion. I may not have a priori before I start, I may not have the data points in hand for me to fit function okay. So I can decide that I am going to use linear interpolants of sometime, I can make that decision.

But is it possible for me to say beforehand and get an estimate as to what is there that I am likely to make or they are related is it the same? Does this question make sense? see in one case it should be clear to you that if I give you the data that you are able to use the data and interpolate, it is all a matter of representing the function. But we saw the motivation that we used for why we are sitting out to represent function on the computer is that, we do not know what the function is.

And we want to come up with an algorithm to hunt for the function, there are different ways by which this can happen, in fact it is possible to use this, we will see those we will see how to use those functions later on. But right now as it stands, right the problem that I posed you is if I had a differential equation which I am integrating out in time, I deliberately choose time because that is in the future I am saying you do not know what the value is right.

So if I deliberately integrating out in time is there a way for me before I start the integration to say this is the nature of the function, this is the representation that I am going to use, this is how it is going to, this is the error that I am going to make in my representation fine. So there is a different kind I want to march off, I want to step off. Given that I have some initial condition X0, I would like to find X1 somehow.

Which is very different from a situation that we had there which was basically given data representing the data okay right. So what we are going to do is we are going to take the slightly different track, we are going to see here we have already seen especially for the box functions, we have already seen that the error was of the order of 1 over N, which basically means that there was of the order of h.

And you will see, when you do this you will find out errors again error in terms of h okay, so this may not be an obvious trigger for you. But what I would say is we will see whether we can approximate a function at time t+delta t given the value at t fine okay. So I use an example that may make you uncomfortable, but what they. So you have you all been taking quizzes, exams and all of that kind of stuff so far right.

So 6 semesters are done, next semester you go for interviews, people look at your grades, they say at this point in time after 6 semesters this is your cumulative grade point average, this is your grade. What does your grade likely to be at the end of 8 semesters? That is the question that we are asking. Given a time t that I have a function value your grade point average at a given time, what is your grade point average likely to be at t+2 semesters.

Someone interviewing you giving you sitting across the table and asking you questions, they look at your score sheet, and they want to make an estimate fine. So give me a good estimate what would so if you say have a grade point average which is 8.5, what would you tell them? What would I do if I were interviewing? What would you do if you are interviewing me? What is a good estimate? It is likely that it is the same right, one good estimate is it likely that is the same.

Okay, you made this, this is a cumulative grade Point average over the last 6 semesters, it is likely that after at the end of 8 semesters this is going to be your CGPA, am I making sense. So it is likely, so one thing one way by which we could do it is so if you give me x of t, it is likely that x of t+delta t is like approximately x of t, that is an approximation right, and I have change the language that I am using now.

So far I have been talking about the representations on the computer, but our experience so far I am now admitting, it is an approximation, it does not look like we are going to be able to represent functions exactly on the computer right, so it is going to be an approximation. So now I come where you have live with this reality, I come now to the point where we say it is an approximation.

I want to now ask the question, what is the approximation? What is the value? And what is the error in the value? Okay, and I posed this question in this fashion, because it immediately motivates if you look at something like x+t+delta t what do you want to do? Expand using Taylor series, am I making sense, you look at something like x+t+delta t you are so why do not you just expand using Taylor series okay. So that is basically what we are going to do.

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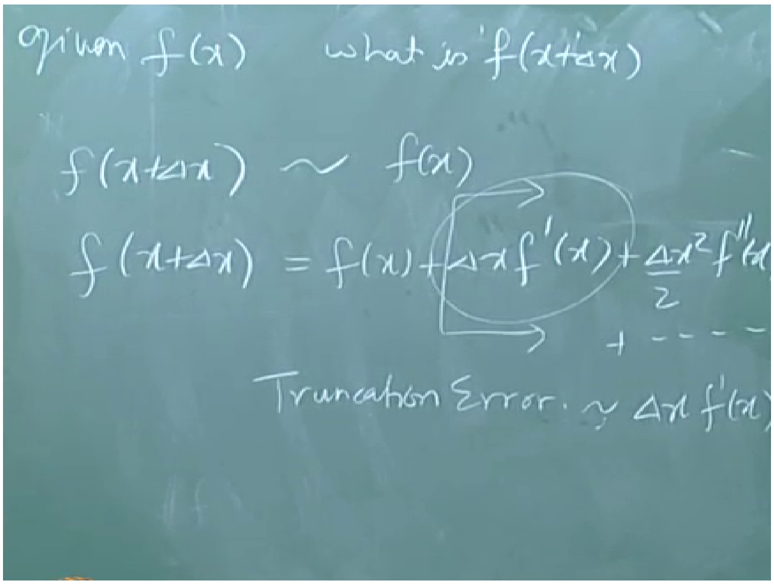


Essentially, what we will do is we will now sort of reset right, that was the notation we introduced so far, but we will be going to standard notation. So if I have f(x) given f(x) what is f(x)+delta x, I introduced the notion of t and time just to motivate the fact that you may not know the function ahead of time that is the only reason why I used t. The question is given f(x) what is the good approximation for f(x)+delta x?

Well, you can say f(x)+delta x a good approximation is a f(x) right, so if I am getting something today, if my salary is f(x) today, good approximation is f(x)+delta x right, it is the same salary. So it is reasonable approximation with may not always be right, but it is the reasonable approximation. So what is the nature of the error?

Use Taylor series f(x)+delta x is f(x)+delta x times the derivative f prime x the prime indicates differentiation with respect to x delta x squared/2 f double prime of x+ an infinite number of terms and I am assuming that f(x)+delta x is f(x). And in making this assumption I would throw away all these terms in Taylor series okay, all of these terms infinite number of terms I throw them away.

I have an infinite series which I have truncated right, the error that you create by taking an infinite series and truncating it is called the truncation error right. If you have an infinite series representation for say a function, the error that you make by throwing away whole bunch of these terms is called the truncation error, and we will represent the order of the error by the leading term which is delta x. So the truncation error here is of the order of delta x f prime of x, is that fine right.

So we tend to call this a first order error, because the exponent over delta x is 1 right okay. So we will come back to this maybe on Monday, I will try to do a demo of some kind, then I would suggest that you try using hat functions and representing various functions. There are many different types of functions as you can okay. And I will come back in the next class, and maybe try to do a demo, so that you can see. There are some issues still with respect to represent in functions fine, I will see you in the next class.