Introduction to Computational Fluid Dynamics
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Lecture - 04
Representing functions - Polynomials & Hat functions

So in yesterday's class we looked at box functions right, and we saw that we could use the whole set of box functions to represent any given function right. I was originally planning to do hat functions today, but firstly maybe we look at polynomial functions as a means of generating basis vectors say on a given interval right, and see what is the problem with that and then we will go onto hat functions, is that fine.

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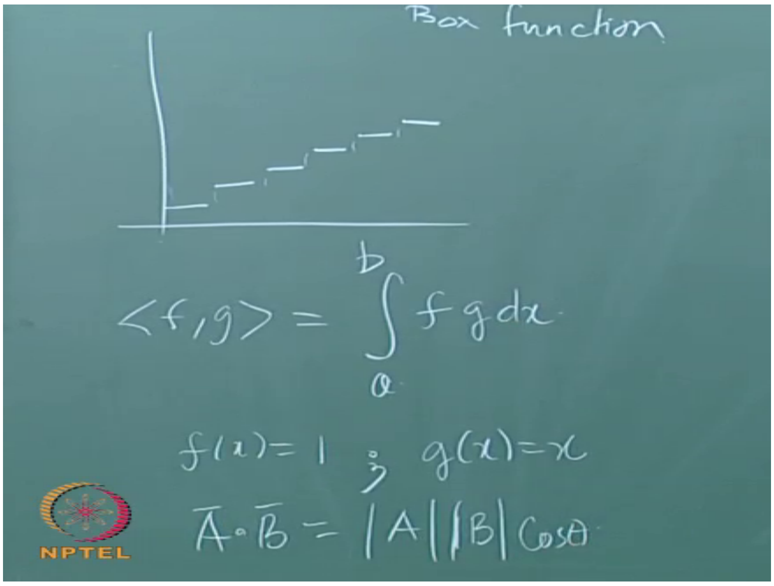

So the question is the problem that we had with box function was that even for a straight line, if we wanted to approximate the straight line we got because our function is a constant on a given interval, we got a representation which was piecewise constant right, and we could get as close as we want to the function that we are trying to represent. But then as a consequence the jumps that we get the number of jumps that we get in the function representation increases right.

So we are trying to ask the question, is not there a way for us to get something that is smoother? We have defined the dot product yesterday as the dot product of f, g as the integral if the functions are defined on the interval a b f g dx. We will see what this means if you just take an interval again says 0 1 or something of that sort, and what it means to the standard polynomials that we deal okay.

Let us see whether we can use those to represent our functions, so consider the functions 2 functions f of x=1 and g of f x=x to start with okay, are these functions orthogonal to each other or what is the angle between these functions does that make sense. So once you define the dot product once we have defined the dot product, we can use the definition of the dot product that we had earlier from geometry where we said that A dot B is magnitude A magnitude B cosine of the angle included angle between them, you can use a similar idea.

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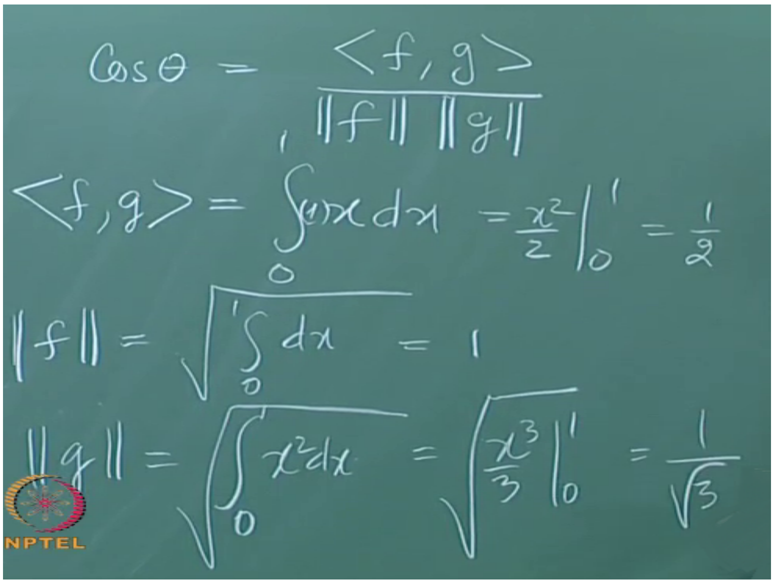

And basically say that cosine of the angle between 2 functions would be the dot product, g/norm of f and norm of g right, analogous to what we did with vectors you could define a theta in a similar fashion. So the question that I am asking, so you already use actually we already used the property that theta is pi/2 and said that 2 functions are orthogonal, we have already done that yesterday. The question that I am asking is it possible for us to find the angle between the functions f and g as given there okay.

So what is f dot g? What is the dot product? Integrals x let us say the functions are defined on the interval x belongs to the interval 0, 1, we take x from on the interval 0, 1, the function is defined on the interval 0. 1, so it is 1 times x dx, which gives me x squared/2 between the limits 0, 1 which 1/2, is that fine. So clearly they are not orthogonal, what is magnitude f? Magnitude f is the square root of integral 0 to 1 dx which is 1.

And what is magnitude g? The norm of g integral 0 to 1 x squared dx square root which gives me x cube/3 between 0 to 1=1/root of 3, is that fine okay. And therefore, of course all of these numbers look familiar, because we just dealt with something very similar in the last class.

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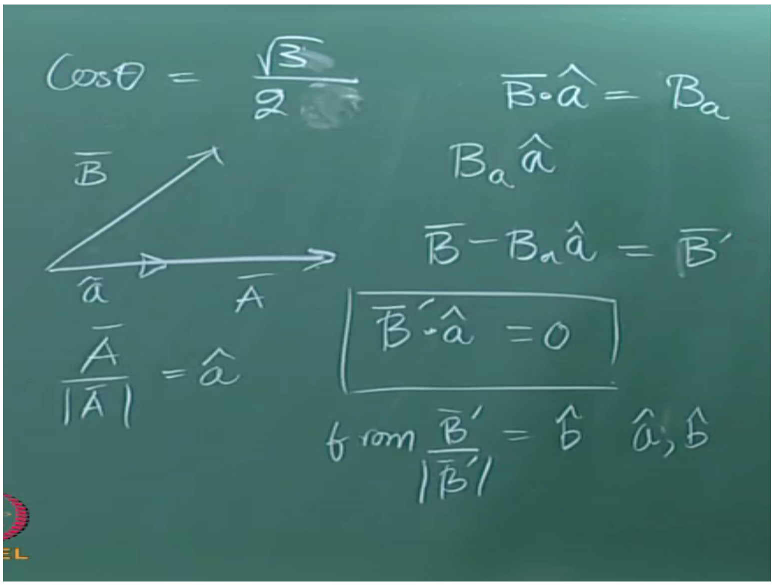

And therefore, what do we get for cos theta 1/2 root 3, and you can consequently find theta. So it is just an interesting exercise, you can consequently find theta, what does it mean? Is there a mistake so I am sorry root 3/2, it gives a much better number right, gives you much better angle okay, right? So but I am not really interested in finding theta here right, or I realize that theta is not pi/2 okay.

And we have already seen earlier that is very convenient to have these functions or the basis vectors that we get to be orthogonal to each other right. We have already seen that it is convenient to have these vectors orthogonal to each other. So given this we will follow the same process that we did earlier and try to get 2 sets of vectors, 2 vectors which are perpendicular to each other from the 1, x, so 1, x are linearly independent, 1 and x are linearly independent.

How do I get 2 vectors so that the angle is pi/2? So what I do is I repeat what we did earlier, so if this is A and this is B right, so from A of course I can get the unit vector A relay which is a hat right, and I can find the projection of B on A. How do I find the projection of B on A? I take B dot a okay, so B dot a gives me the protection the component of B that is along a, right B component of B that is along a.

Let me just give it a subscript B sub a, B sub a times a hat, what is that? That is a vector representation of the component of B along a right, so if I subtract this from B, B-B sub a a hat and I call that, what shall I call that? I just call that B prime, what is B prime? B prime is a vector B with its projection from a removed from it okay. So now the question is what is B prime dot a? B prime dot a is 0, so we have managed to construct the B prime which is orthogonal to a.

And from B prime I can get a I will just call it B hat by dividing by the magnitude of B okay, so you gave me A B to start with and I come back with a prime b prime I am sorry a hat b hat which are unit vectors which are orthogonal to each other okay. So you have given me one set of vectors they are now managed to extract from that a hat b hat which are orthogonal to each other. So if I had a third vector C, I could repeat this process.

If I had a third vector C, what I would basically do is from the third vector I will subtract out the b hat part, I will subtract out a hat part and I will be left with something that is purely C right, if I left with nothing that means C was linearly dependent on the other 2, is that fine okay. We will do that now to the functions, we will do that now to the vectors 1 and x.

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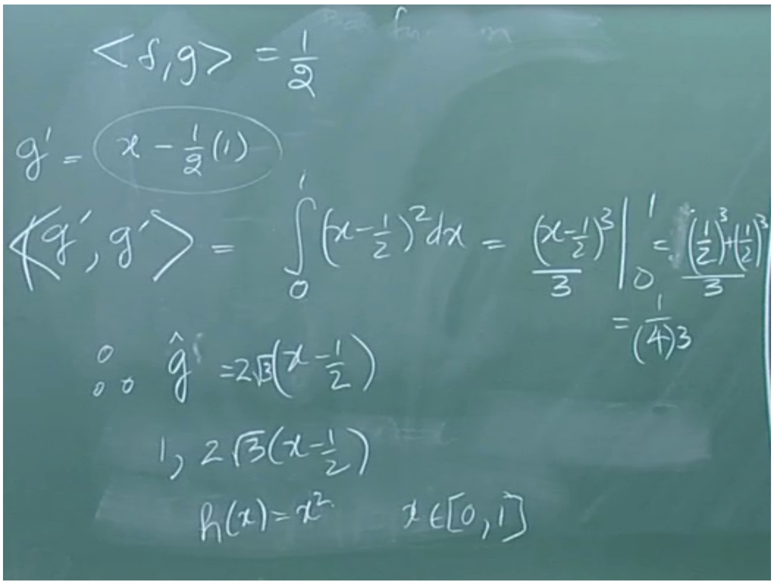

So back here you ask the question what was f dot g? f dot g was 1/2, so this is this times 1 so the function that is along that is the component of f that is along g or g that is along f is 1/2, if that is g along f right, and f is a constant. And therefore, if I subtract out if I repeat the process B-B prime a B-B dot a if I repeat that process, what I am going to get is I am going to get x-1/2 times 1, the basis vector is just the constant function 1 okay.

And this should be now my new improved g prime, is that fine okay. So can I make this a unit vector is it possible for me to make it a unit vector g prime, how do I make g prime the unit vector? So what is g prime dotted with the g prime x-1/2 squared dx on the interval on which it is defined, which is x-1/2 cube between 0 and 1, and what is this 1/2 cube+1/2 cube I am sorry which =1/4 is that right I am sorry by 3, by 3 fine, okay.

And therefore, g hat g if you want to call it g prime hat if you want to call put a hat on it just like we did for the other unit vector or just call it g hat, g hat could be x-1/2 I have to divide by the magnitude, the magnitude is 2 times square root 3, are there any questions, is it fine. So now we have the vectors 1 2 times square root 3*x-1/2 is that fine, these are orthonormal magnitudes are 1 and they are orthogonal to each other okay.

Let me add a third vector into the mix what if I added x square, I want to add x square, so define an h of x which is x squared, if I define an h of x which is x squared and this is still on the interval 0, 1.

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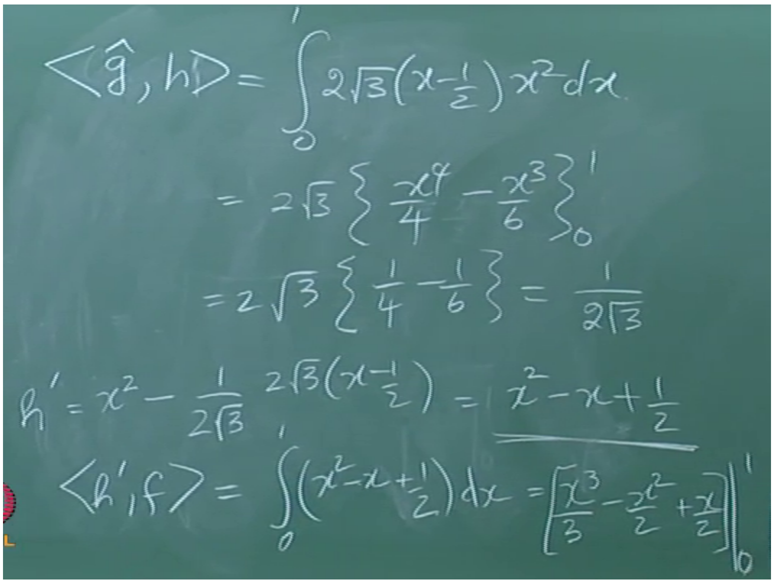

If I define an h of x which is x squared, what is g hat dotted with h that is 2 root 3 x-1/2*x square integral from 0 to 1 dx, is that fine. So what does this integral give me? This gives me a 2 root 3* that gives me a x to the 4th/4-x cube/6 between the limits 0 and 1. So 2 root 3 times 1/4-1/6, what does that give me? 1/2 root 3. So that is the component of h along g hat, so we repeat this process, so from h that is x squared I subtract out this component, so that is 1/2 root 3 times what is the basis vector 2 root 3*x-1/2.

I am repeating the same process that I did hear B-B a times a, I am repeating the same process that I did there which of course gives me x squared-x+1/2, and there is something wrong we have to okay that is fine. So what is the other thing that we have that is x square-x+1/2 that is fine. So what do we have now? So we have removed the g hat component this is has removed the g hat component.

So what I will call this is h prime, now I want from this I want to get rid of the component that belongs to the 1 the function 1. So what is h prime dot f? That is integral 0 to 1 x squared-x+1/2 times dx which gives me x cube/3-x squared/2+x/2 between the limits 0 and 1.

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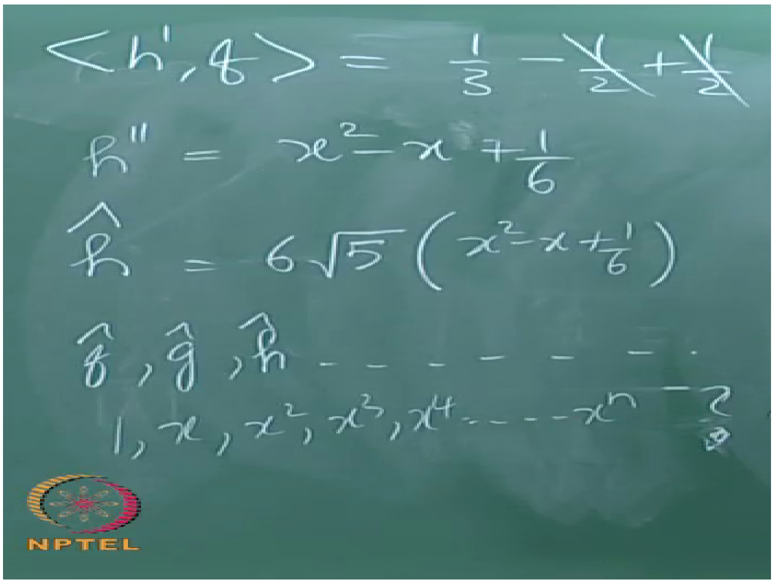

And that should give me the projection of h prime on f which is nothing but 1/3-1/2+1/2 which is 1/3, is that fine. And as a consequence, if I subtract this out from here I am going to get h double prime which is x squared-x+1/6 okay, if I subtract the 3 from the 1/2 I get 1/6. And I let you verify, you can verify that h hat in fact is 6 times root of 5 x squared-x+1/6, and just check that out, find out what the magnitude is.

"Professor - Student conversation starts" (()) (17:50) If I am taking the no I think you are saying g instead of h prime f through h f, g prime and, whether I take h prime from there or h it does not matter, you are saying why did not I just do x squared times 1 whatever that is fine. The order does not matter, what you have to basically do is you have to make sure that if you have a set of vectors that you are subtracting out this and this from x that is true.

You could what you are basically saying that the expression is a lot easier, I did not have to include that x+1/2 because anyway it is orthogonal that is true, because the part that I am subtracting out you have to understand the point that is making the part that I have subtracted out here it basically comes from is already orthogonal to f, so there is no reason to include that in the calculation is that fine okay, are there any questions. **"Professor - Student conversation ends."**

So you can see that now I have I will call them f hat, g hat, h hat, you can see that I can actually generate the whole set of these functions okay. And in fact I can just take the set 1 x, x squared, x cube, x to the 4 and so on, and I can generate a whole set of the basis vectors that are orthonormal orthogonal to each other and the magnitude is 1 right based on our definition of the dot product, is that fine okay.

Now why do not we just use this to represent our functions, right why do not we just use this to represent our functions, so that is one question that we have, why bother with box functions, these are nice and smooth functions, why not just use these to represent our functions, is that fine okay. Of course if our function had a cubic variation and we took only the first 3 terms right, you are unlikely to pick up the cubic variation properly, am I making sense.

You can see want you can go check to see what kind of an error that you would make as one. The other problem is just like in Fourier Series maybe we look at for Fourier Series a little right now, just I am not sure how many of you are familiar with Fourier Series, there is the issue of how do we when we are hunting remember do not forget why we are doing this, why we are constructing these orthonormal functions right.

Just to recollect our governing equations or differential equations the solutions are functions, we want to write programs that will systematically hunt for the solution which means systematically hunt for the functions. So we are trying to create a set of functions which have some structure in them, so that we can search them in a systematic fashion okay, so such a set of functions which have this nice structure, what is the distance between functions and so on, such as set of function we will call it as space function space with space of functions right.

And we are basically trying to construct a function space that is essentially it okay. So let us see what happens, let me we will get back maybe what we will do is we will do Fourier Series also we go along and then see what is the issue with, what is the issue why cannot we why do not we use this right, and we do use it sometimes. Let us see what we get? Now one of the points that I want you to check out you can try this out now.

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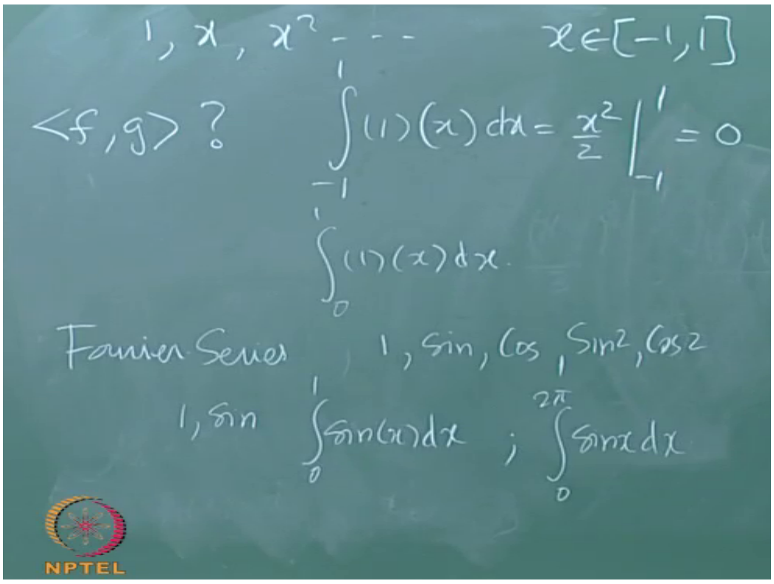

If I take the same functions 1, x, x squared and so on, if I take the same set of functions, but now they are defined on the interval -1 to 1 does anything change? So what happens to f dot g? Becomes -1 to 1 1 times x dx is x squared/2 from -1 to+1 right it is 0, they are orthogonal. So the orthogonality depends on the interval in which the function is defined, the dot product is defined over that old interval, it may be obvious right when you look at it.

But the number of times students make a generic statement saying sine and cosine are orthogonal to each other and then integrate on the wrong interval right it happens it happens to all of us right, so you have to a bit careful it is orthogonal from -1 to+1, but not orthogonal on the interval 0 to 1 okay. I would suggest that you try to do a few of these on the interval -1 to+1 just like I have gone through right.

So for on the interval 0, 1 I would suggest that you try x cube or something like that sort just to make sure that you are able to get through on that, and then do this from -1 to+1 do a few of them okay. If you have had ordinary differential equations if you had a course on ordinary differential equations before pay attention to the functions that you are getting, and ask yourself have you seen them before right.

You should get a familiar set of functions right especially if you have done a course in ordinary differential equations this interval -1 to +1 should give you should yield a familiar set of functions right. Let us look at I am not going to do Fourier Series in great detail here, so Fourier Series the functions that you are looking at out of the form 1 sin, cosine, sin 2, cosine 2, and so on, we have functions of this nature okay.

So obvious question from this discussion the obvious question is, are the functions 1 and sin orthogonal to each other right, so now you should always remember you have to ask the question on what interval are we talking right, what is the dot product definition of the dot product? And what is the interval on which we are talking? So if you say that on the interval 0 to 1 sin of x dx not 0 right. On the other hand, if you say on the interval 0 to 2pi sin of x dx you do get 0 fine.

So the functions 1 and sin are indeed orthogonal on that interval, and of course you can figure out how to go about normalizing it and so on. So the point of this discussion is that the interval the domain on which we are defining the functions is important right, for our idea of orthogonality just as it is important for our idea of the dot product.

And of course when we are actually solving problems, when we get to the point where we are actually solving working on fluid mechanics problems and asking questions and answering questions in fluid mechanics, you will have you will know what is the extent of your domain and therefore, you will know what are A and B fine. So why do not we just use this? Why do not we just use 1, x, x squared and so on? Why not just use Fourier Series right to fit?

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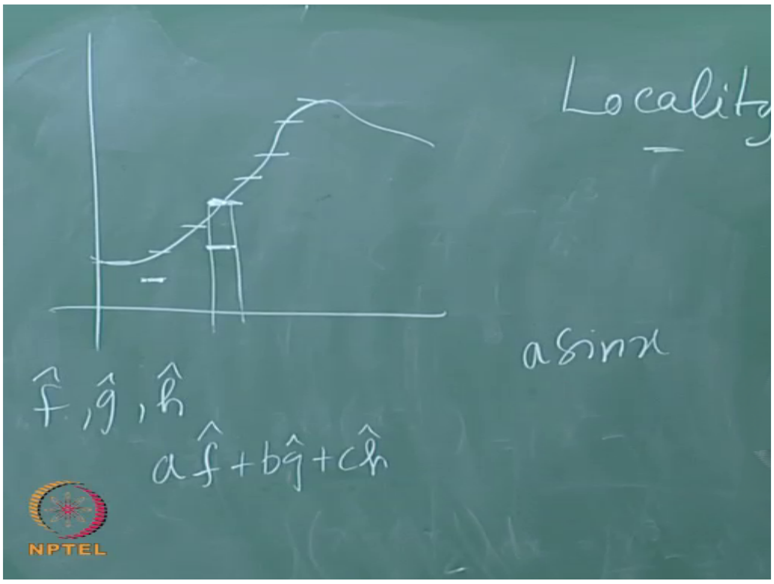

So if I give you a function, if I give you a graph, if I give you some arbitrary graph why not just use these polynomials in order to approximate this function to represent this function, why do I go through this headache of trying to find piecewise constant that will approximate the function okay. The idea here is that if I have my f hat, g hat, h hat that I got earlier that I got here. If I have the f hat, g hat, h hat, I can try to find a f hat+b g hat+c h hat, I can try to find the coefficients a, b and c in order to approximate this function.

And then adjusting these values while I am hunting for the functions, any change I make to a will translate the whole graph up down, any change I make to b will cause the slope to change, any change I make to c right will cause the curvature to change everywhere, keyboard is everywhere right. So if I am now if I have this complicated function, and I am trying to fit a curve to it, every time I adjust 1 coefficient to fit some part that trouble is going to get spoil somewhere else, very likely that it will change somewhere else.

I do not have what is called the property of locality, I do not have this power where local changes in coefficients cause only local changes in the functions, I want to be able to rise and lower this. Whereas, if you take this box function, it is possible for me to take this and lower it there independent of what is happening elsewhere, I can just change the level of just at 1 interval right. So the box functions clearly has this property of locality I can just change in 1 interval.

I can change the level of that function just in 1 interval without affecting anything else that is happening elsewhere okay, so this property of locality is very important, this property of locality gives me this freedom for me to adjust my solution as I go along okay, in particular without affecting other parts of my solution that I have already possibly adjusted to my satisfaction right. Whereas if I do not have locality, then I change any changes that I make is a global change right.

So if I say a sin x, if I change a sin x is going to change in fact If you think about sin x actually define from -infinity to +infinity, so you are changing everything from -infinity to +infinity just by changing this a right total global change. Whereas, what I would like to do is I would like to keep it local fine okay, so for that reason right now I am going to reject using f hat, g hat, h hat right having gone through this effort, I basically says it is nice I know how to do it, but I am going to reject these functions okay.

You can come back later and see whether it is possible for us to use them or not, what we would really like is we would like the smoothness that these polynomial bring and the locality that the box functions got, we would like to have the smoothness that these polynomials bring with tight to the locality that the box functions okay. And therefore, as a consequence now merging these 2 are going to come up with hat functions okay.

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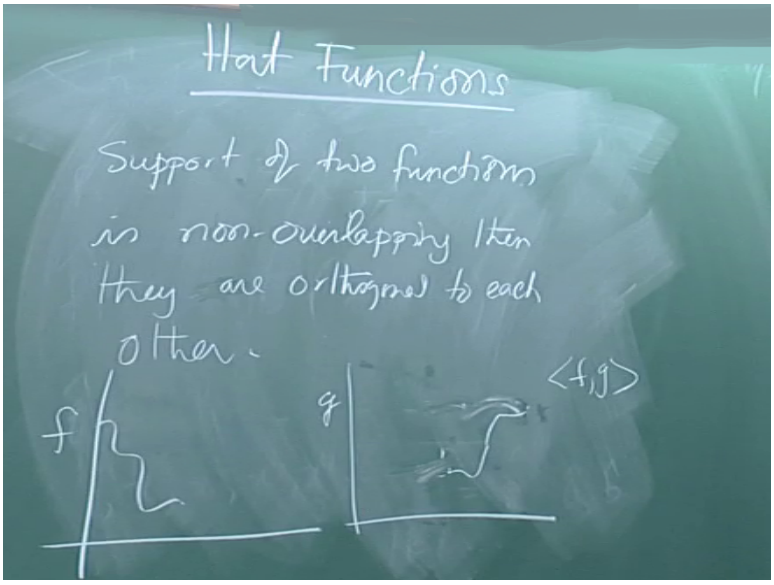

You are going to defend this as follows, what we realize is if you have so from the yesterday's class we know it is the support of 2 function is non-overlapping is not overlapping then they are orthogonal to each other. There is a reason why I repeat this, there is a reason why I write this out and repeat this, so it has only to do the support right, remember the support basically means the function is non 0 and that in that part of the domain right.

So it has only to do with the support, it does not actually have to do the function value. That means that if I have in yesterday I chose functions f and g in a fashion such that the support was non-overlapping, this was f that was g, and yesterday we basically said that f dot g because the support is non-overlapping it turned out that f dot g is orthogonal to each other. The question is does it have to be a constant there, it could be anything as long as it is not 0 right, is that fine.

So since this is true, since it is coming from the support what we will do is we will try to pick one of those polynomials to do it, and to keep life easy we are going to pick the linear, we are going to pick x right, we are going to be a bit careful here so we pick x.

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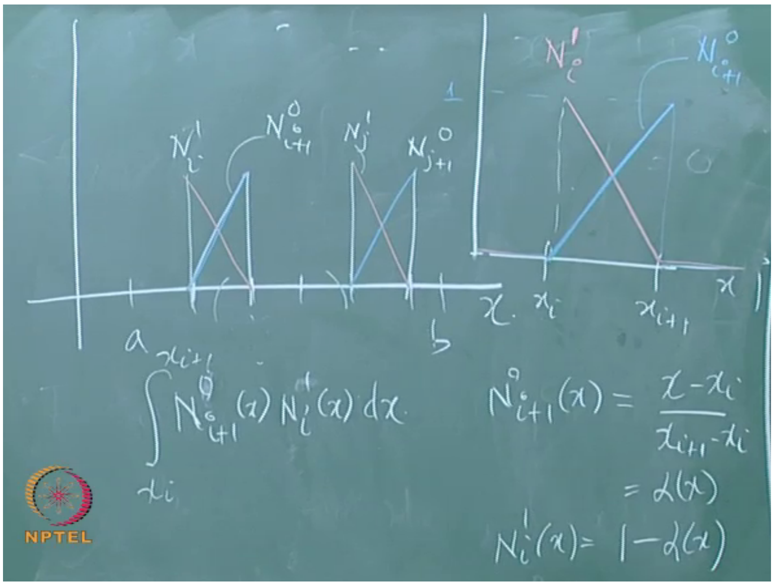

So let me see if I can construct these functions in a systematic fashion, so we have some interval a, b that we are interested right, and as we did earlier we are going to break up this interval into sub intervals. I am going to look at focus on one particular sub interval xi, xi+1 okay, so I will zoom in on that particular interval, so what I am going to do is I am going to just zoom in on this particular interval that is xi, xi+1.

And I will define 2 functions here, one is a function it is basically 0 everywhere, and at xi it is 0, and it then rises from xi to xi+1 it goes from 0 to the value 1. I will name this function N is 1 at i+1, i+1 0, because I am going to define 2 such functions. The other function is also 0 everywhere right, just make the box function, the only difference is that it drops from 1 to 0 going from xi to xi+1.

And since this function is 1 at xi, I will call it Ni, and it is the second function so Ni1 is that fine, do you understand what the functions do. The blue one Ni+1 0 is 0 everywhere, so it is going to be 0 from a to b everywhere except on the interval xi, xi+1 where it starts at 0 and in a linear fashion rises to 1. Ni is again 0 everywhere over the whole interval a, b except at xi, xi+1 it starts at 1, it goes down to 0 fine.

Now clearly if I have 2 such functions on 2 different intervals, if I have 1 interval here and I have another interval here, and if I have 2 such functions on these 2 different intervals, 1 function here the same blue, 1 function here, 1 function there and 1 function here, 1 function there. I have 2 such functions right, this is Ni+1 0, Ni 1, this should be Nj+1 0, Nj 1 right 2 such functions. These 2 functions are orthogonal to those 2 functions, each of these functions is orthogonal to each of those functions.

If I define such functions over all the intervals xi, xi+1 the any given interval 1 of these functions will be orthogonal to functions in the other interval, because intervals are non-overlapping fine. So we use the fact that the supports are non-overlapping therefore, they are orthogonal that is a good part. But what about the 2 functions to each other, the 2 functions obviously to each other are not orthogonal right, they are obviously not orthogonal.

So what is that dot product? What is Ni 0 and Ni 1, I am sorry Ni+1 dotted with is a function of x Ni 0 function of x dx the integral it is 0 everywhere except from xi to xi+1. What is this integral? So in order to do this, we have to get a function from we have to get an algebraic form for Ni and Ni x okay, I have to get an algebraic form for Ni+1 x and Ni x okay. So we come back here, what is Ni+1 is a function of x x-xi/xi+1-xi right, is a positive slope over the length xi+1-xi it goes from 0 to 1 right.

So at x =xi it is 0, at x=xi+1 is 1, it is a linear function so it satisfies right, again we are hunting even here if you think about it we are actually hunting for functions right, I have given you graph when we hunted and found that function okay. And what is Ni? So if this is some alpha of x I name it alpha of x simply because what is Ni 1-alpha of x okay, are there any questions? Fine. So what we will do now is we will find out what is the dot product?

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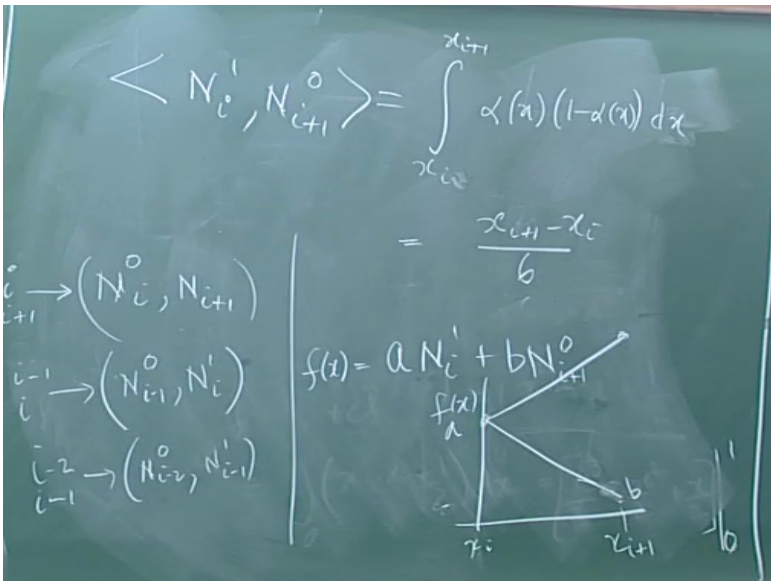

So Ni 1 dotted with Ni+1 0 gives me integral xi to xi+1 alpha of x times 1-alpha of x dx, what does this turn out to be? You can just check this, you can just verify that that is true, so is that fine okay. So what do we have now, what we have now is I have defined a bunch of functions Ni 0 and Ni+1 this depend of course on the interval, so obviously if I go to i-1 this is on the interval i+1, if I go to the interval i-1 to i what will I get corresponding functions Ni-1 0, Ni 1, am I making sense.

If I go to intervals i-2, i-1, I will have the functions Ni-2 0 Ni-1 1, fine on each of the intervals I will have 2 of these functions. Whatever you manage to do so far, what can we do with these 2 functions that we have defined on the interval, so first let us just focus on this interval alone, we will just look at this interval alone. So if I take a i or a times Ni 1+b times Ni 0 as some function f(x) defined only on xi, xi+1.

What does this give me for any given a, b, what does the graph of this look like? It has to be a straight line if the sum of 2 linear elements, and if I were to graph it if I were to just graph f of x, remember this is the interval xi, xi+1, if I were to just graph it at x =xi=Ni is 1, at x =xi+1=Ni+1 is 1 the other one is 0. So this is going to be a function that goes from a to b in a linear fashion, so if I change my b value.

If I change the value of b if I were to raise the value of b from here to some value here, then I would get a graph that looks like that right. So I have a linear interpolant, but I also have locality in the sense that if I raise this value a, b it affects this interval, and we will see if it affects anything else, is that fine okay. So a Ni, b Ni+1 allows me to give you get a linear interpolant in the interval xi, xi+1.

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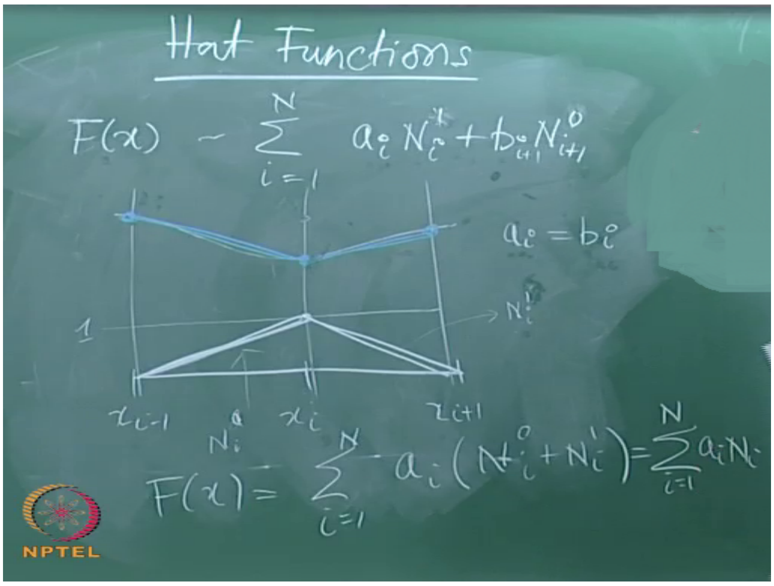

So in general, if I had an arbitrary function f of x, I should be able to write this if of x as summation i=1 through n, I just write it as 1 through n, we have to figure out what happens at the intervals elsewhere towards the beginning and end, anyway we will go through we have n intervals i=1 through n. What I am going to get ai Ni 0+bi Ni+1 1, is that fine. The intervals are non-overlapping; they are orthogonal it should be possible for me to represent any function f(x) in this fashion.

On any given interval the ai, bi will give me the straight line interpolants between those 2 points okay. Let us take 2 intervals and see what happens, so this is xi-1, xi, xi+1 okay, so I have some function and I want to represent this function using my newly developed right linear interpolants. So what I am going to do is I am going to use let me use some coloured chalk here, on the interval xi, xi+1 if I take these 2 values to be a and b, I will get a linear interpolant that looks like that.

On the interval xi-1, xi what will my a value be? What is a? a is the value here, and what is the b? b is going to be what was the a and the xi, xi+1 interval. So in fact though it looks like I have 2 coefficients definitely it is possible, we could come up with the scheme just like in the box function, if you are willing to allow discontinuity at this interval at the edge of this interval at the interface between the 2, if you willing to allow discontinuity, then a's and b's can be different.

But if you want the function to be a continuous function the representation to be accounting which is where we are going right now, then the ai corresponding to xi right has to be the bi corresponding to xi okay. So the a and b at this point have to be the same, is that okay everyone okay. So in fact what you have to do is we have to recombine these, so you are saying ai, Ni, bi, Ni+1, so maybe what I should have done here was I could have made this bi+1 Ni+1 okay.

If I make that bi+1, then what does that do for me? Then I can actually make that statement that ai=bi, it allows me to do that. So coefficient here is the same, and what are the basis vectors Ni that it multiplies, this coefficient ai what does it multiply? It multiplies this is 1 or let me lower let us say this is 1, it multiplied this function, what is this function? What was this label? Ni 1 and it multiplies this function, what was its label? do I have done I switch it around Ni 0, Ni 1.

I have it write the first time okay fine Ni 0, Ni 1, add it write the first time okay, there in the integral okay I made a mistake okay that is fine, you guys should tell me immediately as soon as you catch it, I flipped it around everywhere thank you, you should not let me go through with this, well fortunately what is going to happen if I am going to get rid of those superscripts, so okay fine.

Look at the function Ni 0 and Ni 1, Ni 0 and Ni 1 are non-overlapping I can actually combine these 2 functions I can add them up, I can literally add them up since ai and bi are the same ai=bi the same this summation i=1 through n right, since ai and bi are the same I have to just shift this, so if I go back to when this was i-1, i=1 or if you want me to I can write it out but anyway it is okay.

If I factor out the ai's since ai and bi are the same, I am going to get what I have seen here is I am going to get Ni 0+Ni 1 that is summation f of x, you can open out the summation for a few terms and see that this is true okay. It is actually possible for me to factor out the ai since ai=bi, it is actually possible for me to factor out the ai for Ni 0 and Ni 1 and these can be added, this actually add, and if you can add them this is the function that you get okay.

So this is summation i=1 through n ai Ni, I will rewrite this summation I will rewrite it at the other end.

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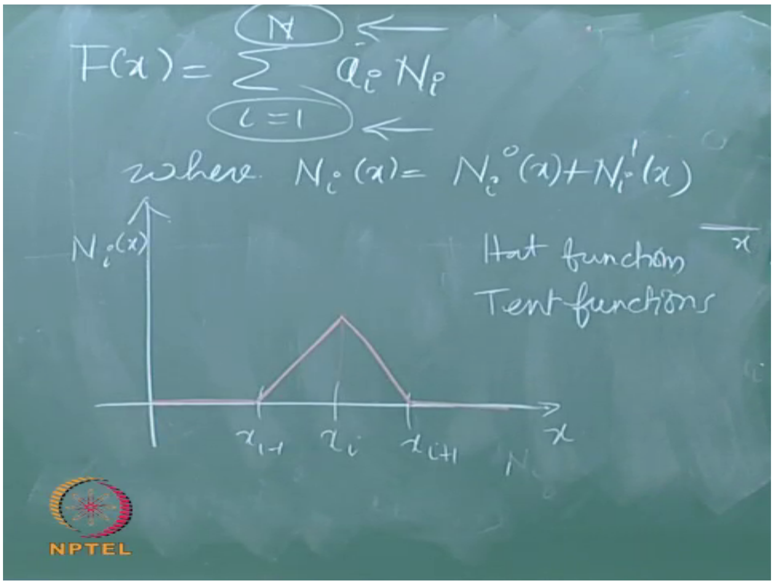

So f(x) can in fact be written as summation i=1 through n ai Ni where Ni of x =Ni 0 of x+Ni 1 of x, is that fine. 0 to N-1 as I said so we will have to look at what happens through because there is an issue. So I leave this we will investigate I just write this just for right, so we have to really look at what happens for all these intervals okay. So now for so the endpoints we look at what happens at the endpoints, you will have to a bit careful at endpoints okay right. So what is this function Ni? What is the graph of the function Ni?

So if I have xi, I have xi-1, I have xi+1, this is Ni as a function of x, it is 0 from a to xi-1, it rises to 1 linearly drops to 0 at xi+1 linearly, it is again 0 okay, because of the shape of the function these functions are called either hat functions or tent functions. You can imagine that if you construct functions like this in 2 dimensions, they look like the tent they are called tent functions or hat functions is that fine okay, has any questions okay. So then we will continue with this in the next class.