Introduction to Computational Fluid Dynamics
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Lecture – 39
Overview and Recap of the course

So today we will basically summarize what we have been doing in the last 40 out classes and we will start by before we recollect collectively recollect what we have done so far. I will just sort of make an observation on what we have been doing so far. So though it is supposed to be computational fluid dynamics, an introduction to computational fluid dynamics we will realize that basically what we have done is.

We have looked at ways by which we can solve differential equations and variational problems, how to represent them on the computer and how to get solutions numerical nature. So I want to put that in some kind of a context. In the sense now I want to make it a very context. So far we have been acting as though it is fluid dynamics, but if you think about it I have not really mentioned fluids that much, maybe I mentioned little gas dynamics along the way.
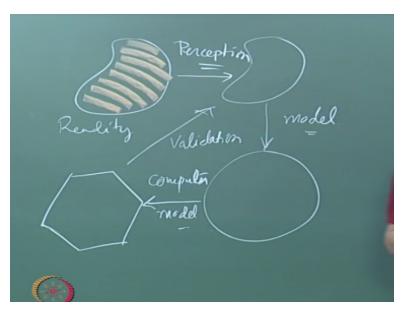
But we have not really spent, we are not really the equations were not like the full flow, full fledge fluid equations or whatever, we have looked at Euler equations and so on. So just to put that in context I want to draw graphically show you something about the nature of how these things work. It is a big picture.

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So on the one hand I have given I will maybe even colour this. I will colour it, I will fill it inside and then one hand that picture is supposed to represent reality as it is. What we are trying to model as it is and if I am able to reproduce it, if I am not able to reproduce it exactly it does not matter. So I will say is that make sense and from here to here I am going to write processes. This is perception. This is what we see.

Whether you use microscope to see or whether you just look when you talked about deriving the governing equations in one of the classes. I sort of waving my hands around and said molecules what molecules, no molecules. I do not see the molecules. Am I making sense? We may be able to infer that they are there, but the phenomenal logical attitude would be continued. It is continuous media around.

So that would be perception. This is what we say and the trouble with perception is that there is a lot of detail, benches, carpeting, lot of detail, lot of fine grain detail, we cannot handle that much detail. So even for what we so there are things that we do not see that there are things that we just cannot see I mean there are unlikely to see locations of every molecule at every point at a given instant. It is not going to work.

So we come up with we throw away what we think as a details. We throw away the details and come out with mathematical model and for a mathematical model of course what I am going to

choose is a geometrical shape as close as a circle I can get. So we come up with a mathematical model. So this is a big picture what we are doing as a reality then what we see cannot handle the detail mathematical model and very often as in the case of Navier-Stokes Equations.

And so on cannot handle the mathematical model. So then we go, where we go? Then we go to the computer. So this is deliberately of course it is clearly geometrically supposed represent the fact that this has an infinite number of sides that has only a finite number of sides. So, one other thing that we looked at is, for when since asked the question of consistency. So, the number of sides keep increasing thus this becomes the circle.

There is a solution go, there is a model go, they are request since that we have asked. So there is an issue of verifying that this is indeed this. You have written a program. So actually I will just say computer model this itself you can add levels. There is the chalk dust that we write of the finite difference scheme or whatever then there is the actual program which is a representation of that algorithm actual representation of the algorithm on that particular computer.

And then there is an issue of whether does that represent the algorithm that you set out implement. Does this become this. You understand? Or does it you know want to verify whether that is correct. So you can verify whether this is correct. This is called very often this is called validation. From here to here we actually called it consistency. You understand what I am saying. So this is a sort of a big picture. This is what we have. This is what we see.

This is what we are hoping we can do in the mathematical language. This is the abstraction from this. Abstract basically means to extract. You abstract out the essentials and throw away the nonessentials and by chance you throw away something that is essential that is a different story, but you throw away you abstract out the essentials and you know what is there because that is how science works. The science works on generality.

Science works on things that are common. The particular is the more interest to science. We want to make a general statement. We are not particular as of no interest to us. So we abstract out the general and very often we find that this we cannot found this. This is then we are. So actual fluid

what we see Navier-Stokes Equations, finite difference method. It is one way to look at. Is that fine?

So as I said I just wanted to talk about this so that we had this general context in which we can look at the course and possibly some of the material that you have seen elsewhere. So why do not see if we are able to do a review of the material that we have covered so far. Where did we start? We have about 40 minutes in which we can cover the 1 minute per day it is essentially rush. so where did we star or what does the start.

"Professor - student conversation starts" representation of numbers. Yes. We basically said that we started with the idea that the problems that we are solving have solutions which are functions do you remember. The problems that we are solving have solutions which are functions so we need to able to represent functions in the computer. **"Professor - student conversation ends"** So we sort of realize that we have to able to represent things on the computer.

So towards the end we basically said that let see how things are represented on the computer that is where we so that was the motivation. All the equations that we have solved typically end up the solution end up being what we mean when we solve. It is not like x square = 2 therefore x square root of 2, so you get a nice number where what we got our more like functional equations the equations for which the solutions are functions.

And we basically set out if I want to solve that in a computer then I need to be able to represent the solution of the computer, I need to able to represent the problem on the computer. So how does one represent things on the computer and we started by saying how do we represent well integers. So computers are good and representing integers. How do you represent the real line? And computers are not good at representing the real line that is what we found.

Right at the real line we found that the computers are not good at representing the real line. So the real line was the first time that we encountered represent a form our representation error called floating point round off error. So of course there were 6 point and floating point I will just

mention that since I in advertently it is a floating point so, a fixed point representation and floating point representation.

Fixed point representation is just like dealing with integers it is just like having a integer in assuming the decimal point at a set point. Floating point of course the decimal point in theory moves because you have exponent. So the representation error so one another thing is that you have to take from this course the representation error in numbers has a special name. It is called round off error.

So anytime you are basically asked for what is the error that you get? and the round off error. What is the round off error? Round off error is the difference between the number and its representation on that computer. So that computer could be a piece of paper. You decide that you are going to represent a number using 3 decimal places then you have the computer. Then that is 3 decimal places is what you are going to add and therefore that would be the difference.

Now, if you read old books, then they say the computer should be careful, they really mean you. So you go back and get numerical analysis book from 1940s are whatever, when they are saying computer should be careful they really do mean you, you as the individual doing the computer. So that is as far as the representing numbers goes and once we said we can represent numbers it obviously we look at other mathematical entities and the obvious mathematical entity that we have is a vector or an array.

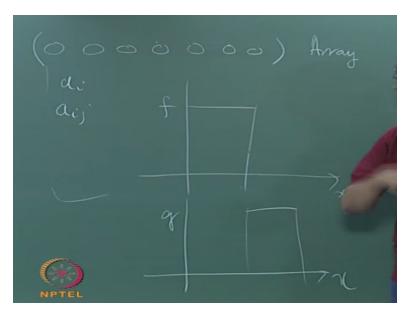An array is a general idea in computers so whether you can represent vectors and matrices. You can represent matrices, but if you want most programming languages will not do the matrix algebra. They will not do the matrix addition. You have to do it. Most programming languages will not do that for you they may be a few that would, but most will not do it for you. So it is possible and how did we represent matrices so.

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We represent matrices and vectors basically well vectors we represent it by basically locating a memory in a sequential fashion. We were able to say that there are so the 1, 2, 3, 4, 5, 6, 7 elements placed adjacent to each other could represent the vector in a one dimensional array inherently so we are using the term array and of course if you want to go make get a matrix you could then stack1 on top of the other that was the idea.

So it is actually possible to get a single subscript object ai or a double subscript object aij or even a triple subscript object and so on and then what you do is you take whatever it is that you have stacked in the second dimension you would stack it in the third dimension. So then you know that you can actually create multidimensional arrays. It is actually possible to create multidimensional arrays. Is that fine?

So by now what we have done is we have managed to represent integers, numbers, real line. We are able to represent then using those integers and real numbers so to speak real line. We were able to represent matrices or arrays, 1 dimensional arrays, 2 dimensional arrays, multidimensional arrays. So the second thing to do is now are in the state where we are talking about representing functions. So what did we do after that?

What was the how did we go about that. We did a little we set up we looked for function basis so we need a little review of lecture of vector algebra. We showed and how we constructed vectors

in the normal vector algebra. When I say vector algebra, the algebra that you learn in your college we did a little vector algebra and shown that well if you had the dot product so it is enough for you to actually construct.

So it is enough for you to actually construct a set of basis vectors and so on given a linearly independent set of vectors that span that space that you are interested. So what we did was we constructed Box function and here this is the first thing that we did. We constructed Box functions. So the Box functions were of 2 kinds. We constructed a Box function f, m, g I am not going to do all the details as I said though it is a quick recollect.

We constructed Box functions as f and g and basically I said that on non-overlapping intervals if the Box function is 0 in this case it is 0 here, 0 here and the intervals do not overlap. Remember wherever the function is nonzero it is called the part of the domain where the function is nonzero it is called the support to the domain. So the support to the 2 functions are non-overlapping and the functions turn out to be orthogonal. That is one way to get orthogonality.

The other way that you get orthogonality is that the functions somehow change signs appropriately. Right. trig functions sine x and cosine x occupy the same interval. They are defined in the same interval 0 to 2 pi, but they are still orthogonal to each other. So we were able to show that we could represent functions using function basis, using the Box function and we saw again that there is a representation error.
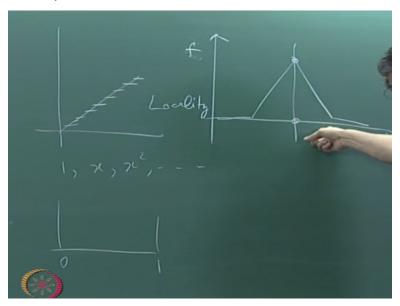
We again saw that there is a representation error. For the Box function we asked ourselves the question what is the nature of the representation error. The reason why we ask ourselves what is the error? That question is important. Can I get an idea what is the error because if you know what is the error? You can then try to get an idea to where the error is coming from, and then there is a scope for improvement.

So I keep repeating. It is not just still of that what we do that those are skills, but you should also see there is a higher level skill that you dame at. You have to look at the process that we are going through. So you always ask you do something then you have to ask the question how good

is it. Is there a way that I can get an idea is how good is it? because the minute you are able to quantify it then there is scope for improvement.

So that is very important. So we asked the question this representation error and our analysis basically showed these are constants. So you can represent constant functions exactly and nothing else. So basically what we then did was we said we will go to higher order polynomials. Of course, one of the disadvantages this had was that a representation using Box function what is the bad thing that it had? You had lot of jumps.

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So in fact I think I used a straight line is there lot of jumps they are non-overlapping. So I have to make sure that you get a lot of jumps. It gets a function is very jumpy. You can get closer and closer to the solution, but the function just gets jumpier and jumpier. The alternative to this was saying that well we got orthogonality this way, but we know that using something like the Gram-Schmidt process or whatever we could make functions.

You can top get come up with an independent set even if they are overlapping. So we looked at the set of polynomials 1, x, x square and so on. On an interval we have to specify the interval on an interval. So on - 1 to + 1 for instance 1 and x are orthogonal to each other. On 0 to 1 they are not. If they are not orthogonal to each other you can subtract from x, its projection on 1 and get an orthogonal set.

You understand what I am saying and then you can build up an orthogonal set using this. That is what we saw however just like sine x, cosine x and so on this has a problem. So along the whole interval on which you are trying to represent the function, you represent the function using a series 1, x, x square and so on or an orthogonal set form from it. Any change in coefficient will change the function on the whole domain whereas Box functions are something called locality.
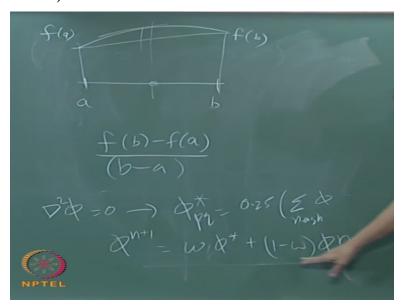
So I want to recollect that locality. Box functions has something called locality which allows you to locally change the value, locally move the function value of the representation without effecting anything elsewhere. So now we are able to represent function. We basically combine this idea this with the non-overlapping what should I say intervals or supports to construct Hat functions.

To construct functions that look like this. To construct Hat functions. The Hat function was 1 at the node at which it is defined and 0 at the adjacent grip points. So we were then able to get linear interpolates however what is the disadvantage? that you have overlap of adjacent which can create interesting things like in the last, but one class take interesting situations. But have overlap.

There is always a problem. see orthogonality that seems we seek orthogonality because otherwise you do not have, but on the other hand you will get linear and we saw that you could not only linear, you can do quadratic, you can do cubic, am I making sense. So your basic functions can be any order that you chose there are many ways by which we can choose it. So I think I did I mentioned quadratic and I did cubic in class if I remember it.

What else did we do, what did we do after that? We got an estimate of the representation error here. Then we basically turned around and said well if I know the function I can use this for the representation, but if I do not know the function ahead of time that was the motivation that I used. If I do not want the function ahead of time what do I do? So then we used Taylor series and we came up with a finite difference representation. The finite difference representation of course basically made use of the fact.

I am not going to use is and js as the indices I am going to just make a general statement made use of the fact that if you have the function value f(b) and you have the function value f(a) then what is f(b) - f(a) divided by b - a. This looks like something like for mean value theorem. So mean value theorem says that, there is some point unknown location psi at which, this is the exact derivative for the continuous function there.

Now what we know is that if I represent this using Taylor series if I represent this if I say see f(b) - f(a) divided by b - a is a number, just say it is 2. So you calculate this and you get the answer 2. What it basically says is if you say 2 is the derivative there the representation is first order representation the error is proportional to b - a. If you assume that it is the derivative there if you say 2 is the derivative there that is the approximation that you are making.

Then the representation is first order, the representation is still first order. The sign of the error changes and if you assume that it is the representation of the derivative, the same number 2 if you assume that it is the derivative at the midpoint then the error becomes second order b - a whole square. So here it turns out that the derivative corresponds to the function being represented by linear, linear, and there is actually quadratic.

Okay, you look at the term it is also the function has represented the quadratic even though it is actually you have drawn a straight line. So, the same number and as I said unfortunately mean value terms as that at some unknown points it is exact, but we do not know that is the whole deal. So at some unknown point in between it is exactly the right and we are looking at graphically you can just basically say what is most probably at that point, graphically yes, but in reality we do not know for what it is.

So that is basically the idea of so finite differences and we found the truncation error using Taylor series. Truncation error, so I introduced the term truncation error if you have an infinite series you truncate, you throw away, you eliminate all the higher order terms and retain only the leading terms. The error that you make doing that is called truncation error. What else did we do after that?

What was the once we have finite difference represent, so you can represent first derivative, second derivatives we showed that we could represent higher order derivatives, then we turned around and said okay why do not we apply to solving an equation and we look Laplace's equation as the start and Laplace's equation was relatively easy and we basically showed that Laplace's equation nabla square phi = 0.

This is an equation that I have written many times in this class so you could do. phi at pq = a quarter of basically the average of the neighbor's. Summation neighboring phi. So it is the average of the neighbor's. Laplace's equation turned out to be averaging of the neighbor's and we used that fact that it is averaging 2 actually show that what did we showed that the solution is unique we understand that 2 of you cannot get 2 different answers.
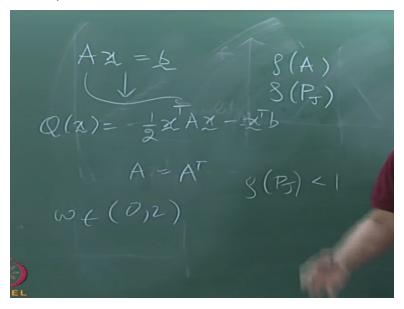
And so on and the solution is unique. We use that averaging to show that there is a principal called maximum principal that it satisfies that the maximum and the minimum in this case actually occurs in the boundary. We proved the consequences are maximum and minimum occur on the boundary. Are they make any sense? And use that then show that the solution is unique for the numerical case. You may have seen for the continuous case, for the discrete case.

We showed it only for the discrete case. So what else did we do it Laplace equation? What did we do next? We looked at some acceleration schemes in particular we looked at SOR. We basically said that well if you have a phi that comes from this instead of calling it phi and + 1 we called it phi * and it is possible to take a linear combination of the proposed solution and the current solution.

So to get phi n + 1 you could get omega * c star + 1 - omega * n and we looked for wave by which we could find omega, the optimal omega like we said this is a situation where we are looking for non-uniform convergence. If it is uniform convergence, then omega it does not help. You want as a function of we want the convergence rate to change as omega and we found that there is actually way to find the optimal omega.

By systematically hunting there is no analytic method by which you can do it, but you can systematically hunt for the optical omega. We also showed that solving this was the same as solving a system of equations. Solving this is the same of solving a system of equations so we had what?

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We showed that it was the same as solving x tilde = b tilde which is the same as minimizing a function Q of x tilde = 1/2 it depends on which way you want to take it. x tilde transpose A x tilde - x tilde b x transpose b because I think that I did that at least twice because we went

through some interesting errors there, but any what it is fine. So we basically showed that this was minimizing this.

It is the same minimizing this is the same as solving that. A, remember is symmetric that is an important property here. A is symmetric. A = A transpose. A is symmetric and there is some interesting properties, but anyway we will not worry about it and we showed for SOR as a consequence why omega has to be in the range 0, 2 that we got an idea we have to pick omega within the range for this linear equation omega should be between 0 and 2.

And you looked at hunting we looked at a few demos, you have tried it out and we found that for Laplace equation it is that omega is very close to 2. What is the other thing that we did anything else? Then we looked at what is the convergence rate basically. We looked at how quickly does it converge and indeed we found that because again using the same principle we found key word that you used as error.
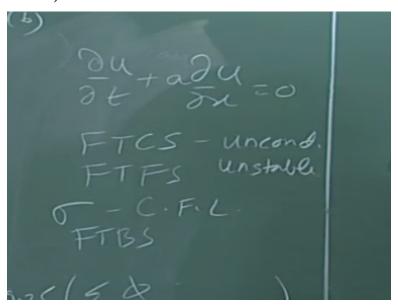
We found out what is the error and we basically once we got the expression for the error we asked the question how quickly is it decaying and that gave us an idea of convergence. In a funny fashion I introduced the ideas of Eigen values and Eigen vectors and the spectral radius the largest Eigen value. So rho of A or rho of I think if it remembers it I called the iteration matrix P sub J for Jacobi iteration. Rho of P sub J is the value of the largest Eigen value.

Or the largest Eigen value. The magnitude of the largest Eigen value it is called the spectral radius of that matrix and we basically showed that you want the spectral radius of the matrix to be < 1 for convergence that is fine, but that is not enough. That is okay if you are saying does it converge eventually, but eventually is not good enough. We found that actually for this class of problems rho of PJ is extremely large. It is very close to 1.

So there are some issues there. So are one of those things that sort of help us fix the issue. But there are some issues there. Then what did we do. We looked at we change gears and looked at linear wave equation. We looked at linear wave equation. Linear wave equation, Laplace equation is averaging.

Right averaging basically means if there are differences you eliminate differences. Linear wave equation is different. Linear wave equation is propagating. Whatever there is, is carried. So it is like a stream. The example I gave is stream of water flowing add chalk dust, chalk dust is carried at that speed. So, linear wave equation basically looks like.

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And we tried a variety of schemes for this just like things worked for Laplace equation you just try a variety of schemes for that that was a sort of deliberately set up because I want it to make sure that since Laplace equation everything worked I wanted to do the same thing and we obviously know that they are not going to work.

We tried forward time central space which was unconditionally unstable. So the stability analysis we looked at the von Neumann stability analysis we basically linearized stability analysis anyway it is already linear so it does not matter, that we added we looked at the same times of exponentials that we looked at when we are looking at the error term here and showed that forward time central space was unconditionally unstable.
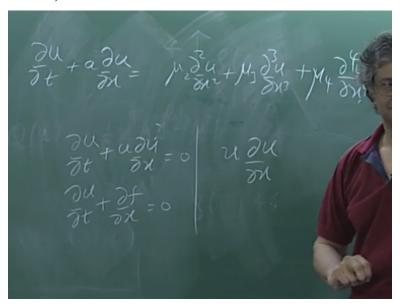
There is nothing that you could do. Forward time central space was unconditionally unstable. In all of this discretization of course we came up with a parameter which we called sigma this was Courant–Friedrichs–Lewy number and this number turned out to be pretty critical so maybe I am

getting a little ahead of myself we also did FTCS fashion to get that. So we also unconditionally and then finally we found we did FTBS for time central backward space.

And basically showed that, if the Courant–Friedrichs–Lewy number is < 1 that we had a condition. We got a condition. FTFS in fact basically gave us the idea and from here we got the idea of appending that is FTFS did not work but FTBS work because a is positive and if the sign of a changed if a became negative then FTFS would work and FTBS would not work. So the equation is propagating you in a certain direction then you use the word some information.

But the equation is propagating you in a certain direction and your scheme also it looked at needed to propagate you in the same direction. Is that fine? To which we asked a question so what is the difference between FTBS and FTCS? Well I guess before we did that what did we do after that? You looked at the modified equation. We asked ourselves the question what is the problem that we are actually solving and we got from that the modified equation. The modified equation looked something like this.

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It had terms which were of that form higher order derivates so asked ourselves the question what would each of these things do? In fact, we looked at it up to the fourth derivative and we concluded that the odd derivates are dispersive in nature for this equation and even derivatives

are dissipative in nature. Whether the u decays or is amplified depends on the sign of u and the nature of the derivative so in this case nu 2 has to be positive.

In this case nu 4 has to be negative that is what we conclude and this is just dispersive. So we saw we say decay and we saw dispersion. Dispersion basically means high frequencies and low frequencies travel at different speeds. So it is a very important phenomenon. Different frequencies travel at different speeds. So then we came back and ask ourselves a question what is the difference between FTCS and FTBS how come this works and this does not work.

And we realize the difference was that in 1 case the nu 2 is negative and other case the nu 2 is positive for the modified equation. So the natural question was cannot I just use FTCS and add a little amount of artificial viscosity which we have tried it actually works and that FTCS the relationship between so these scheme may literally can be written as FTCS + and appropriate correction, if you choose to it.

So that as far as from here did we do anything else with linear wave equation then we looked at the quasilinear wave equation and showed that and of course the general form there refers a function of u and we showed that in this case where there was no discontinuity to start with and the original function is smooth, the usual condition is smooth the discontinuity can form. So it is an interesting combination.
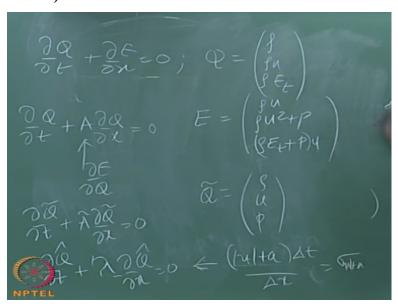
On the other hand, you have diffusion or Laplace equation kind of which is averaging out and eliminating discontinuities, high frequencies are decaying fast as low frequencies as one of the outcomes of this which I forget to recollect important outcome, high frequencies decay fast than low frequencies. So that being the effect of the right hand side this is just interesting that this is creating high frequencies and I pointed out that if you have in a different context.

I had pointed out that clear u dou/dou x and if you substitute sine theta u is like sine theta then dou u/dou x is like cos theta so sine theta cos theta is like sine 2 theta. So there is an increasing there is a mechanism that increases the frequency that doubles the frequency which is very critical. So this quasilinear this term seems to do that. It seems to increase the frequency. On the

other hand, this seems to try to decay and that combination is what of course makes fluid mechanics.

So interesting! So then from here what else we do of course we derive the, if we get a discontinuity it is called a shock and we derived the Rankine-Hugoniot conditions for the shock speed and so on. Then what else did we do. We looked at 1 dimensional flow. We derived the governing equations, looked at 1 dimensional flow. We tried FTCS + dissipation for it. We tried first of all to make the equation look like this linear wave equation so we wrote it 2 forms. We wrote it a conservative form and a non-conservative form may be I just write the equation.

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Dou Q/dou t+ dou E/dou x = 0, Q was rho, rho u, rho E total energy and E of course is rho u , rho u square + P, rho E + P * u. These variables are said to be conservative variables. It is written in a divergence free form, mathematically said to be written in a conservative form. For us of course from gas dynamics we know that across a shock these quantities are conserved. So you can give any reason you want us to why you want to call in the conservative form.

But you may call conservative variables as a consequence even if you write it in a non-conservative form. So you can use, you can still write this in a non-conservative form and sort to a desperate attempt to make it look like the wave equation linear wave equation. This term the

flux Jacobian is dou E/ dou Q. The flux Jacobian is dou E/dou Q and then we said this system is couple, is it not there something that we can do to decouple it.

We tried to do change the variables from Q to Q tilde which was rho, u, P and that did not help either that still gave a couple system of equation dou Q/dou T + A tilde dou Q tilde dou x = 0. In terms of Q tilde which are non-conservative variables. Certain combinations of course you have rho, u, P also. So it turned out the still couple.

I can delete these 2 each other with a similarity transformation. I can transform this equation to this equation and A and A tilde are related to a similarity transformation then we asked ourselves a question is there is a transformation that will diagonalize matrix. So if you use the module matrix of A or you use the module matrix of A tilde the matrix made of Eigen vectors of A then in fact possibly to diagonalize it.

And we got the characteristic form where the system of equation was diagonalized and it was decoupled. So we got 3 equations that were basically propagating like the wave equation. Nonlinear wave equation because the happen to it is a quasilinear wave equation, but at least our analysis would work and the CFL condition for this if you do a discretization and we go the Courant number at least, the Courant Friedrichs number would be of the form delta t/delta x that would be sigma you could say u + A or mod u + A.

Or whatever that will be a largest Courant number and typical stability condition would require that this is < 1. Using module because u can be positive or negativ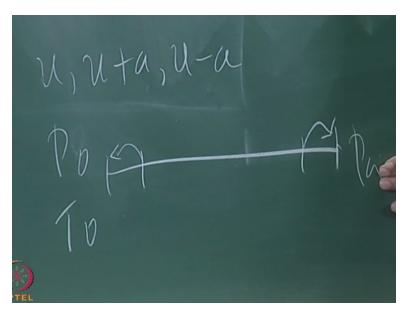e a is the speed of sound. Are there any questions? Anything else what did we do after that? We looked at the critical part of applying boundary conditions and applying boundary conditions we decided to use the fact that this equation is propagating in a certain direction determined by a so at subsonic inlet so the Eigen values are the lambda and.

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That equation is u, u + a, u - a where a is the speed of sound so I had a subsonic inlet these 2 are positives that is negative. So at inlet these 2 are propagating into the domain and this is propagating out to the domain and we use that to determine boundary conditions. So typically at the inlet we apply for flow through a pipe P0 and T0 in the inlet and P ambient at the exit because at the subsonic exit this is negative, these 2 are positive.

So these are propagating out of our domain at the exit whereas it is propagating back into the domain am I making sense is that okay. so we need 3 quantities we have got 3 quantities, but actually at each point we need to get 3 each so we have to get 3 more which we did by extrapolating so if I actually write the domain and show the grip points the first grid point and the penultimate grid point and the last grid point.
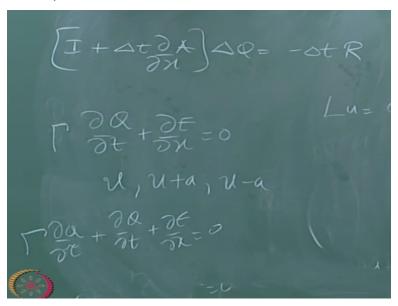
Then we basically said that there are certain quantities that had to be extrapolate. We explored various things that we could use to extrapolate to the boundary and it is very important. So the emphasis here is there are boundary conditions that are required by the physics of the problem. Your pressure vessel, there in it has a certain P0, it has certain T0 these are measurable quantities. You have a pipe, you have valve, there is an ambient pressure.

You open the valve the ambient pressure and the P0 basically determine what is the speed with which the air is going to flow and they are making sense and the T0 will help you determine the

other parameters that is what physics requires. The mathematics basically says that you need 3 quantities because + and initial condition because you are single time derivative and you have a single spatial derivative, first degree spatial derivative and you have 3 quantities.

But the numerical algorithm insists that you need Q here and Q there and therefore you have to generate. The numeric requires no boundary conditions. So we have to actually generate those quantities we have no choice. Is that fine? So then of course we also wrote one important thing that we did was we wrote this in the delta form and this is a.

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We wrote the equation in delta form I will just write 1 as I said none of these and these are just to recollect where R is the residue so I have not defined the recollection and I have not defined the residue so far. The residue if you have an equation and you substitute a potential solution into the equation if the right side hand side does not match whatever is left over is called the residue. So if you have $Lu = 0$ and you substitute a candidate u and you do not get a 0.

But you get some value that is called the residue what it leaves is called the residue that is the residue from this state Euler equation and this is the delta form. This is the correction to the correct state and Q that you have, current candidate solution that you have. The big thing that we want to take from here is that given this R will determine if we reach the solution or not. If the residue is 0 your current state is correct.

If the residue is not 0 your current state has a problem needs to be corrected. You can use to solve the system of equations to determine the correction. If the residue is 0 the correction will be 0. If the residue is 0, the correction is 0 as long as this is not singular it can be anything and therefore it is possible for us to choose something that will make the convergence faster that was the other deal that we had.

So we do this carefully because this determines when you reach the solution. You do this carefully because that will determine how fast you get the solution is that fine. Right and then we did a little more involve boundary conditions application of boundary conditions here using converting it to characteristic coordinates and so on. Is that all? Did we do anything in the then I wrote out the equations for the quasi 1-D and that was basically about.

What did we do after that? We looked at unsteady flow? Or what did we do after we looked at. I think we spent a little time on unsteady flows. We will look at the preconditioning the unsteady term is that the deal is that what you said. So we used to see there is a certain freedom that this gives this argument gives that is the reason why I want emphasis this. This determines that you have the solution and therefore this goes to 0 the correction goes to 0.

And therefore whatever multiples it goes to 0 say this gives you a certain freedom. When we look it and say if I am looking for the steady state solution dou E/dou x goes to 0 if you look at for a steady state solution dou Q/dou t goes to 0 and therefore if I multiple dou Q/dou t by some matrix it does not matter so why would I do that. I will do that because the Eigen value is happened to be u, u + a, and u - a and the problem gets difficult and even u - a is close to 0.

That means you are going after a trans-sonic speed or u itself is 0, the problem becomes extremely stiff. The Eigen vector, Eigen value is becoming very desperate. The propagation speeds become very desperate. So by pre-multiplying this by gamma it is actually possible for us to precondition the problem and since I am looking only for the steady state this pre-multiplying by gamma dou Q/dou t does not really affect the steady state, but affects the rate at which you are going to reach it.

The same idea is that. There are something going to 0, I can multiply by whatever I want along it changes algorithm so that I get there faster, but where I go is the same destination that is the key. Is that fine? Then of course we looked at unsteady flows by adding a pseudo time term. Of course you can solve this directly using I think post my demo I did Rankine-Hugoniot method. You can solve this equation without the gamma of course.

If you are looking for the unsteady solution, and solve this directly. So if you are looking only for the steady state solution this is called a time marching scheme. If you are looking for the transient, you are looking at a time accurate computation. You could use Rankine-Hugoniot or whatever it is, but what we said is we have built up so many machineries for steady state solution that if you add a pseudo time term dou Q/dou tau.

And you could then because that is going to 0 multiplied that by a precondition. You could actually converge to the steady state in tau and get the unsteady solution in the real time that you want. So we are then talking about acceleration schemes and then we went back and basically remembered that high frequencies decay fast as low frequencies.

We recollected all of these things and of course in a review I think I left out representation of functions in the demo that impart critical demo that I did on high frequencies and low frequencies but it does not matter. So high frequencies decay faster than low frequencies and we basically came up with the multigrid scheme.

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The demo that I am taking about is, where we represented sine x using the demo I recollected the demo, I will go back to the demo when we talked about literally representing sine x using Hat functions that were the critical demo for us because it showed that for a given grid there was the highest wave number that we can represent it is very important or turning it around.

Whether the wave number is high or low depending on the grid that you are talking about and if you say high frequencies decay fast than low frequencies then you are interested in using a coarser grid because the wave numbers then become higher frequencies. What are low frequencies in large grid become higher wave numbers n a coarser grid hence the multigrid scheme. So what you basically do is you use grids of different sizes.

We use grids of different sizes h, 2h, 4h, 8h and you transfer the residue this is the critical part. You transfer the residue from the fine grid to the coarser grid and you transfer the correction from the coarser grid to the fine grid. So it is possible for you to actually go through this process of transferring the problem in a sense by you transfer the residue you are transferring the problem, transfer the correction you are transferring the solution in a sense you think about it.

So it is actually possible for you to run your program on a very coarser grid to eliminate so that you convert what seem to be low frequencies on h to a relatively high frequency on 8h. So multigrid scheme is of course when we said that there is a possible actually in fact why would we

do that we would just basically start with the coarser grid and transfer them in fact when we talked about Laplace equation in the beginning that is one of the things.

We suggested that compute the solution on a coarser grid and transfer it as a initial condition on the final grid. So you could actually do this here. You can start on the coarser grid get titrate a few times for Laplace equation or take 10 times step for Euler equations or whatever and transfer it to the next final grid so you could actually go from 8h, 4h, 2h, 2h and then start your go through this process and typically the critical thing that you want here is.

What is the thing that you have to recollect here work unit that is what you always work units? That 1 sweep on the final grid right in 1 dimension is equivalent to 2 sweeps on the coarser grid is equivalent to force sweep if you go to multiple dimensions of course that gets even better so we would expect that in 3 dimensions multigrid would extremely well right the actual convergence rates that you will get in wall clock time.

How quickly you get through the solution is much better. What did we do after that after multigrid methods? Calculus variations. We looked at so calculus variations basically come back and says so we are back to this business I am representing a function. The only thing is now you actually have a measure or something so you say that I have if I think the example that gave is if you are coming from your dining hall to this class room.

What is the shortest path that you have? So you can come up with some kind of a measure or a matrix to see which is the function that you want? You are looking for you have a functional, you have a measure and you want to minimize that measure and from there we derive the Euler Lagrange equations. We showed then that the relationship between the variational problem and the differential equation because it involves optimization.

There is a differentiation and setting = 0 kind of a process that you do to get the Euler Lagrange equation. So, in that sense the Euler Lagrange equation is like a derivative. The Euler Lagrange equation is like a derivative. So you differentiate find the first variation and you get this

derivative and we showed that Laplace equation we found the limited the variation for Laplace equation.

And showed that you could get Laplace equation directly which was the analogous to the earlier ax = b, b being a minimization of a quadratic cube. It is analogous to that and we basically said that it is possible for us sometime to solve it in the variational form. If you get the variational form, the amount of smoothness required as a function is not as much the smoothness required as a function is not as much.

You do not need as many derivatives so it is possible for us to reduce the derivative requirement and very often the expression in the variational form is simpler than done in the differential form, very often that is so, but it is not always easy to get the variational form given the differential form. You do not get something for nothing so there is a difficulty in doing that and finally of course we end with today's class while we will recap put it on the back to the big picture.

So the big picture possible is that this is the reality as we see it, so you have and of course I have put this little brown to indicate and from here reality you get this is the perception this is what you see. So this is what it is. I just draw, I can draw anything there because we cannot see it what I am saying this is the perception and from this perception we can abstract model it would be ideally in a figure that I do an exact circle because mathematical precision.

The abstract of the model and then of course we cannot even solve the model, we cannot even get this model so we end up doing a computer program which is discrete the idea is that is pictorially that is what I am trying to represent here. So this modeling, this is discretization if you want. you can check consistency whether this model is consistent with this as the number of sides of the polygon increase as it become the circle that is consistency and you can do validation. You can check and perform an experiment and you can check where the result is.

We cannot do that. The scientists and everyone is constantly trying to do that and of course here there is a test that you would do here also you can add terms, move terms, where the flow is inviscid, the flow is viscous so all of that stuff is happening here, but once you decide to do the

Euler equation you go back here. So there are 2 possibilities either you say this solves this and therefore I validate there, but you look at this you know they are not. So validation is validation that is it and no more than that.