Introduction to Computational Fluid Dynamics
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Lecture - 37
Calculus of Variations - Application to Laplace Equation

So we looked at a little introduction to calculus of variations, right, and I think we saw one application which was the shortest distance, okay, in flat surface, the usual thing at this point would be that in the course on calculus of variations the usual thing that you do at this point was derive an expression for geodesics or something of that sort, shortest distance for 2 points in the surface of a sphere.

Right, that would be the, so you can easily pick up some book and go through, I mean, the derivations are pretty straightforward, it is all just calculus at that point, okay, so I am really not going to be doing that today, since the amount of time that I am going to spend on this topic is really not that much, we have only spent a few classes on it. I will look at applications directly, okay, just to first recollect, right, to pull the problem.

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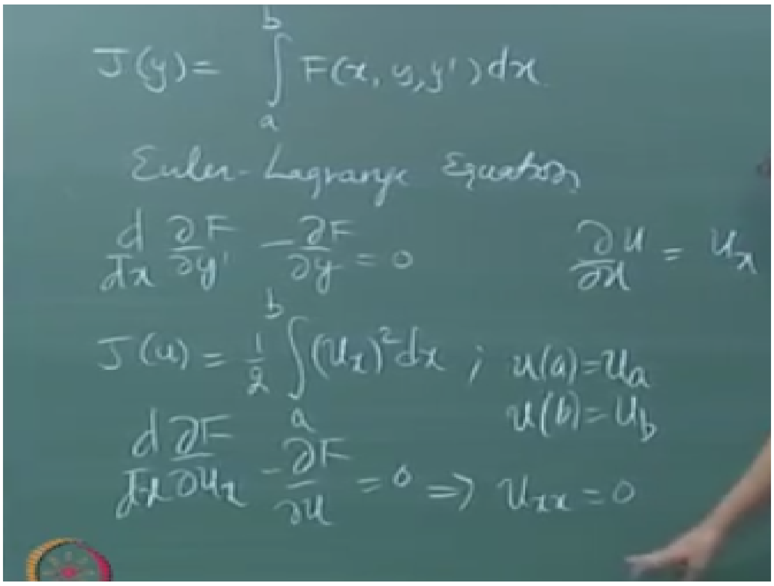

So J of y is the functional, it takes the function as an argument and returns a number, a real number in this case, okay, so this is the map, just like our norms and so on, even the norm took a function and returned a number, you understand what I am saying, so we have seen this before, and if you want an extremum for this, either a maxima or minima, then we saw that you could take the first variation, okay.

And to find out just to put it in context to find out whether it is the maxima or minima, if it were a regular function what would you do, you take the second derivative and check the sign, in this case you would take something called the second variation, which also I am not going to do in this class, but I just as a piece of information right, they are the first variation, you could look at the second variation and so on.

So we are not going to look at the second variation, you can go look up, read up something on calculus of variation if you want to look at it. So you can set the first variation to 0, right and get the corresponding Euler Lagrange equation, right, which is what we derived in the last class, we applied it to the shortest path between 2 points in a 2 dimensional Eulerian, in the plain as a blackboard, it is a 2 dimensional Eulerian plain okay, fine, okay.

So can we do something interesting with this, fine, so I will look at a simple problem first, first in one dimensions, we will see what we get, so consider this problem, so in this class I am going to mix my notation a bit, so I want you to be a little careful, okay, I am going to use some kind of mixed notation, those are subscript here means differentiation with respect to x.

Yes, I know I used prime here to mean differentiation with respect to x, but in this case there was clarity, it was clear. You can say well even in this case it is clear, but you will see that as we go along there maybe ambiguity, so I choose to introduce this notation at this point, right, so you will see me use both of these. You will see me use both notations, okay, the subscript indicates differentiation with respect to whatever that parameter is, is that fine.

So what is the Euler Lagrange equation for this and if I am given so right because like we said the paths, the end points maybe fixed, you may be given U of a, you may be given that, you may be given those auxiliary conditions, right, just like I was saying earlier if I want to walk from your dining hall to this classroom then the end points are fixed, the path may change, but the endpoints are fixed.

So I have to prescribe to the boundary conditions and the Euler Lagrange equation for this is dou F/dou y, in this case it is u with this notation just so that you see the chalk dust is just chalk dust, right, so this is, okay, the Euler Lagrange equation would be that dou F/dou u = 0 because it is not a function of u and only that is there and that gives me, is that fine, which is 1D Laplace equation, Laplace equation in one dimension, okay.

So this problem extremizing, finding the extremum for this problem is the same as solving this, okay, the big advantage so far it looks like other than the fact that you have to find the extremum, the big advantage here so far is there a difference between the 2 formulations, are they identical? In the sense there is a difference, right here we are really not asking for the candidate function to have a second derivative.

We are not asking that the candidate function have a second derivative, I can substitute any path that has a first derivate defined, okay, and I am going to integrate it, it can have finite number of continuities, countable number of discontinuities, whereas this requires that the second derivate exists, so there is a difference, it looks like there is a difference, this seems to look for solutions in a larger set of problem than does this, okay, is that fine.

Okay that is something that is of interest, we will pay attention to that, that is something that is of interest, but okay so I know how to solve this, how do we solve this numerically? Yesterday what we did was we did analytic solution, I knew there was an analytic solution so I sort an analytic solution, even this has an analytic solution, because it is an OD, but the point is not to use the fact that we know that it has an analytic solution.

How would you solve it numerically, okay, so recollect we defined in the beginning of the semester we defined Hat functions, maybe we will use Hat functions, you may be wondering other an interpolating for what purpose we define the Hat function, we will try to use the Hat functions, right, what was the Hat functions.

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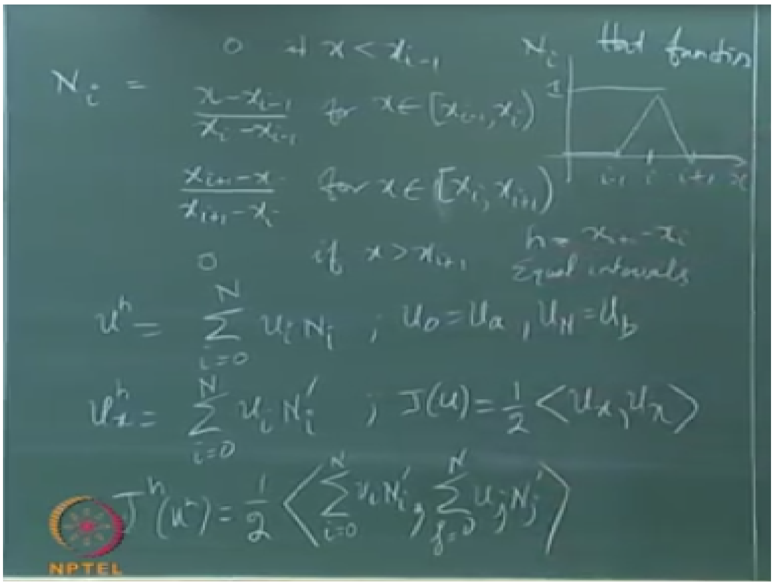

Ni what was it is definition do you remember? 0 if x < xi-1 x - x1-1 for something like that, I hope it is exactly, I hope the definition I am giving is the same, just open and close, we have to be careful that we did is the same, but it does not matter you can check it out to make sure I am being consistent, okay, that was the definition and of course a small graph just to remind you so that is i i-1, i+1, the function looks like that okay, it is called the Hat function.

Just to recollect, and this value is 1, this is x that is Ni, okay, right, so let us say the function u can be represented as Ui Ni okay and clearly U0 will be from the boundary conditions, U0 will be U sub a and uN will be U sub b that is given, these are known. These boundary conditions were given, okay, so we will bear that in mind, right now I am only looking for a general expression right when you can work it out properly.

I am only looking for a general expression because I just want to give you a drift of how these things work okay. So what is dou u dou x what is U sub x and do not get upset that I am using a prime here and a subscript there, as I told you, I am going to freely mix the notation right, there is clarity here, there is a reason why I am doing this you will see as we go along, the reason why I am doing this.

Right, both of them are differentiation with respect to x, there is no confusion right now, okay, and as it turns out and I want this also, that functional can be written in deed, can be written as is that right, it is Ux squared integral which is the same as this dot product, right, we happened to be fortunate at least for this example I happened to be fortunate that I am able to write it like this so I make use of that right.

Because I want to keep life simple, but otherwise actually it does not matter, you would just substitute there, I just want to keep it, you understand I am just making use of the fact that that worked out for me, so now what I want to do is I want to use so the early I should not instead of saying U you remember I should actually say Uh to indicate that it is an approximation where what is h?

I will make it identical right for any i they are all equal, equal intervals as I said in the beginning of the semester in this class, I am going to assume they are all equal intervals okay we can worry about non-equal intervals elsewhere. So what is the approximation representation for the functional, see now we can represent the functional. The representation for the functional, okay this is the point where you can make a mistake.

You have to be careful, okay remember when you are using subscripts and you are doing these kinds of things you have to change the subscript when the second term comes along right, this is the potential location for error, so I changed it to j, it is the potential location for error, okay, fine. Now in order to go on with this so this is just, this machinery will now just roll, there is nothing, it is just a matter of manipulation, okay.

What we need to do is, we know that we are going to get dot products of things like Ni Nj so we have worked this out before, but we will let us just look at it.

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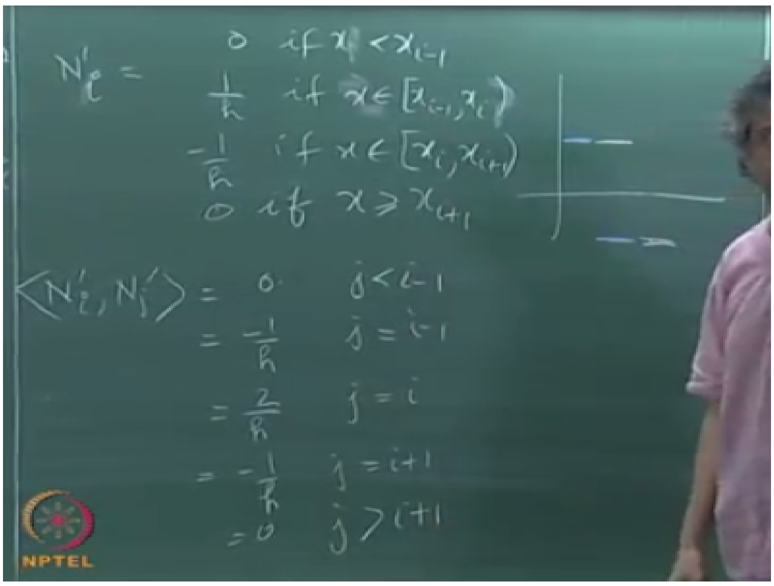

So if this is Ni, if Ni is defined in this fashion N prime i or j it does not matter = what is it 0 if x < xi-1, the slope is positive 1/h if x less than, okay, is that fine, okay, now we just take the dot product, now we know this Ni Nj, now we just take the dot product. Prime i dotted with N prime j, so what does this give me? what are the possibilities that we have? See the graphs basically look like that where this is the height 1/h.

The graph basically looks like that, okay, so you can have the negative of the j, that is j = i-1, if j = i-1, what is the i-1 function look like, let me take a different coloured chalk, the derivative of the i-1 is this, this is the derivative of Ni, this is the derivative of Ni-1. They overlap only on this interval, one is negative, the other is positive, so the product gives me -1/h squared if I integrate one h will go away, so it will give you -1/h, right.

So it is = 0 if j is < i-1 right, it = -1/h, if j = i-1 = well if it overlaps exactly j=i then this is positive and that is positive because -1/h * -1/h is still 1/h squared okay, there are 2 of them and the sign is positive, so it becomes 2/h, j = i and again -1/h if j = i+1 and 0 if j > i+1, right, if we have got the dot product, so in this summation, this summation of j, the only sensible values of j that we take are j is i-1, i and i+1.

All the other dot products are 0, is that fine, is that okay, the only terms that we take are i, i-1, i, i+1, right, so in fact I could replace that, I can maybe I will rewrite it, it is okay, so I can rewrite that now, right.

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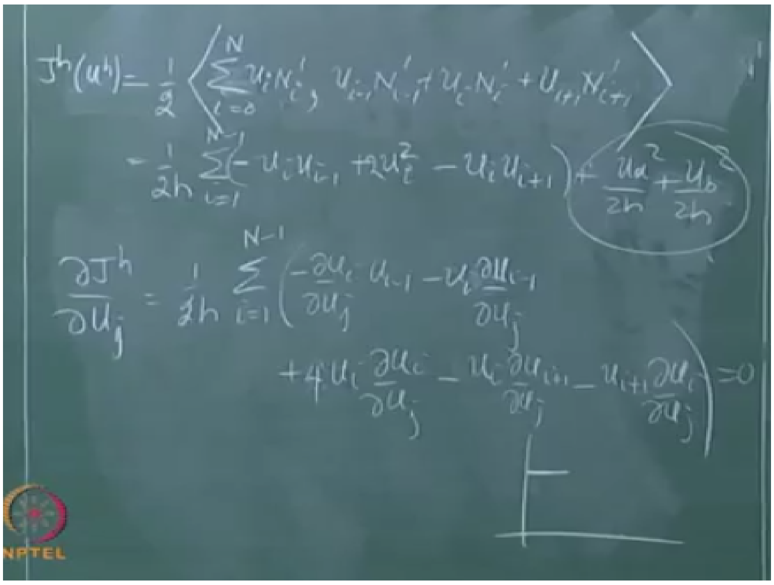

I can rewrite that as Jh Uh = 1/2 is that fine, okay, “Professor - student conversation starts” please, where is that, I am sorry, which one, no no but you are going to integrate so you will get a kind of a thing, it is essentially 0 to h that is what you get, so this will give you a xdx/h squared xi-1 to xi so that xi+1 that will give you an extra h is that fine okay “Professor - student conversation ends”

So we will come back here clearly I can take the summation out okay, so this is going to turn out to be or I can write it as 3 terms, so this is going to turn out to be 1/2, we will write it as 1/2h I think, fine, I can factor out the h also 1/2h summation i = right now I will write it from 1 through n-1, okay, because I know the first point and last point are basically boundary conditions.

The first interior point will also have a boundary condition in it, but I am not going to take care of that right now okay, right, just to indicate you that boundary conditions had to be taken into account, I will only remove the first and last points. So what does this give me, Ui Ui-1, what is the dot product of N prime i and N prime i-1? “Professor - student conversation starts” -1/h “Professor - student conversation ends” so that is the minus sign, there is an h there.

Okay + Ui squared and there are 2 of them because, that was 2/h, Ni prime dot Ni prime, Ui Ni prime dot Ui Ni prime that gives you Ni prime dot Ni prime which was 2/h, okay, so 2 Ui squared + what is the last one, or minus rather Ui Ui+1 and plus there will be these as I said boundary conditions which will be like Ua squared/2h + Ub squared/2h and so on, okay there will be 2 terms, 2 extra terms for either end, okay, which I have just knocked out.

I mean I could have left it in there, but I just took it out just to show you that they will come out remember that when i = 1, the Ui-1 will be Ua and when i = n-1 this Ui+1 will be Ub just bear that in mind, okay, the boundary condition still there, it has not disappeared, okay, now what, differentiate and set it equal to 0, that will be easier. All of it within the summation = 0, did I forget something.

Now you have to ask the question what happens, how does it work, so what happens to these thing, when i not equal to j it is obviously 0, right when i = j this is obviously, so if I am differentiating it with respective j you have to figure out all the ones when this is i, am I making sense.

"Professor - student conversation starts" yeah your question, yeah, I mean that is what I was saying when i=1 you are back here in this line, when i=1 if this is U1 this will be Ua and when i=n-1, Ui+1 will be Ub, you have to take care of that, I am not writing that, then I would have to write it from 2 to n-1 and write a liner term, these are product terms there will be 2 liner terms and then there is this, 2 first order terms.

No no listen to what I am saying, so you can 2 to n-1 and then you will get a U1 times Ua when i=1 you have to do something special that is what I said earlier, when i=1 you have to do something special. This term when i=n-1 you have to do something special, Ui+1 will be Ub, you can write it as a summation going from 2 to n-2 and just like I wrote these 2 secular terms you will have 2 other terms which will be U1, Ua Un-1 Ub.

It is not clear, from your face it is not clear whether you understand what I am saying or not, just write out the summation, but the U's do not figure in the dot product they only work with the Ni's there functions are Ni's the U's are just coefficients, that will be this function, only the first one N0 N0 will be nonzero all the others will be 0, N0 N1 will be the one that I am talking about here, which will give me U1 Ua.

It cannot be, how can that be, how is it possible you take each one of these, there is only one summation now, it cannot a double summation anymore, I opened out the other summation, this is the other summation, no these are the actual terms, when i is 1, oh you are saying I will have an n-1 term here, I have an N0 term here is that what you are saying, do all of them duplicate that means you are saying this 1/2 goes away, all of them will duplicate?

So this coefficient which I am anyway I guess I got away with it so far because I always set it = 0 is that fine, you are saying that when i=1 here this is Ni-1 which is N0 and when i=0 here you have a N0 here, but then what the Ni-1 does not exist, but when i=1, but you do not get another Ni-1 Ni-1 you get only one Ni-1 Ni-1, there will be only one, all the others there will be 2 and that is going to show up here actually.

It does not matter you can actually evaluate this term when you will see that it should come out the same way, does it come out the same way? There will be 2 of these, there is a 0 term here, there is a 0 term here, no if you take i=1 see this 1/2 multiples only the summation anyway we will work it out, I think I do not want to we are already 5 minutes into this, so what we will do is now since we have had this discussion you can check out these coefficients set this equal to 0.

You will now pull the plug on this whatever right because we work it out. **"Professor - student conversation ends."** So you can work out the details there, I think that is something, because with this counting you always have to be careful, so, right you can work out the details of, it is a matter of counting fine, so you have to make sure that you count right, but I will do it in a different fashion because I want to as I said.

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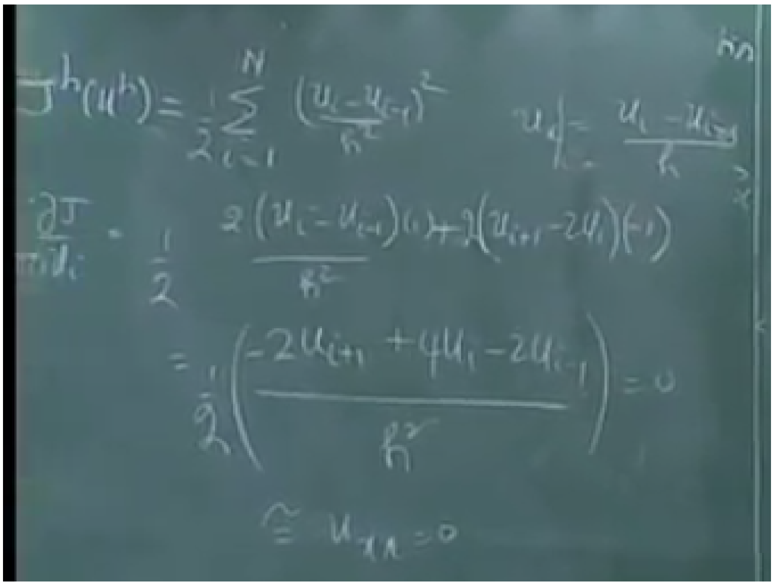

Let me do J of h the same thing if you want taking about Hat functions and this is the reason why this doubling occurs that you are talking about so to give you an insight into where possibly that doubling occurs let me give you summation i=1 through N Ui - Ui-1 squared/h squared, is that okay, we did Hat functions first then we did Taylor series representations of derivatives directly right.

So U sub x is Ui – Ui-1/h right this was the other, this is the other representation that we had Ui Ui-1 at h, am I making sense this is the other representation that we had. So you can differentiate this dou J/dou U you can do the same thing or we can, I will cheat, N and what does this give me, you have to be a bit careful here, okay, normally I would do it with J because I want to be careful but today I am going to we will push it slightly.

So what do you have here, you have 2 U_i U_{i-1} times $1/h$ squared, anything else? + when i is i+1 you have to shift it right, that is why I said doing it by J is easier, if you are differentiating with respect to 2, U_2 for instance U_2 then when i is 2 this is U_2, when i is 3 this is U_2, this will contribute + what does this give me $U_{i+1} - 2U_i$, 2 of those * -1 + is there any + fine, okay, so this gives me what?

If I set this = 0 so this is not, actually there is no summation here anymore, so if I set this = 0 what do I get? Actually even there is no summation anymore but it does not matter, we will get back, what do we get? $2U_{i+1}$ - $2U_{i+1}$ $2U_{i+1}$ $-2U_{i-1}$ so you will get $-2U_{i+1} + 4U_i - 2U_{i-1}/h$ squared, of course there is 1/2 outside, and I set it = 0, which is really a point that I was trying to get true okay.

So effectively you get the discretization, you get the discretization of which is the representation of $U_{xx} = 0$, am I making sense, it is a representation of $U_{xx} = 0$, okay, so all of this was leading somewhere, let us see where it goes, that is in one dimension, maybe I will just quickly do this, so if you have $U_{xx} = 0$ or $U_{xx} = f$ or whatever it is there is a game that what you call it that we play, that is one way, this is calculus of variations part.

I am going to come back to the calculus of variations part, but I want to take a slight detour before I go there, okay, so we can choose any other functions instead of choosing, I chose N here being Hat functions, but it could have been N_0 N_1 N_2 corresponding to quadratics or N_0 N_1 N_2 N_3 corresponding to cubics, we have done higher order representations also. Just so that we do not get confused with that.

I will change the bases functions maybe to say something like B and B has enough derivatives for me to do whatever it is that I am going to do now, so what am I going to do now?

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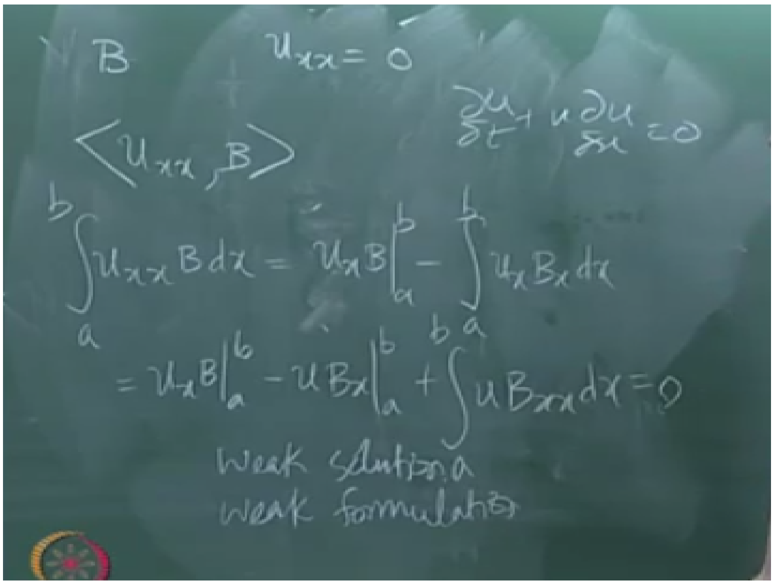

I have Uxx = 0 or Uxx = f whatever, I can project this equation directly on to these functions okay, so I basically say that Uxx dotted with B, I project that, right I project this operator on to this fine. So what does this turn out to be integral, this is a dot product, but this actually is Uxx Bdx on the interval a, b and that is supposed to be 0, now I will do integration by parts, integration by parts gives me.

You understand when I said, B has enough derivatives okay, that is what I meant, I am going to differentiate B, so I can do integration by parts one more if you want, once more, right, I can do integration by parts one more time, so you get, is that fine, right, all I am doing is integration by parts, so if B is sufficiently smooth I can transfer those derivatives and again like we did earlier so there I did it as a variational problem like we did earlier.

The requirement on the number of derivatives on U has now decreased, the smother I take my B, the less derivatives I need to insist on my U, am I making sense, you can admit a larger class of solutions so if you are talking about say something like wave equation, nonlinear wave equation, wave equation, where you can have a step function. You cannot substitute it into the differential equation and check whether it is a solution or not.

Because you cannot take the derivatives, okay, whereas in a formulation like this, you can actually substitute it in and if your B is 0 at A and at the end points that term goes away anyway, you understand, if the Bs are 0s at the endpoint that term goes away anyway or you can choose it so that something happens to Bx, I mean it is up to you, how you apply the boundary conditions, right.

So it actually turns out that so the U that you get could satisfy this but you may not be even able to verify whether that is true or not, it may not have a derivative, if it has a shock it does not even have a derivative, right, the beginning of the semester I said wait a minute here we have dou u dou t + u dou u/dou x = 0 even though we gave it smooth initial conditions the discontinuity appear as part of the solution.

Then the question was how do I substitute it back, if there is discontinuity I cannot differentiate, so in the classical sense is it a solution, the question does not make sense, whereas if you were to convert it in to something of this form you could still ask that question and answer it, okay, right, this is just for lingo, just to get the jargon, so the solution to this is called a weak solution, this is a weak formulation and a weak solution, okay fine.

Let us get back, this is just as I said, this is just an aside let us just get back to calculus of variations, what if it were in multiple dimension, you should have suspected when I went to the notation use of x that is I am going to talk about multiple dimensions right, obviously, so what if it is multiple dimension so it is a 2D, so the independent variables are x and y, okay.

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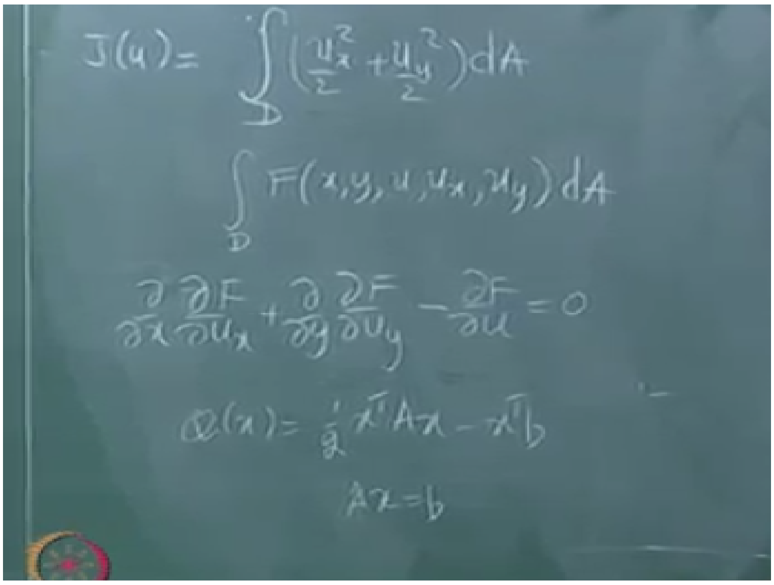

So J of U over some domain D right, so this is something of that form and I am not going to derive the Euler Lagrange equation, I am going to just squint at the earlier one and write it out, right, so it is likely to be, okay, is that fine, now you know why I went to the subscript formulation, the form, notation, so if you use this what happens to this, this becomes 2D Laplace equation, right.

So you have a variational representation of 2D Laplace equation and this will correspond to, this will give you Laplace equation directly, this is a variational form, so when we are solving Laplace equation I basically said that look solving ax = b for Laplace equation was the same as minimizing something. Solving the Laplace equation, the differential equation is equivalent to minimizing that, this is the continuous equivalent.

I showed you a discrete equivalent earlier, discrete version earlier, right, saying that if a was symmetric that Q of x = 1/2 x transpose Ax, you remember this, x transpose b. Minimizing this Q of x right, was the same as solving the system of equations Ax = b where A was symmetric, this is the analog okay, solving this is the same as minimizing that, am I making sense, okay, we are sort of tied the two together.

So if you were to discretize this or you have to discretize that you would suspect that this is discretization that will get you the same in both of them right which is what we did in one dimension, am I making sense, is that clear? So there are times when doing the variational form is easier than doing the differential form, you may look at Laplace equation, say wait a minute I say I am giving you Laplace equation as an example.

You could look at it and say that it is just averaging of the 4 why should I go to an minimization problem, minimization problem looks more difficult right, but that is because it is Laplace equation, okay, so what if it were a different problem so I will just write out the equation for a different problem just slightly messier problem.

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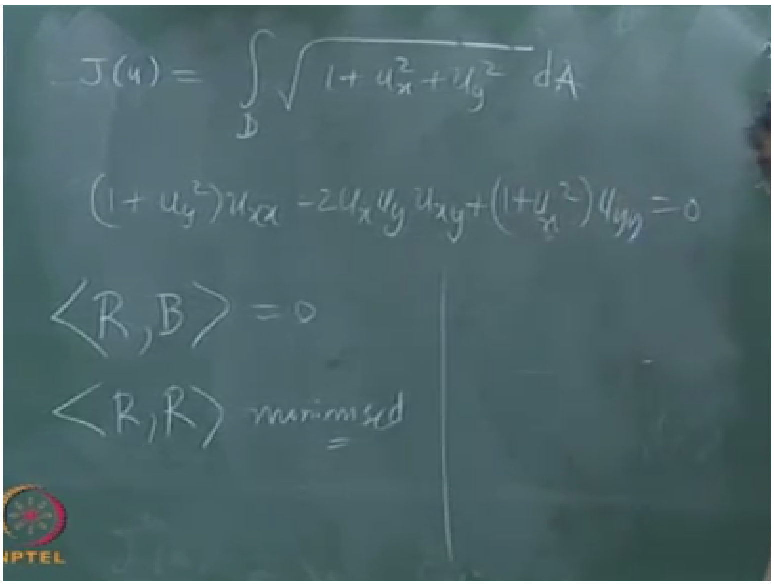

Consider this J of u, you know what this integral is? Surface area, right, if u is the surface, this is the surface area, right, this was the length of the curve, this is a surface area okay, so if you were to minimize this you will get a minimal surface so this is the typical soap bubble right, you take a wire frame of some kind stick it into soap water, take it out, a film forms on it, you want to know what is the function and because the thickness is almost 0.

You can equivalently tie the energy in that system to the area and so minimizing the areas like minimizing the total energy, right, you can actually tie the two together and it turns out that this minimal surface problem as it is called is a very classical problem, lot of people have studied it. The equivalent differential equation if you want so just to encourage you to think of variational problems occasionally is this.

If you are wondering why do I remember this, because it has a nice pattern to it, right. That is the messy differential equation that is not Laplace equation you understand what I am saying, this is the pay into discretize this works right, this I would rather really discretize this and throw it on some kind of an optimization problem, there are still issue, right there are still issues with respect to this.

We do not really have the time to discuss it, but I would rather discretize this rather than set up the discrete equations that come from this, am I making sense is that okay, so I would rather write a function that evaluates this and then say hang an optimization routine on top of it and let it go through the optimization process blindly as though it is a black box and come up with a solution rather than actually working out the differential equation that comes out.

The nonlinear algebraic equation that you get, n-dimensional nonlinear algebraic equation, that is a mess and in order to solve it anyway you will have to do some kind of Newton method or whatever it is and then you are committed in this optimization context to a steepest descent or something of that sort, right, instead of going there go directly to the optimization form, it is easier to formulate right.

So the variational techniques have a place, so far Laplace equation problems that you have never seen before in this class, how does it fit with respect to solving the Euler equation or Navier-Stokes or whatever, how do you do this with respect to these schemes, well, one is you could follow this path and it could be done in various ways, one possible way would be that, you take R which is the residue, R is the residue of whatever equation that you are solving.

See right now I am talking about it in a very general context, R is the residue, so you can take R and you bases functions and try to set that equal to 0, right, you could go through that process, at generic situation, am I making sense. R dot B will give you the components that go with this, the other possibility is that you could look at what is this it looks like the norm of R and you could try to minimize this or set it equal to 0.

Right you minimize this, so now I have a functional representation, you minimize this, if this R is minimized okay, you have a variational formulation, this could be Navier-Stokes equation or it could be Euler equation, it could be any form, of course there are special techniques the whole class of special techniques if you are talking about up winding and so on, right, I do not want to go there.

There are special things that you have to do if you want to get into this business of up winding, but if you leave that out for further research for you guys to do later, so what you could basically do is without even writing the differential equation so if you have a generic differential equation and R is the residue you could write the expression for R, take R dot R and minimize it.

Am I making sense and within round of error you should be able to get the minimize it to, you should be able to find the minimum, you actually wanted to be 0 because it is a residue, right, you actually wanted to be 0 and this could be minimized, so we have generating a variational problem is relatively easy, if you want to generate a variational problem directly from let us go back to Uxx = P.

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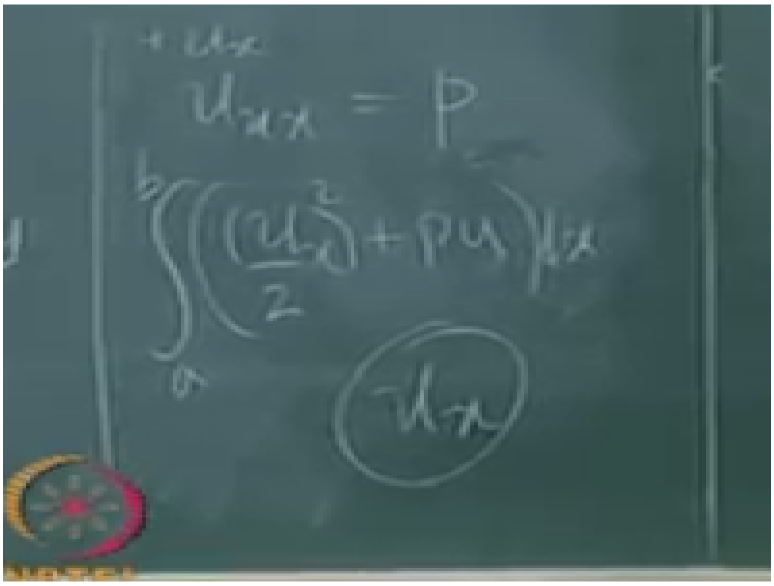

If you want to generate a variational problem directly from the differential equation one way to do it is, do it this way, but what if you wanted to act as though this is the Euler Lagrange equation or some other equation of some variational form, what is that variational form? How do you find that? I have picked something easy here, right. What I have picked basically is if you take Ux squared by 2, okay.

If you take something like this it works, I picked something easy, you can ask me the question how did you get this? well I guessed it just like all other integration. I thought about it, I said I want a dou f, dou u to be P, so it should be P times U, okay right, fine. The only derivative for which there is a headache it is the problem is the first derivative. So the first derivative, if you have a first derivative here that is the problem.

As always those first derivatives are headache okay, right so I think that about covers what I want to say about variational techniques that indexing thing is something that we need to, alright, we can just checkout, I will see whether Monday I will come back with the correction or I will just leave it for you guys to fix it, is that fine, okay thank you.