Introduction to Computational Fluid Dynamics
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Lecture - 36
Calculus of Variations - Three Lemmas and a Theorem

Today we will start a new topic which given everything that we have been doing so far that this basically we have been looking at one dimensional flow, will be a little different it is in some fashion connected to the material that we did right in the beginning of the course when we talked about representation of functions and so on, okay, so I want to give you a flavor for variational techniques.

Some of you may have encountered these variational techniques in different courses earlier, variational principles are used, right, have been used, principle of virtual work and so on, you may have seen these before. So there is whole area of study called calculus of variations, if you happen to take the elective it is good, but I think the last time I asked nobody is really dealt with calculus of variations before.

So what I will do again in the spirit of this introductory course I will try to give you a flavour for calculus of variations, it is, I am not going to do any finite element method as I said in right of the class of techniques I have basically looked at finite difference method I have mentioned, hand waved little on finite volume method, but this is in a sense of foundation for finite element method, okay.

In the sense it is basically the foundational material for finite element method. So the idea is very simple, what I am going to do is I am going to set up the relevant theorem, the important theorem that we need, so I am going to do it in parts, okay, so mathematics normally the way we prove something is that we take intermediate steps which are called Lemmas and then you prove your theorem based on that Lemmas, it is a logical sequence.

So the other reason why I want to do this is also to give you in case you have not had this before a flavour of proving something okay, so you may have seen, in calculus maybe you have manipulated, perform manipulations and so on, but just so that at more mature stage again you see right when these techniques of proving something. So that is basically the driving motive for these set of classes is that fine.

So there are 2 ways that I could do this one of course I could give you the motivation right up front, or we could move at the Lemmas, I do not know what is the, we will try out what the Lemmas like and then maybe we can see whether motivation is required I will give you the motivation, is that fine, okay.

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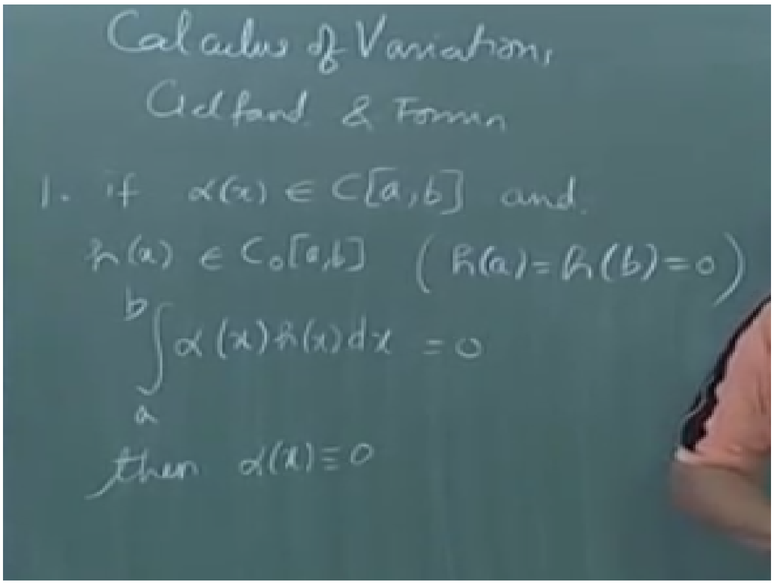

So the first Lemma, this is by the way I am following calculus of variations by Gelfand and Fomin, a very readable book, you can check it out, our library has it. So the first Lemma, okay, basically says if alpha of x is the continuous function on an interval a, b, so I would write that as alpha of x belongs to a, b, so this is shorthand notation, right that is the whole point about mathematics that learning language, I hope all you are familiar with this.

So C a, b is like this big bowl and the bowl contains continuous functions, you can stick your hand and take out a function right from that bowl, am I making sense. The a, b indicates that it is a bowl of functions defined on the domain a, b, on the interval a, b is that fine, is that okay, right, this is just matter of notations, alpha of x belongs to C so I am taking an alpha from this and I will have another function h of x which belongs to sum over notation.

So I am basically going to be introducing, I am using this opportunity to introduce this kind of notation, so C0 a, b and what I mean by this putting this 0 in between is that h is not only continuous on in the interval it is defined, it is a function on the interval a, b but it is 0 at the end point, what I mean by this is h of a = h of b = 0.

The integral of over a, b, alpha of x, h of x, dx if it = 0 for any h normally if you look at Gelfand this statement would come afterwards, you basically say if alpha of x belongs to this and integral a, b alpha of x, h of x, dx = 0 for any h of x coming from here then alpha of x is identically 0 okay, fine. So the way mathematics works, if you are taught about this is basically a conversation between 2 individuals, right, all you have to have a split personality, right.

So essentially what I am saying the statement of this theorem what I am saying is look I will give you an alpha of x, the alpha of x will be defined on the interval a, b, it will be continuous, right, so I am going to give you a continuous alpha of x such that alpha of x a, b integrated this integral will be 0. Now the other individual, you can pick any h of x that you want, you pick any h of x that you want which is 0 at the end points and it is continuous.

And I guarantee right this integral will be 0, fine, to which then you say at no, your alpha must be 0, am I making sense that is the conversation, right, so it is like a discussion, I am basically saying, look, I will pick an alpha of x, I do not tell you what and I am guaranteeing this is 0 and you get to pick the h of x, you pick any h of x that you want, as long as it is continuous then ha = hb = 0 okay, right.

So it is like conversation to which you basically say, if you tell me that then I insist that you must be picking an alpha it cannot be that oh I will just pick some alpha, your alpha has to be 0 that is what you are insisting, your alpha has to be 0, is that fine, okay, so you know mathematicians will look for different ways by which, the straightforward way would be that you assume that, you give me the benefit of the doubt.

You see okay Ramakrishna, impossible, you can actually find, we will give you right, there is a place, there is alpha is not 0 everywhere, it is not identical you see, okay, alpha is not identical, we will go with you, let us see where you go, right, basically you say I will give you what you want, we will get you in trouble, that is the idea, right, so alpha of x is not equal to 0.

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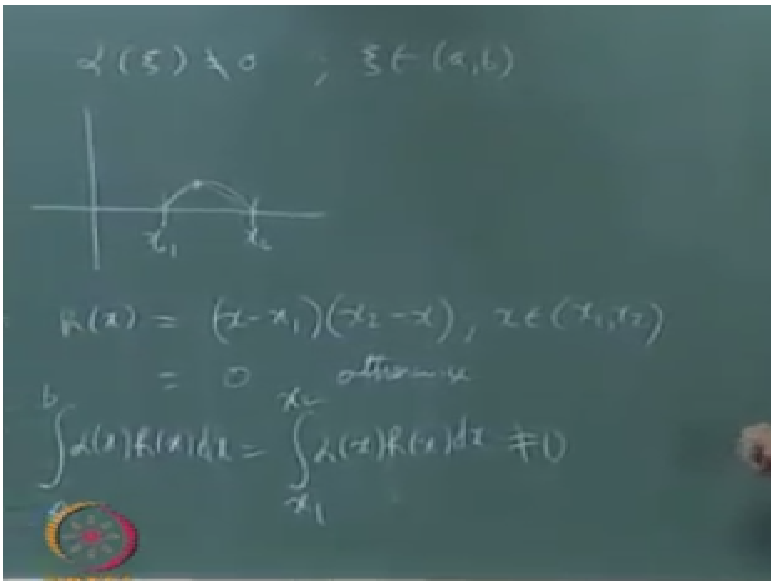

At some xi, there is some point xi in the interval a, b, right, where alpha is not 0, alpha being a continuous function there is a small neighbourhood around that point where it is not 0 basically okay, that is the idea, there is a small neighbourhood where it is not 0 that is why the continuity component is important, you have to make sure that I use every bit that I have stated here, right.

So if the function is continuous you cannot just have a nonzero value here and zero everywhere else that does not make sense, right, so there has to be a neighbourhood in which it is nonzero and we can assume that without loss of generality as they say then it is positive, you can make the same argument assuming that it is negative, is that fine okay.

So I finished my part of the story, now your part of the story, now you have to get me into trouble, so your part of the story would be, okay, we need to pick an h, you get to pick the h, right, see what I am saying is for any h this is true, you have to just pick one h for which it fails, that is all that you need to do. Your hunt now is to find an h that gets me into trouble, is that fine.

So you pick an h in such a fashion that it is 0 everywhere and nonzero on this interval, that is the idea, am I making sense, okay, so you can pick an h so if this happens to be say x1 and that happens to be x2, there is an interval over which it is nonzero, right, so if I pick an h of x which equals and there are different ways by which we could do this, of course we know other way by which we can construct it.

But anyway I will just write x – x1 times x2 – x, I picked this cleverly for x and x1, x2. You understand what I am saying right, so in my mind, I said well if the point is in between, I want something positive times something else that is positive, right, so x2 is > x if it is in that interval, x1 is < x if it is in that interval so x-x1 times x2-x is positive, right, that is why I picked that, am I making sense and it is 0 at the end points x=x1 it is 0 x=x2 it equals 0, okay.

You could have used our HAT functions also, you could have put a small HAT function there instead of this but it does not matter equals 0 otherwise. So then what? I substitute integral a to b alpha of x h of x dx = integral x1 to x2 where it is nonzero alpha of x h of x dx is not equal to 0, violating my guarantee, you understand, which means that I cannot guarantee, the only way I can guarantee this is if alpha of x = 0 identically everywhere, is that fine.

This Lemma is just a sort of warm up to get you a feeling as to where we are going okay, I am not going to use this directly, but the next Lemma we will use directly okay, fine. Questions? Okay, let us try one more, see where that goes if alpha of x is in C a, b same thing and h of x is in now I am going to change it, I will say C1 a, b, the superscript 1 indicates now h has also got derivatives okay, the superscript 1.

Now I am not talking about functions that are continuous right, the function can be continuous, but there may be points where the derivative is not defined, okay, now I am saying no the derivative is there everywhere, this is the bowl of function which can be differentiated everywhere, okay, this earlier this is the set of functions which were just continuous, now we are basically saying no that is not it, this is the set of functions which is not only continuous, but also has derivatives, okay.

It is just notation that basically tells us see, you have to get comfortable with it, very quickly it looks like oh it gets messy, but if you imagine look at how compact that notation is, defined on the interval a, b, 0 at the end points has derivatives, you understand, right, it is continuous derivatives, fine. Okay, if alpha of x h prime of x where h prime indicates differentiation with respect to h.

Then what do you expect? We will turn out that alpha of x is identically a constant. Okay, we will turn out that alpha of x is identically a constant, is that fine. Now we have the same strategy now, we have to figure out a strategy before we start off, obviously what you are going to say is, we have the same discussion, right, it is obvious that I am saying, I am going to give you an alpha from this bowl of continuous functions.

So I am going to dip my hand in and take out an alpha and I will take that alpha and I guarantee to you for the alpha that I have got, I take a look at that alpha and say you know what, for the alpha that I have this will always be true, you get to choose the h, this will always be true to which again your response will be if that is the fact then your alpha must be a constant, okay, right.

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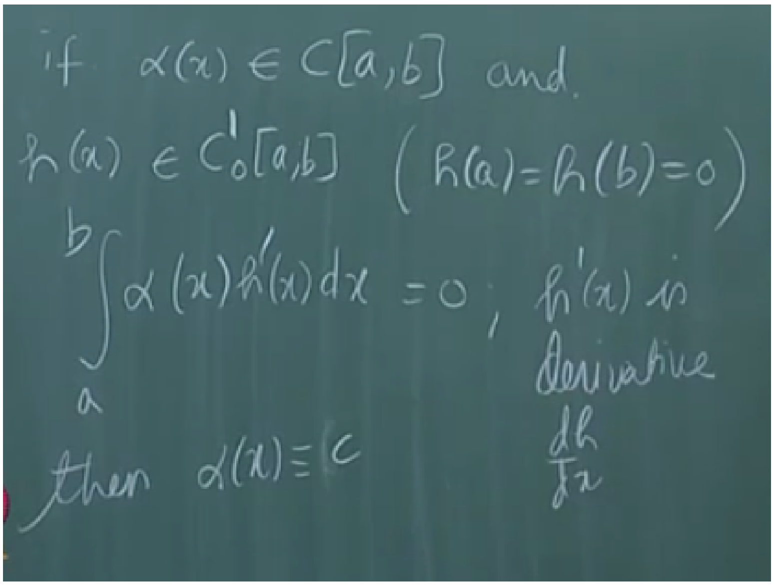

And again we go through the same argument saying well okay may be my alpha is not a constant, right, you give me the benefit of the doubt one more time, and say okay alpha is not a constant, alpha is the function of x, so if alpha varies your business now is to give me an h that will get me into trouble, right, so that it will force me to concede that alpha is a constant, am I making sense, okay, is that fine.

So our objective is now to construct an h further, not only construct an h, h which has a derivative, we need to construct an h which has a derivative, with what we have in our hands, what we have available to us we need to construct an h which has derivatives okay, so it is a good idea to define h as the integral of something, see there is a logic that pushes us, okay. So these are standard tools that you would use when you are looking for something.

There is inspiration it strikes you saying oh I see the proof, the other is that you sit down and you actually work through systematically, so these are clues that you have, so it has equal of constants, so we will start by defining a constant as the average of the alpha, okay and I do that the average value of the alpha, right then if I say alpha is, you understand what I am saying, alpha is a constant it will be the average value obviously.

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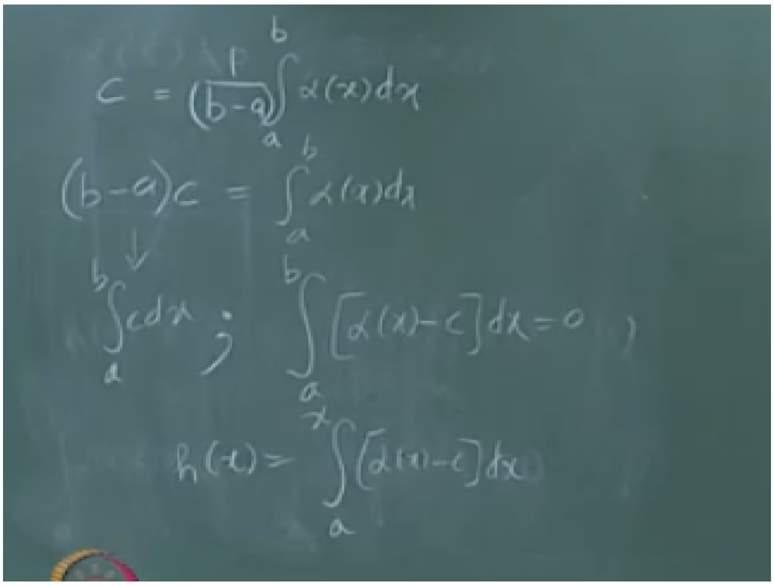

So the average value of alpha, I call that, I will say constant, I will define C as what is the average value integral a to b alpha of x dx 1/b-a that is the average value. We will start there, we will start with that constant, okay, so if your alpha is identically, so what I want all we have now come to is, I have to prove that alpha is identically that constant, fine, so this can be rewritten actually, this is b-a times C which suspiciously look like integral cdx between a to b, okay.

So b-a times C, so this looks like integral cdx a to b right, which I take over to the other side, the average in the sense I mean you may have seen this normally people will just do it direct, so you say the average and we put a semicolon there, so alpha of x, so I define a C in that fashion that is just the average which is the same thing, I have just manipulated, I started off with the average, multiplied by b-a.

Recognize that that is the integral, average that is essentially what it means right, the integral is the function and the integral of the average are the same and that is essentially what we are saying, integral of the function and the integral of the average are the same, fine, right, so I said oh I need an h that is differentiable, I have an integral, right, so I can define an h now, so I say h of x, okay, so what is h of a? 0, what is h of b? 0.

And the derivative exist, you understand what I am saying, so we have got what we want, is that fine. Now what, now I have to somehow involve this in this integral, I have to get an integral of this form, right, I have this, so I say okay, let me look at into h prime dx, okay, which is nothing but integral, okay.

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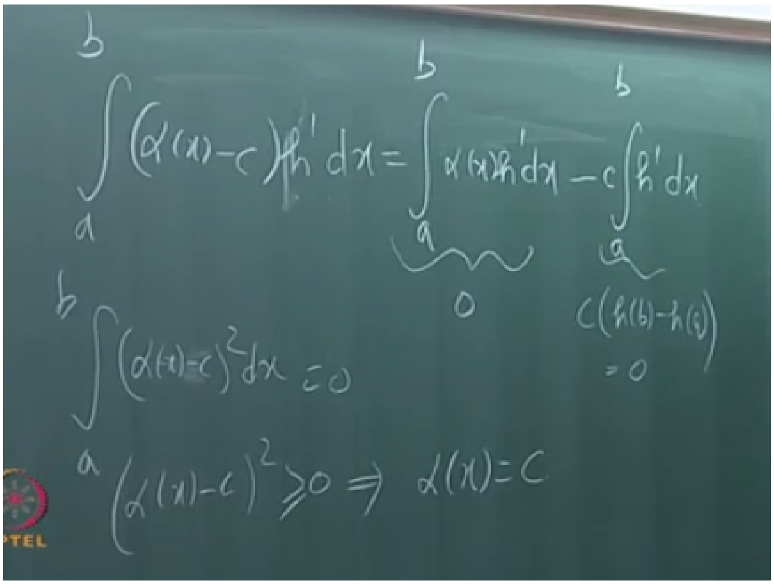

So this is 0 because this is hb-ha this is just hb-ha, C times hb-ha that is 0, this is what we want, now what is this? this I am guaranteeing to be 0, this is what I claim is 0, I have given this to you, I said oh you pick any h I guarantee this will be 0 right, so this is 0, fine that is given it is 0, that is my guarantee, and what is this, what is h prime, alpha – C so now I have integral a to b alpha of x – C whole squared dx = 0 and this is the positive quantity.

The integral of this positive quantity is 0 therefore the thing itself has to be 0 there is no choice. Integrand is positive, is >/=0 alpha of x – C squared is >/=0 this tells us that that is possible since this righthand side equals 0 alpha of x has to be a constant, has to be that particular constant, is that fine, so it is not bad, I mean we can work through and always when you are reading these theorems you should always look at it as a conversation right.

At least it helps me I do not know whether it helps you, it would always help me right, to look at it as a conversation. Okay, now let us see if we can get to the third Lemma, Lemma 3 before we get to the actual theorem okay, the third Lemma.

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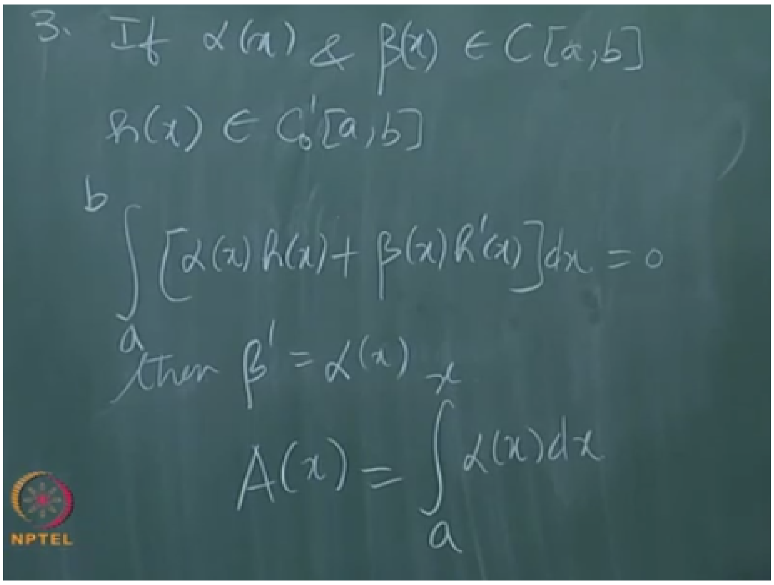

Says I have 2 functions now, and if alpha of x and beta of x belong to C a, b there are 2 functions that I have, alpha of x and beta of x that come from C a, b, right, so and h(x) again may be I need to erase this, belongs to a, b which is 0 at the end points first derivative exists. The integral a to b alpha of x, h of x + beta of x, h prime of x dx = 0 okay, so this is neat relationship then this is remembered that I have picked alpha and beta from bowl of continuous functions.

If this is 0, right, then you can assert you can tell me look you must have picked beta which has actually got a derivative and that derivative is alpha, if I guarantee this and you can tell me you must have pick the beta that has a derivative, right and that derivative is alpha, okay, is that fine, so again we look here we look for clues just like we did last time clearly will have to figure out something to do with this integral, now it has h and h prime.

So we either have to convert it into something that has h or something that has h prime, we have done Lemma before which had an h prime, right we had done a Lemma way before that which had an h so if you could somehow convert this to something that is completely h prime or something that is completely h we could apply one of the 2 previous Lemmas, okay that is one thing that is clear, am I making sense, see this is strange.

The first one had only h in it, the second one had h prime in it, this has both, okay, so I am taking well, yah if I want to convert this to an h prime I know one rule that will introduce derivative when we are doing integration, integration by parts, we have products of thing right, so I know one mechanism by which we can do it. Integration by parts will get me a derivative.

See we are working out strategy now, right, so integration by parts will give me a derivate so that is fine, okay, the second thing is I want beta prime to be alpha so it looks like I need the integral of alpha in some fashion, so we start by defining a function which is the integral of alpha so A of x is integral a to x alpha of x dx, okay, so that is the potential candidate for beta, you see what I am saying because I look at this and I say okay.

Let us lay the foundation first, right, so we have that then what do we do? Integration by part so we start with alpha of x, h of x.

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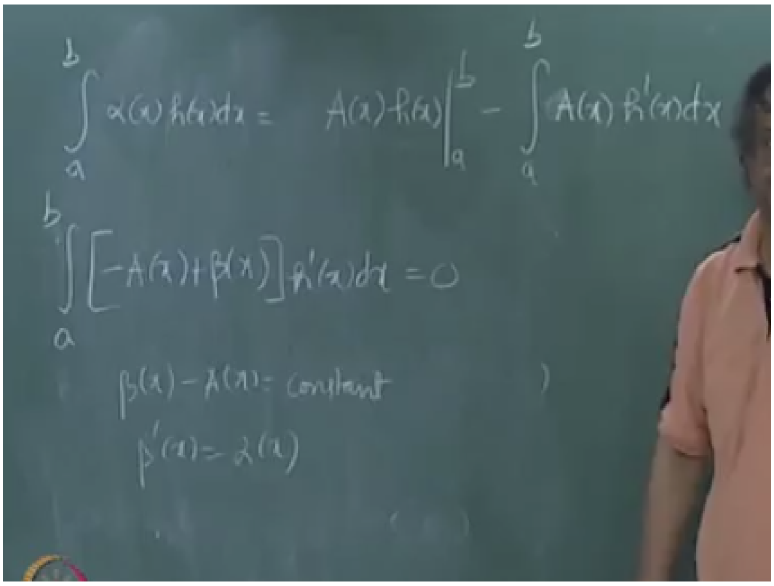

So the integral a to b alpha of x h of x dx = integral, how should I do this, A of x, h of x between the limits a to b actually I want that but you understand what I am saying, I basically want and minus the integral a of x h prime of x dx, right, that is not bad, we can just go back now and substitute for alpha of x h of x, I can just make this an indefinite integral if you want but anyway it is okay I will leave it as it is since I have written it I will leave it.

And therefore we apply Lemma 2, you understand, I have some function of x times h prime of x, dx = 0 and our second Lemma basically said this quantity must be a constant, okay, am I making sense, see that is the idea of, that is why you do these Lemmas have small, small results along the way, they are like they are the programming equivalent of writing small functions that you can use to build the reach the bigger objective, okay, that is the idea.

So this tells me that beta of x – a of x = constant or beta of x is if I differentiate beta prime of x is alpha x, okay, that is 3 Lemmas, this belongs to as I said the segment of the course that I call 3 Lemmas in the theorem, so we have done the 3 Lemmas what about the theorem? for the theorem I will give you motivation, right, why have we done all of this what is the point, okay.

So we had said in the beginning as I said, I will tie this up to the material what we have talked about in the beginning of the semester as I said in the beginning what we are looking at is functions, the solutions that we are looking at, right, are functions, the problems that we have at hand, the solutions that we seek are all functions. A simple example now would be you are at your dining hall or whatever having breakfast this morning, you wanted to come to this class.

There are any number of paths that you can take from your dining hall to the class, am I making sense and each one of those paths is the function of in this case x y z, right, since we are on the third floor in this case x y z, right, you start off, you walk along or you bike along but we assume that you walked along, so start walking, there are different ways by which you can get here, okay.

Some of you may forget make a turn near chemistry building and head out in the wrong direction and then maybe come through humanities and science block and come here so it is a long winded path, right, so the question that we have now, nice, so there are lots of path, right, it is nice to know that there are lots of paths, so one block path is blocked you can come by another path.

So but the question that we asked now right which makes this interesting is, is there a shortest path? Is there a safest path? right, is there a most energy efficient path? summer is coming is there is a path that is the coolest path?, see you have a metric, you have a measure, this is like our residue, to say that yes I have what I want, okay, so you can ask this question, so there is an issue of optimality here.

What is the optimal path, what is the best path in some sense okay, so if you say that you have the best, what do you do normally in calculus if you say I have the maximum or minimum, what is that we normally do? "Professor - student conversation starts" no no before you get to the derivative, derivatives comes later, we will go towards something that looks like the derivative, derivative comes later.

You perturb it "Professor - student conversation ends", you disturb it so if you think that you have a minimum, you disturb it and the disturbance should cause the value of whatever the measure that you are looking at to increase, am I making sense, so if you say I am looking for the shortest path, if you disturb the path, then the length should increase, it should be longer than the shortest path, any other path should be longer than the shortest path.

So the length should increase does that make sense, okay, but clearly you always want to start whatever the disturbance that you do you always want to start at the dining hall and end up in this class room. If you create a disturbance that takes you from the dining hall and takes you toward the wrong classroom that does not help, right, so that explains why h of a = h of b = 0, h is the disturbance, right, to explain why is the function.

What is this peculiar function that I am talking about where I keep saying the end points are 0 and it is a continuous function, it is continuous, yah you are not going to teleport from one point to another point, you are not going to it would be nice, but you are not going to sort of walk along and suddenly zap and then you appear at some other point, right, there is no discontinuity, the path is a continuous path.

Okay so the path is the continuous path, so you are going to start at one point, always end up at the other point these points are the same, you can change the path, but the end points do not change, okay, so what you do is you have h of a = h of b = 0, a is the starting path, b is the ending path, am I making sense, and you have continuous functions then you can perturb with this h that is the idea, okay, fine.

So in calculus of variations the usual way by which we do this maybe I will leave that there, so we look at, we will use the standard notation calculus of variations.

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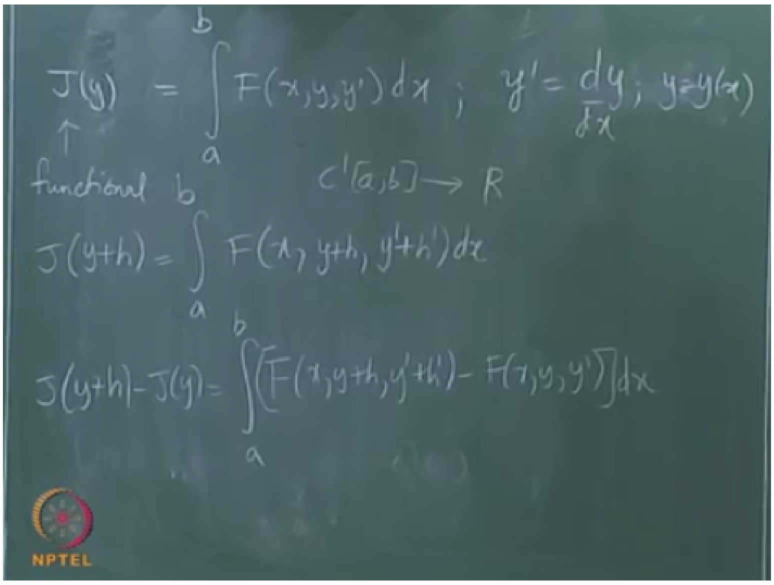

Functional, argument is a function, I am going to give you a path, a whole path, you understand, the argument is a function, so this thing is a function of a function, right, so it is a functional = integral a to b F(x, y, y prime) dx where as earlier y prime is dy/dx obviously y is the function of x, fine, what we want is, we want J(y), we want that y, we want to find a function y so that J(y) is the minimum, it is an extremum or a maximum, right.

This could be some function J(y) could be of profits, you want to maximise your profit, right, this could be some function, it could be the distance between 2 points and J(y) is the measure of the distance, you want to minimize the distance, or it is the time, right or you want to maximize. This is the function, it gives you, J(y) is the amount of time that an airplane flies.

You want to maximize it to get endurance then y is the function that will tell you what is the longest maximize that, you understand, these are all basically they take a function as an argument and it returns a number, it is a map, it maps a function, given a function it maps the function into a number, you understand what I am saying, so if our function y, it is y prime has a derivative.

If a function happens to come from C1 defined on a, b it maps it to the real line, am I making sense, that is what this is doing, it is taking a function and giving you a number, just like I normed it, norm of the function that is what it did, the norm of the function basically swallowed a function and spit out a number, the same thing, okay that is what normed it, so it is not very different from something that we have seen before.

Okay so we want to take derivatives, we think back derivatives, the perturbed J (y + h), what is h? h is our earlier friend h, 0 at the end points, you understand and continuous, a to b F(x y +h y prime + h prime) dx, okay, and just like we do when we define a derivative I subtract this, from this I subtract that, so I get J(y+h) –J(y) = integral a to b F(dx), so far, so good, what can I do now.

Lefthand side there is nothing much we found, we want to find that out, righthand side is the only thing, there is not anything that we can do here, so we look at this Taylor series, you look at this you think Taylor series right, since I am thinking in terms of derivatives, ignore the fact that we are talking about functions, ignore the fact that this is the map from functions to real number, right now.

Since we are thinking about derivatives, what are the derivate, what is the general sort of symptom or definition that I gave for a derivative, it is a liner transformation and a direction. We know the direction, I have perturbed it in the direction of h, we want the liner transformation corresponding to that, getting out of that, right. We want to get the liner transformation, not corresponding to that, but getting out of that.

So we have the difference so what it basically means I am going to expand this using Taylor series and keep only the liner term, right, keep it only till liner because that is what I want, I want the liner transmission.

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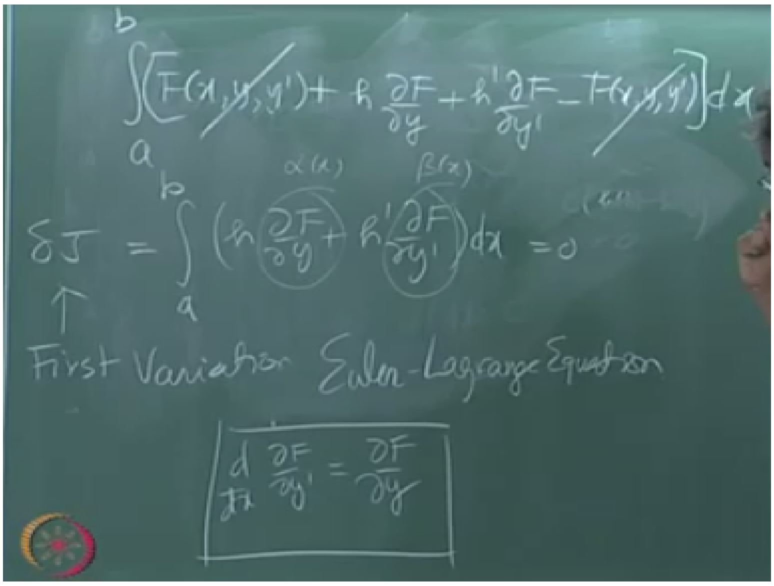

So what do I have? integral a to b F(x, y, y prime) + what is the second term of Taylor series, h times dou F/dou y + h prime times dou F/dou y prime – is that fine everyone. I am just blindly doing Taylor series, just manipulate the ink marks, manipulate the chalk dust do not worry about oh this is the derivative all of that kinds of stuff, remember what we did when we did Flux Jacobian, substituting q1, q2, q3.

If it helps you replace this by x, y and z, instead of y prime, okay, think of it as x, y and z, right so then this would be h times dou F/dou y delta z times you understand what I am saying, dou F/dou z that is it, this is the Taylor series, right, and I have chopped it off at the liner term, this of course cancels, giving me the integral a to b, h dou F/dou y + h prime dou F/dou y prime dx since it is the liner part.

We call it the first variation it is given the symbol delta J called the first variation it is a change, it is a first variation like it sounds like first derivative, like the first variation, and if it is an extremum this variation will be 0, fine. Now we are ready to apply Lemma 3. Lemma 3 basically says that if you have alpha of x h of x + beta of x h prime of x dx = 0, 0 for any h that you give right, any pertubation that you give then beta prime of x = alpha of x.

That is beta prime is d/dx dou F/dou y prime = dou F/dou y, is that fine everyone, these equations are called we have seen this maybe in your physics or something of that sort, Euler Lagrange equation okay, fine. In your physics most probably you heard about it as a Lagrange equation, you would define the Lagrange and then so on, right, so that comes from the analytic dynamics point of view.

So they are called the Euler Lagrange equation, is that fine, has any questions. So we have managed, this is an interesting thing that has happened here. So this is a differential equation, right, in some sense this is like a derivative. So we have managed basically to go from an integral variational form which was here, we managed to go from an integral variational form J(y) which we want to minimize or maximize or get the extrema.

Right get either the maxima or minima, you managed to go from this form through this process to a differential equation, so we have gone from something that looks, so this is like differentiating it in some fashion, like taking a derivative and setting it equal to 0 that is basically what we have done. We wanted an extremum from that functional and we have managed by some process of differentiation to get a differential equation.

Is that okay, everyone, is that fine. Of course, there is the equivalent, you could ask the question if I give the differential equation can I go back to the variational form, okay, and just like differentiation and integration, going from there to here is easier than going from here back there, right, because now you have to guess, you have to come up with the variational problem and then turn around and say, if I take the first variation of that though I get the equation that I have at hand.

So it is like the equivalent of the integration form, again involves guessing, right, so the direct thing going from the variational problem the optimization problem to here is relatively easy, fine, is relatively straightforward, very often there are times, you can ask the question, why would you want to go from here to there, there are times when this is very messy and the variational form looks quite simple and elegant.

Okay, maybe I will give you an example of that a little later. Let me see if we can, I will just set this up, we will see how far we can take this today, so what was the example that we talked about, we talked about distance between 2 points right.

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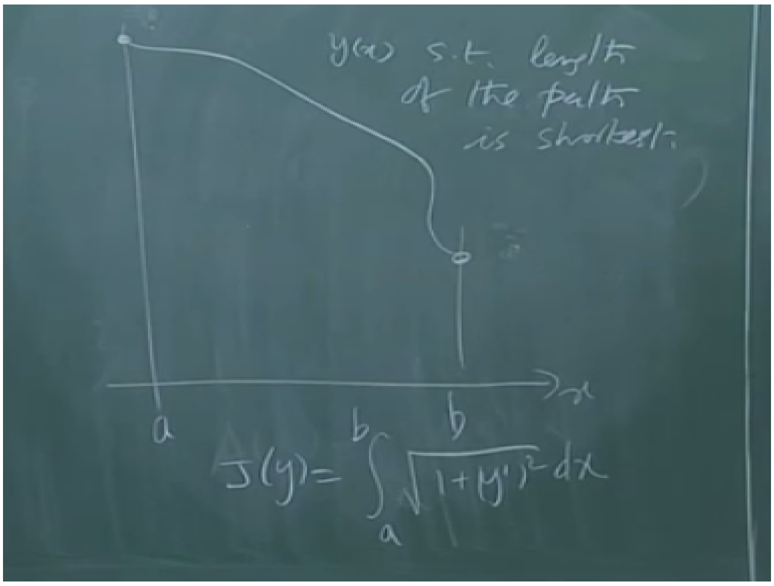

So I will take the distance between 2 points a and b, that is a, that is b this is x axis, right and what we want is we want y of x such that length of the path I should not say distance between 2 points, length of the path is shortest, okay, the shortest path then you could identify as the distance between 2 points and I have actually changed though I gave the example as from your dining hall to here.

I in an subtle way change the problem how I change the problem? there are no obstacles here, this is truly the straight line, there are no buildings in between, there are no obstacles, you can just walk the straight line path which is what you should get, right you can just walk the path, am I making sense, so what is the length of this path, y(x)? what is J(y)? integral a to b, remember square root of 1 + y prime squared dx.

Okay, it is the square root of ds squared, ds squared is dx squared + dy squared, finally write it this way and F happens to be okay.

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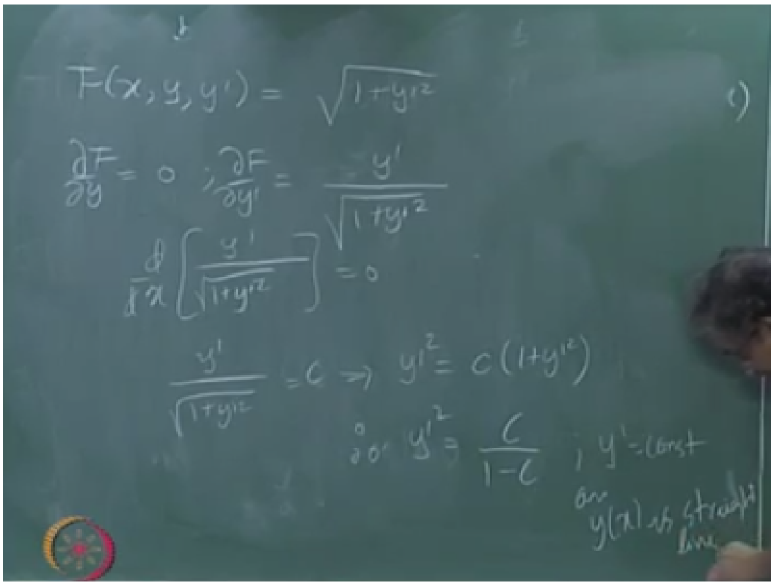

What is dou F/dou y? = 0, fine, what is dou F/dou y prime? = 2y prime/square root of 1 + y prime squared, right the 2 and the half cancel, so the Euler Lagrange equation tells us that dou F/dou y is 0, Euler Lagrange equation tells us that d/dx of y prime/square root of 1 + y prime squared = 0 or y prime/square root of 1 + y prime squared is the constant just say C. There are different ways we could do this.

But anyway we can start from here integrating appropriately or we can do indefinite integrals, then it is alpha y prime, y prime squared equals C times 1 + y prime squared. Therefore, you always be careful when you square things, you allow for spurious roots, therefore y prime is squared if you want, you want me to keep it simple, y prime squared is C/1-C, okay.

The other thing that you can do is you can integrate this between a to x then you will get a lower limit which is this quantity at a and then you can manipulate if you feel more comfortable doing that, so this tells you basically y prime is a constant or y of x is a straight line is that fine, okay, right, thank you.