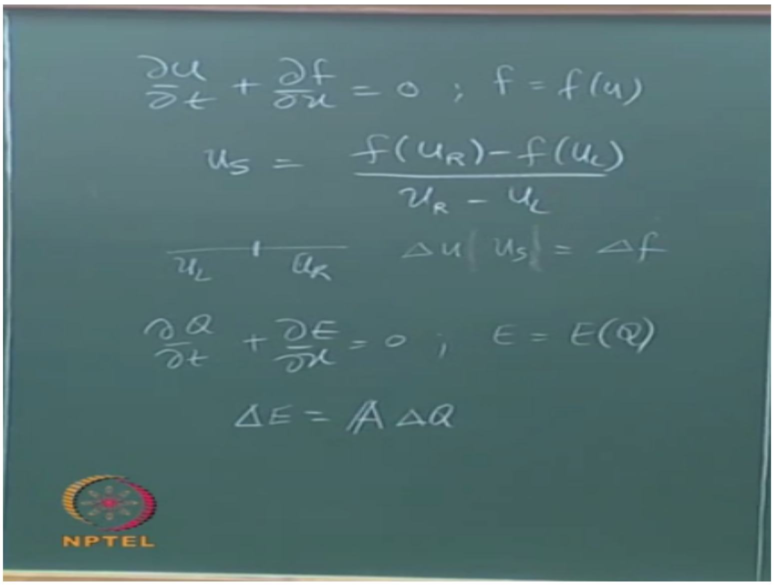
Introduction to Computational Fluid Dynamics
Prof. M. Ramakrishna
Department of Aerospace Engineering
Indian Institute of Technology - Madras

Lecture - 30
Roe's Averaging

So we will continue with our derivation of an expression for the average term okay at the interface. There are again just based on a question that a student asked me yesterday.

(Refer Slide Time: 00:36)



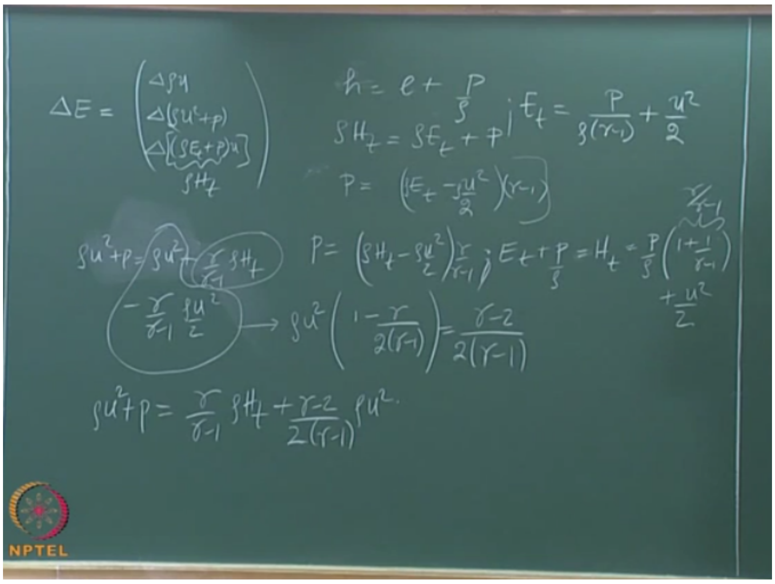
I thought I would remind you that when we looked at the equation dou u/dou t+dou f/dou x=0 f is f of u right, you remember that across a shock as we called it across a discontinuity because I had explained the model in terms of a small shock tube which shows a tiny discontinuity and we are trying to figure out what happens in that shock tube, so just to remind you the propagation speed at that time right.

We found the propagation speed at that time just to remind you was f of uR-f of uL/uR-uL fine so where you are talking basically in terms of you have an interface and on the left hand side you have uL and on the right hand side you have uR and we were basically looking at how that interface propagate okay and we derive this expression for the speed of propagation fine, we are essentially using that.

Now we are asking the question what is this A. In the scalar case, this was just a number but in the vector case then it becomes a little more complicated. This is not just a set of 3 numbers right. So in the vector case, we have dou Q/dou t+dou E/dou x=0, E is a function of Q in a similar fashion. So if you look at this what we are basically saying here is delta u times Us=delta f right.

And we write an identical equation here delta u times this is delta u times Us=delta f and we are writing an identical equation, so the different ways to approach this okay. Yesterday, I gave one kind of motivation. So you can just basically write delta E equivalent to delta f is some A times delta Q. Is that fine? And we want to find out what are the entries in A? Now as I had indicated we will convert this equation in terms of enthalpy okay.

(Refer Slide Time: 03:34)



So what was delta E? Delta E is delta rho u, delta rho u squared+p, delta rho Et+p times u okay. So rather than the delta's it is clearly the same as E. So we will just look at E. So if I want to convert it to enthalpy, there are 2 things one the objective is to get rid of the pressure and the total energy okay so I need to get rid of this p and the definition of enthalpy of course directly gives us the total enthalpy here, so rho Et, you think about it.

Enthalpy is h is e+p/rho, so the consequence the total enthalpy is total energy+p I have multiplied through by rho okay. So immediately if this quantity becomes rho Ht that is nice okay. The other expressions are little messier, so it is nice we will take it right. What was p? You remember the expression for p. We got p through the definition of E total as gamma-1+u squared/2.

Is that right? p/rho is Rt, R/gamma-1 is Cv, Cvt is E okay that is basically what we have done and as a consequence this p in fact turns out to be E total–u squared/2. If you want rho E total you can take the rho inside also gamma-1 okay. Yeah so A you will have to bear with me because we are going make substitutions and all of that stuff, B you will have to make sure I stay honest, make sure I do not make mistakes right.

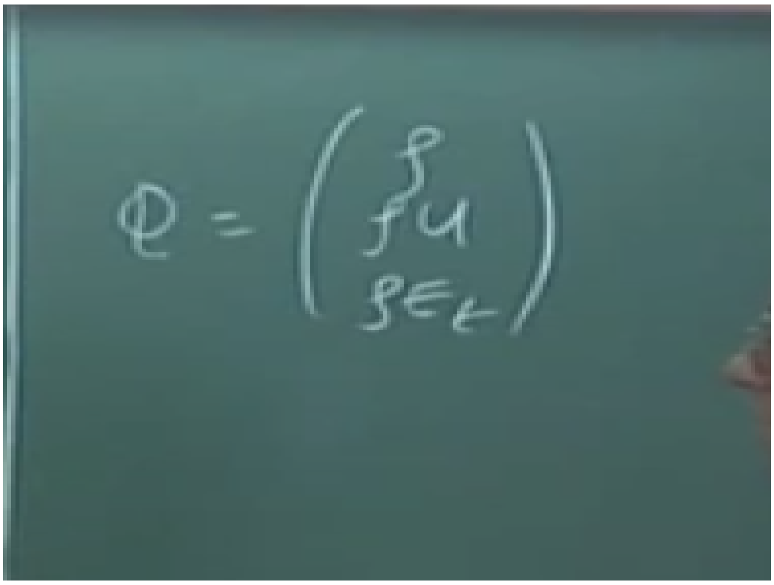
So okay here we have it. What is rho u squared+p? Therefore, is well before I do that I need to do one more step right because if I substitute for rho u squared+p from here then I will get a rho Et right. So what shall I do? I want to eliminate this rho Et. Is there something I can do up here? Okay so maybe before doing this you can either add p to this and eliminate it just like we use this equation. You have to be bit careful okay.

I can add p here so Et+p/rho is Ht. So you have p/rho*1+1/gamma-1+u squared/2. Is that fine? So I get gamma/gamma-1. So in fact the same p can be written as rho Ht-rho u squared/2*gamma/gamma-1. Is that fine? Okay. The only difference is that this gives me a gamma/gamma-1, little manipulation, now come back. What is rho u squared+p? Rho u squared+all of this stuff okay.

So that gives me a gamma/gamma-1 rho Ht-gamma/gamma-1 rho u squared/2 fine, not much we can do with this. We can combine these 2 terms. This gives me rho u squared times 1-gamma/2 times gamma-1. So I will have 2 gamma-gamma which gives me a gamma in the numerator, then I have a -2/2*gamma-1 okay so you forget out rho u squared+p is gamma/gamma-1 rho Ht+gamma-2/2 times gamma-1 rho u squared. Is that fine? Okay.

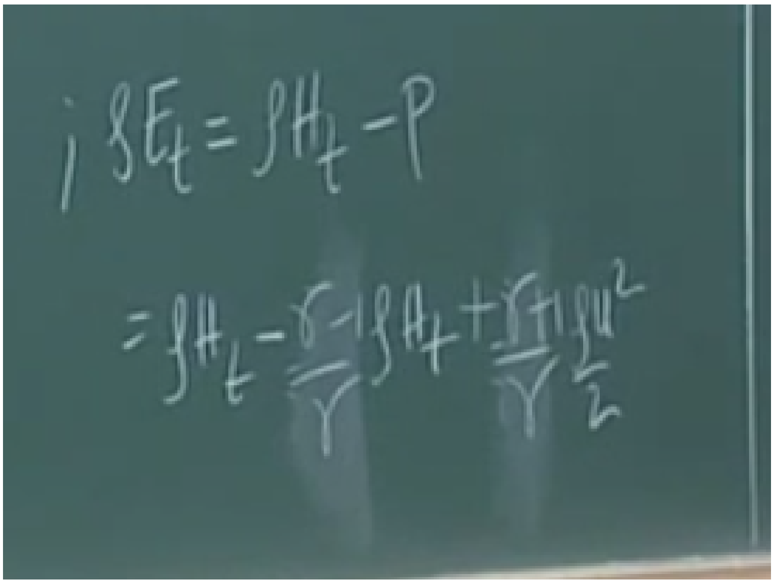
As I said you please check to make sure that there are no algebraic errors fine. So that is delta E. What about delta Q?

(Refer Slide Time: 11:18)



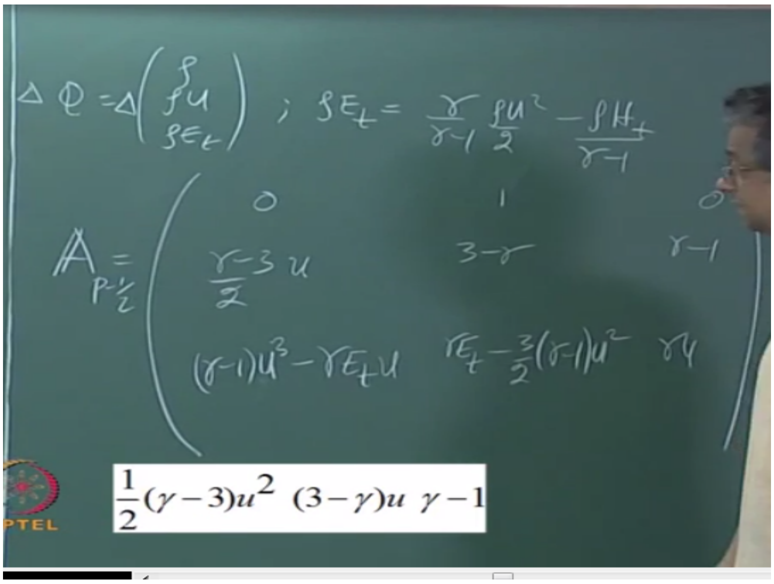
So we need to just look at Q, Q is rho, rho u, rho Et and we have made a substitution for rho Et already.

(Refer Slide Time: 11:30)



Rho Et is rho Ht-p and we have an expression for p in terms of rho Ht. This equals rho Ht-gamma/gamma-1 rho Ht+gamma/gamma-1 rho u squared/2. Is that fine? So here again I get a gamma-gamma that goes away, I am left with -1/gamma-1.

(Refer Slide Time: 12:33)

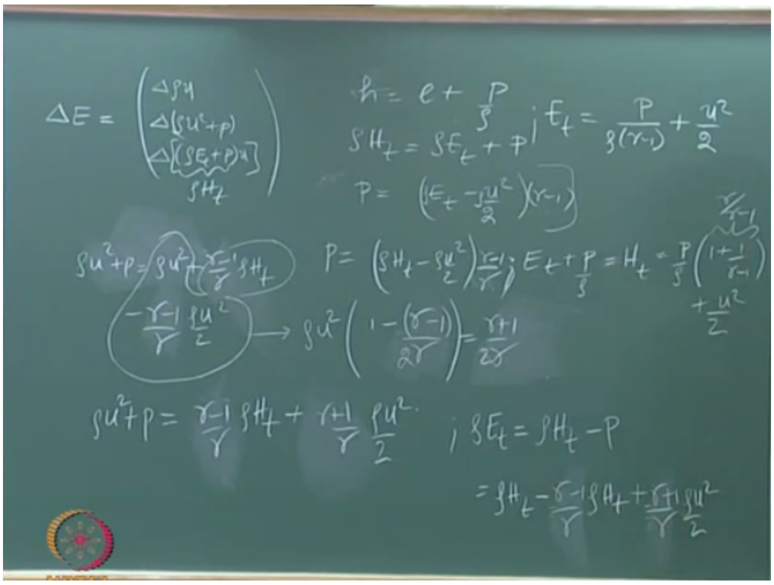


So in fact rho Et=gamma/gamma-1 rho u squared/2-rho Ht/gamma-1. Is that fine? Okay. The only one left. So of course we can stick a delta in front of this, so that we get delta rho, delta rho u, delta rho Et. The only one left is A okay. What is A? 0 tell me the entries of A, 3-gamma, gamma-3, gamma-3/2 times u, what is the here? Gamma-1 times u cubed-gamma Et times u.

Then that is 1, 3-gamma, gamma Et-3/2 gamma-1 u squared and that is 0, gamma-1, gamma u. So really the only ones we have to be concerned about are these two okay. Now I want to say something here right, think back to the previous class whereas this A is an average A where is this A? So this is the A that we are getting at the notation that we used in the previous class p-1/2.

So all of these u's that I am indicating here they are not the same as u's that you see here. All the u's that you see here are different category right. These correspond to delta rho u and so on. Yeah **"Professor - student conversation starts."** It should be multiplied by gamma-1/gamma did I do gamma-1/gamma.

(Refer Slide Time: 15:16)

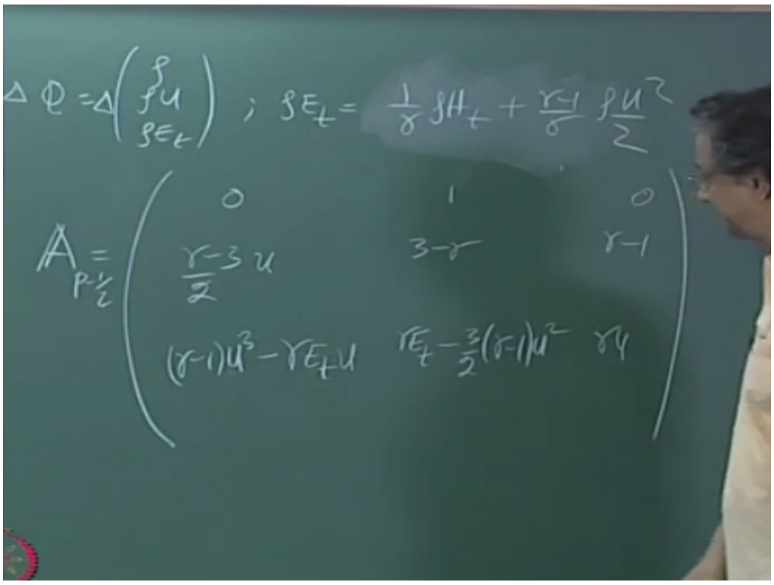


Yeah I have a gamma-1 here that is fine and what is the consequence of that. That is going to ripple through the whole thing then because if I substitute for p then this two did not really look that good but it is fine so understand and give me here. I have a cheat sheet I can actually pull out of piece of paper and write it out but I thought I will work through it. Yeah tell me so as a consequence I have rho u squared+p and p gives me that is here.

What you are saying is this should be gamma-1/gamma is that right? That should be gamma-1/gamma and this gives me a 1/gamma, a gamma-a gamma gives me a 1/gamma. Now I am happy. I do not remember seeing a gamma-2/2 but anyway it is fine 1/gamma fine. So that is 1/gamma rho u squared+p will give me a 1/gamma and this is the gamma-1/gamma. This is gamma rho u squared/this is 2 gamma, 2 gamma-gamma gives me a gamma+1 okay.

I am willing to live with that as I said by 2 gamma as I said right and this should be gamma-1/gamma. **"Professor - student conversation ends."** Is that fine? Okay so what happens to this then? What does that give me for rho Et?

(Refer Slide Time: 18:46)



Rho Et is so this is a gamma-gamma so that is gamma-gamma so 1/gamma rho Ht+gamma this is gamma-1 okay right now I will go with you guys that is why is said I have a cheat sheet anyway I know what the answer is supposed to be so little workout right, so if it does not work out there is a reason why I am doing this. I mean it is very easy to just put up the answer at the end right.

You do not want to get into the habit of just opening a book and say there is an expression in this book I just take this expression, there you can waste a lot, there could be a typographical error, there can be a sign error, there can be something as silly as this, it happens these things happen right. I could take this, I can write it up, I can type it up, I can publish a book and nowadays it is very easy, it is out there, you look at it, it is on the web, it is in the print.

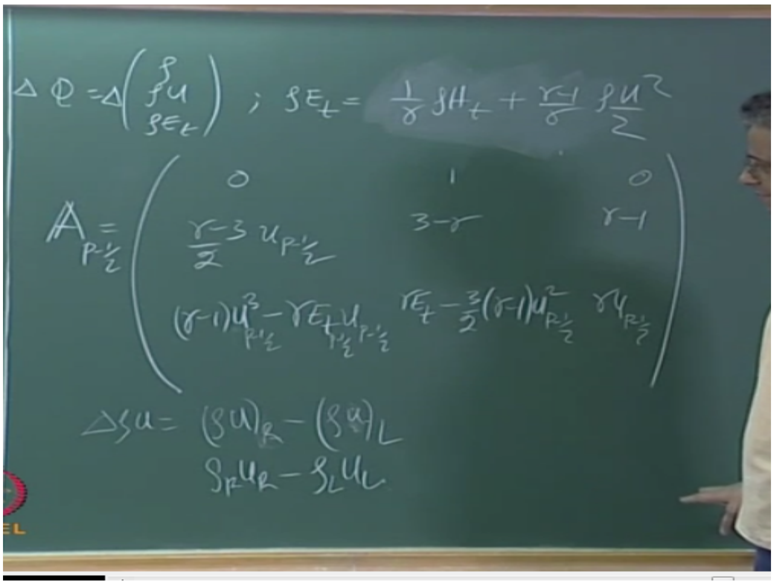
And you say yes this is correct you take it, you implement it, you are spending a month, 2 months, 3 months your code is not working. Things are not coming out right and it maybe all because of a silly typographical error right. So whatever it is just like you done it, you should get into the habit of any equation any expression that you use. I am not just taking it from the book or taking it from a reference.

But make sure that you can derive it right, make sure that you can derive it and that you are confident that the derivation is correct am I making sense okay. So as I said I can very easily write this and I have a little cheat sheet but there is a reason why I go through this process right. I want you to see, I want you to get into the habit of doing, you have to do it, you have to check right, you have to check.

And yeah it is error-prone process, you have to know how to make, you know how to do counter checks and so on fine okay. So one of the things here of course is that you know that this gamma Et that is very nice, so gamma Et will give me you know you can divide through by rho, you can make a substitution here. Yeah okay back to where I wrote the explanation I was giving you earlier.

So these u's are not the same as the u's that you see up here. These u's are different okay. This u of course comes from the state but the minute I write delta Q I am talking about some u left and u right.

(Refer Slide Time: 21:27)



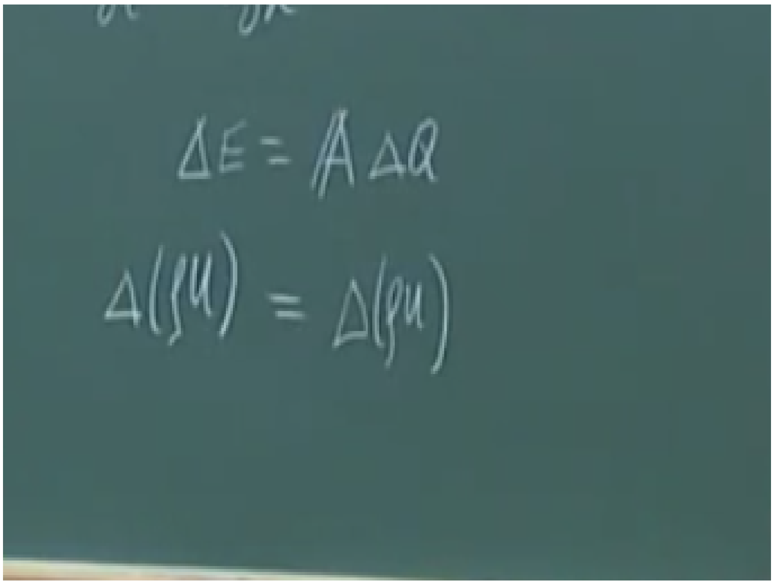
So delta rho u is actually rho u right state-rho u left state. Am I making sense? That is delta rho u okay. So these are actually rho R uR-rho L uL that is where these are, these are all lefts and rights, this is actually at the interface, these values so do not confuse them, these values are actually at the interface. In fact, our objective is to find this u right our objective is to find this u.

So if I substitute for Et and yes I know that this has only Et and u in it but if I substitute for Et here in terms of total enthalpy the only two things that I need to find here are u and total enthalpy so that is a genius here. You understand what I am saying the genius of this derivation is that you are left with only u and Ht to be found right though we started off by saying oh my god there are 9 quantities that need to be found I have only 3 equations right.

What we have ended up with basically is u and Ht at the interface. So these are all actually at p-1/2 okay. So right now for now I will write p-1/2, later on I will make my life easier. So these are all at p-1/2 because once we have called these uR's and uL's right basically so why do not I just leave that as a u, so we will leave that as it is okay. Now we need to do the grand multiplications okay.

The first equation is relatively easy; it requires no effort okay. So we are going to go across the 3 parts of the board here right just to remind you. You look here delta E is A delta Q, so I need to find A delta Q to get my delta E.

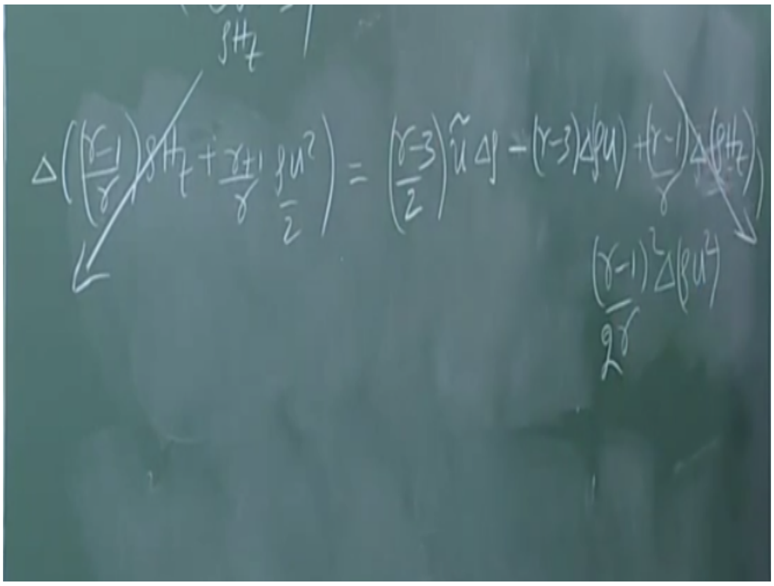
(Refer Slide Time: 23:53)



For the first equation what is delta E? The first equation is delta rho u=what is A delta Q? Look at the equation and tell me, it gives me nothing, it says delta rho u is delta rho u. First equation falls apart okay. The first equation gives me nothing, it basically says delta rho u is delta rho u but fortunately we still have 2 equations and we have 2 unknowns right. That is what I am saying, so the great thing is that we have reduced it to 2 unknown.

So delta rho this is just an identity that gave us nothing fine is that okay? Let us look at the second equation. I have my rho u squared+p here, so I am going erase the lot of this, the lot of the stuff right whatever we need to substitute for rho Ht I presume you have in your notebook so you will help me out.

(Refer Slide Time: 24:53)



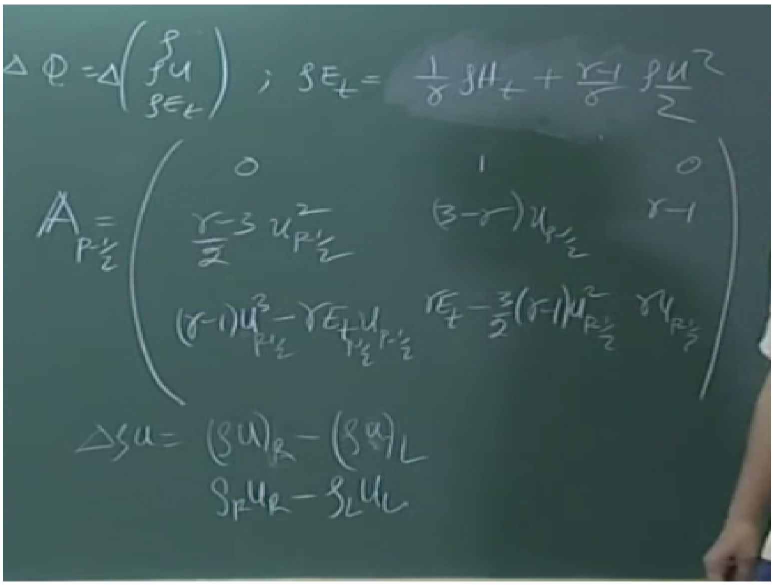
So I am going to erase the lot of the stuff, so all evidence of errors are going to disappear. What does the second equation give me? This is rho u squared+p somewhere I have written rho u squared+p. Delta of rho u squared+p is oh that is here so gamma-1/gamma rho Ht+gamma+1/gamma rho u squared/2 right that is rho u squared+p and this equals that is delta E A delta Q, first element is gamma-3 right u p-1/2.

I want to differentiate I do not want to keep saying this u p-1/2 may be I use tilde's or something of that sort to indicate that it is the value at. So this is gamma-3/2*u tilde and what does that multiply delta rho second one gamma-3*delta rho u okay and gamma-1*delta rho Et, which is this quantity+gamma-1 times that quantity rho Ht delta rho Ht gamma-1* is that fine? Where is the minus? 3-gamma, therefore it is minus okay.

So you should have a sense that we are in the right direction because gamma-1/gamma rho Ht gamma-1/gamma rho Ht. Anytime things cancel you feel good though in fluid mechanics sometimes they cancel and you made a mistake right you always have to keep your eyes open. You still have to keep your eyes open, it has happened to me right. There are times I was thrilled that something canceled and everything came together.

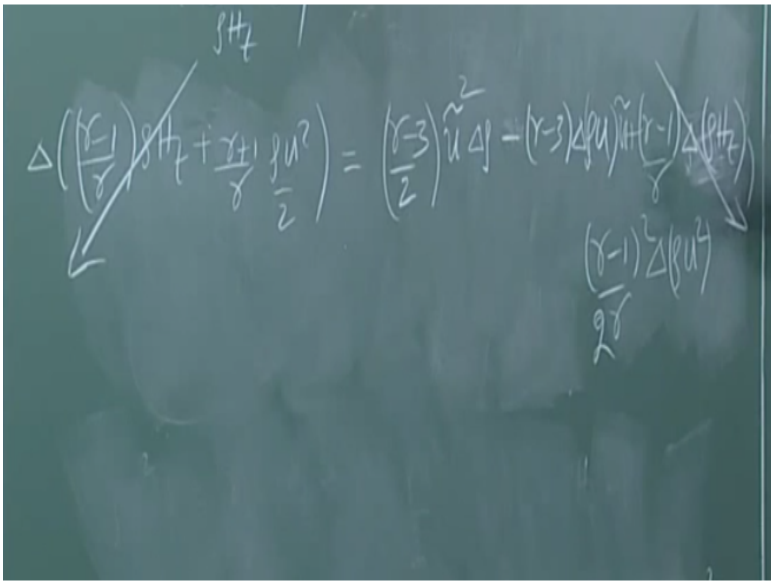
And then you have to understand that fluid mechanics was beautiful because sometimes they do not cancel, they add right such as problem okay good. So you have that you cancel that out right we cancel that out. Tell me now what is the equation that we have? What is the equation that we have? Is this right? **"Professor - student conversation starts."**

(Refer Slide Time: 29:20)



I think there is a u here p-1/2 right because it is coming out a lot easier than it is supposed to. I am not supposed to get a linear equation; I am supposed to get a quadratic. **"Professor - student conversation ends."**

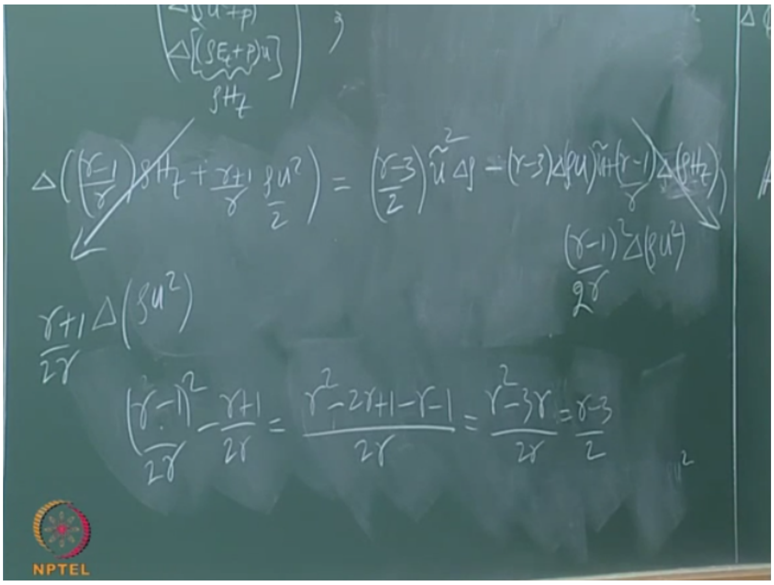
(Refer Slide Time: 29:38)



So that is u tilde squared and there is a u tilde here okay fine. There is a certain pattern to it. See when I step back and look into it I say why I have a u squared, u cube, u, u squared you understand nothing u. There is a certain pattern to it right. One corresponds to energy which is like momentum times du integral am I making sense? That is you expect those kinds of pattern right.

I am up here I am not really paying attention to it but at some point you have to end. I caught it because I know I should get a quadratic equation right but normally you can keep your eyes open for these kinds of patterns to make sure that right, those things really work okay. So what does this give me on the left hand side?

(Refer Slide Time: 30:28)

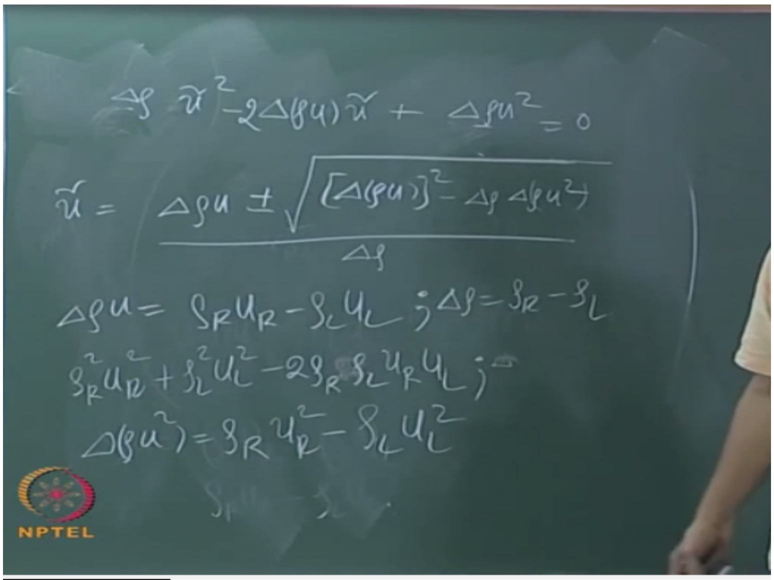


Delta of rho u squared gamma+1/2 gamma and here I have another so I guess I can bring that over to the right hand side okay. So what I want to do is what I need to get the figure out, what is gamma-1 squared/2 gamma-gamma+1/2 gamma just to straighten that out. I guess I will have to expand that out gamma squared-2 gamma+1-gamma-1/2 gamma that gives me a gamma squared-3 gamma/2 gamma, which gives me a gamma-3/2.

Yes, and again I am happy. Why am I happy? The gamma-3 will cancel right. So this is all going in the right direction. We started off with this really messy equation. So this is one of the things about derivations. It starts getting little messy sometimes we panic and we just say let it go this cannot be right but if you persist right if you stick with it then things sometimes start falling into place right.

And things like this make you feel good, everything cancel yes this must be the right direction okay. Now I will erase this right because I am going to leave the enthalpy thing to you okay and I will tell you why it is not complicated and you know it is not complicated. Lots of things are going to be simplified and the expression that you get will be very similar to what we have here.

(Refer Slide Time: 33:15)



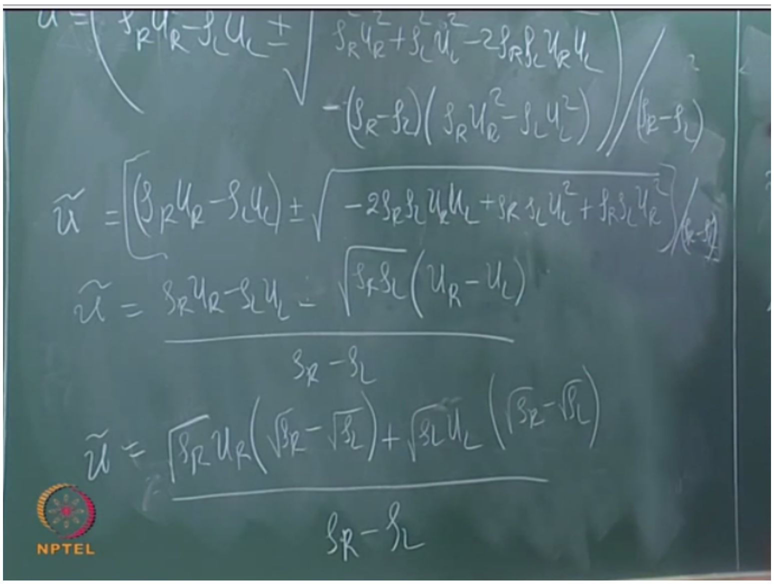
I will leave the enthalpy part to you so the gamma-3 goes away so tell me what we have. We have a delta rho gamma-3 what happens? Delta rho/2 u tilde squared-delta rho u*u tilde+delta rho u squared/2. Is that right? And this equals 0. We have a quadratic equation in u tilde. In fact, we can multiply through by 2, get rid of that 2, multiply through by 2 so you get 2 times, it looks even better okay.

So the roots for this are u tilde=delta rho u+–square root delta rho u squared-delta rho times delta rho u squared right we understand delta is applied to the whole thing being a little lazy here whole divided by delta rho fine okay. This cannot be simplified any further. Yeah the 2's go away, all the 2's will cancel out, that is why I got it in this, this is a canonical form, I got it in the standard form okay.

Is that fine? Now you cannot go any further. Now though it looks messy we have to substitute for the delta rho's and delta rho u's okay. We cannot go any further. So delta rho u is rho R uR-rho L uL. So delta rho u whole squared is rho R uR squared+rho L squared-2 rho R rho L uR uL okay. Delta rho of course is rho R-rho L that is easy and delta rho u squared is I will write it below rho R uR squared-rho L uL squared, substitute right.

We boldly go ahead because it has been working so far, it is going to work now. Substitute, now, what do we get?

(Refer Slide Time: 37:31)



U tilde candidate is rho R uR-rho L uL+-square root rho R uR squared+rho L uL squared-2 rho R rho L uR uL delta rho times this plus maybe I should anyway it is okay. I will stick this under all of this is one numerator+rho R-rho L times rho R uR squared-rho L uL squared. This is the numerator, the denominator of course is this whole thing divided by rho R-rho L will get to that. Where did I miss the minus?

This comes from the quadratic that minus is very critical because every minus is very critical but that minus is particularly very critical. It is critical because I see a rho R squared uR squared that is going to cancel that and rho L squared uL squared that is going to cancel that okay that minus is our friend okay. So how do we get?

This gives me u tilde is if I simplify+-square root I get a -2 rho R rho L uR uL+rho R rho L uL squared, --+, +rho R rho L uR squared square root okay whole again divided by rho R-rho L. Yes what can we do to this? **"Professor - student conversation starts."** Factor out the rho R rho L good. Rho R rho L*uR-uL squared okay that is what that gives you so you can get it out so u tilde equals rho R uR-rho L uL+-that divided by rho R-rho L. Is that fine?

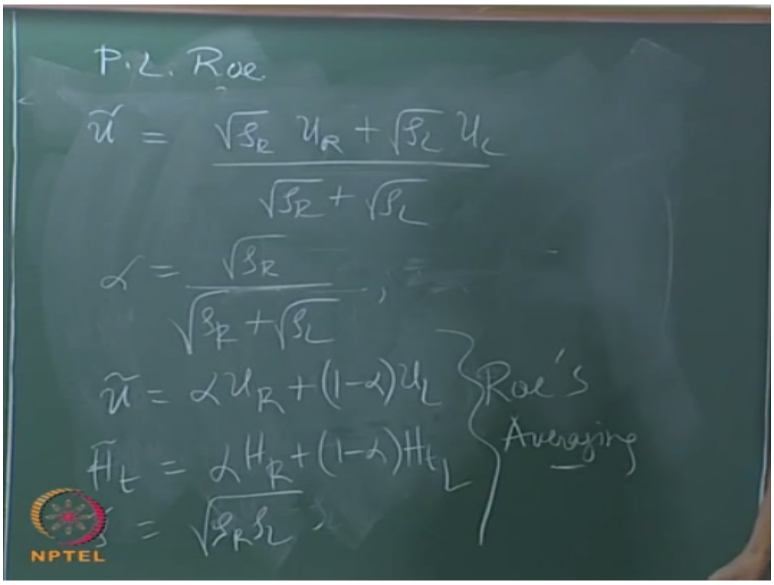
Yes, the + and – will take care of that. **"Professor - student conversation ends."** Now I am going to so there are 2 things that I have already told you I am going to leave the second equation to you right the last equation to you. The plus root will not give us something that is useful okay. If you take the plus root I want, you to try it out okay the plus root if we have the time may be we will get to it but the positive root is not going to give us something that is useful.

So we will take the negative root okay. I am going to take the negative root. You can follow through on the positive root with the argument that I gave for the negative root. You can then go back and look at the positive root and see why what is wrong with it? There is something wrong with it okay. So I take the negative root so I use obviously the first time you derive it you are going to try out each one and instinctively we try the positive root and then try the negative root.

Now what? Tell me a way to simplify this? See we have to live with the square roots. So we will do something so these are the times for us to do bold things right. I am going to write this as square root of rho R squared. So this looks like a squared-b squared thing, we can factor it and then there is hope that we can do the same thing here right. I presume that is the direction in which you are going okay.

So then if you go in that direction so you have uR so you can pull out the uR's and the uL's right you can combine the terms. So u tilde is tell me so what do I get? I can pull out the square root rho R uR that gives me a square root rho R-square root rho L and what does the other one give you? +square root of rho L uL, u is outside the square root. I have –sign here, -sign here that becomes a+*rho R-square root rho L okay divided by rho R-rho L which of course can be factored.

(Refer Slide Time: 46:08)



As a consequence of what is algebra we get this amazing expression right which tells us that u tilde is square root rho R uR+square root rho L uL/square root rho R+square root rho L. I have to be a little more patient just thinking in my mind the convex combination and doing a wrong thing okay. So if I define alpha as square root rho R/square root of rho R+square root of rho L then the other is 1-alpha is of course the other term.

So u tilde is an alpha times uR+1-alpha times uL okay right. I want you to follow through on the positive root well I do not have enough time to do it right now. I want you to follow through on the positive root but I want you to understand that look at that positive root in this context. Alpha is clearly between 0 and 1 you understand. So this is truly a linear combination.

And because alpha is between 0 and 1, u tilde will be between uR and uL okay. If you go through the positive thing you will be able to reason out that you do not have that. Am I making sense? So I have given you 2 quantities and I am saying get me a mean value between these 2 quantities and your answer is yeah it should be between those 2 quantities. Get me an average between these 2 quantities and you expect it as and this satisfies that okay.

The positive root will not okay. This negative root satisfies that it is in between it is an interpolation, we promised an interpolation and we have got an interpolation. The only difference is that this looks a bit bizarre right but remember that we interpolated in some kind of a function space that is E, there is a function of Q and we had 2 values of Q right and we are looking interpolating, we are looking for a mean in that setup not in the xy coordinate system.

So we are willing to live with this okay. So the average you will get a similar expression for Ht. We will get a similar expression for Ht, you can verify it, that you can get a similar expression for Ht. You do not take my word for it, check it right I had indicated you will get a similar expression for Ht. So you know the 2 quantities, we need a third quantity, it is usual to take rho tilde.

It is usual to take rho tilde as the square root as the geometric mean looking at all of these stuff, looking at the expressions that we have got it is usual to take root tilde's rho R the geometric name square root of rho R rho L okay fine right. So now we are able to find this is named after the scientist who derived this after P.L. Roe. So this is called Roe's average. Is that fine okay?

Are there any questions? As I said deliberately I wanted to go through and some of these derivations I want you to see it. So it is not that all of us it is not that we open a book and so and so has done this, I am just going to take it and use it so on. I am very serious, just a simple sign error you can struggle, you know you go through all of this and then find that there is a typographical error, it is not worth wasting that kind of time.

It is worth the investment of a day or two or whatever and checking out the derivation yourself okay. It is not worth wasting time on some printed or downloaded material, which may have a typographical error. It could be as simple or it could be that someone has actually made a derivation so how you are going to find it. So you have to actually sit down and derive and make sure that you are able to come up with it okay.

Please, please do not get into the habit of just opening the book and ripping the formula out of the book. Is that fine? Okay. In the next class, I will see if I can set up a demo for you in the next class and then I am going to try to do if we have time right at least one or two more things with respect to the Euler's equations okay and then we will go on looking at schemes as to how to make things run faster, converge faster fine okay.

So we are sort of coming towards the end of the semester, I want to give you an idea to where we are going and then we will do a little what I call (()) (51:26) I will do a little calculus of variations for you and show you a different way of looking at all of these problems that you are working on. Is that fine? Okay thank you.