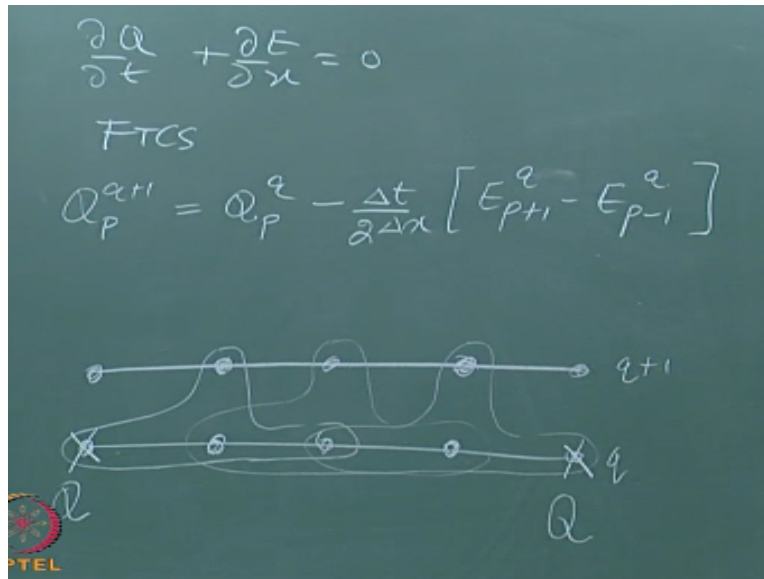


**Introduction to Computational Fluid Dynamics**  
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**Lecture - 27**  
**Applying Boundary conditions**

Okay, so we will continue with right with 1-dimensional flow.

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Handwritten notes on a chalkboard:

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = 0$$

FTCS

$$Q_P^{n+1} = Q_P^n - \frac{\Delta t}{2\Delta x} [E_{P+1}^n - E_{P-1}^n]$$

Diagram showing a grid with points labeled Q and Q+1, and a stencil for the FTCS scheme.

So, the governing equation that we had was this and we basically said that we could discretize using FTCS right discretize using FTCS as  $Q_P^{n+1} = Q_P^n - \frac{\Delta t}{2\Delta x} [E_{P+1}^n - E_{P-1}^n]$  okay. Just to recollect what we did in the last class and then we said that if you look at 2 grid lines this is a Q this is at Q+1 these were the boundary points right.

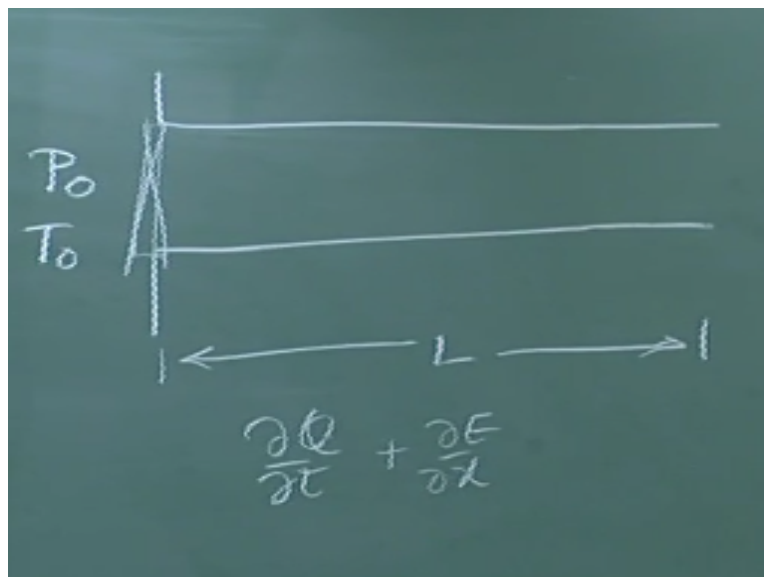
I will draw few interior points just 2 I am not going to draw a whole lot of them just for the sake of this discussion. We will draw a few but few interior points so it is clear that we can do FTCS right using that stencil no problem you have everything at Q and we can determine the value of Q+. The difficulty is when we get to determining the value right 1 short of the boundary on both ends.

On both ends determining the value 1 short of the boundary requires values at the boundary right

it requires values at the boundary fine. And this is clearly the way FTCS works since we are going to do this this and we are going to use them to determine the interior points this clearly is a problem. We have to solve because when we take the next time step when we go from  $q+1$  to  $q+3$  we are going to have the same difficulty.

We are not determining these points okay and we are not finding out what these values are and what do we need at these places we need the  $Q$  I need  $Q$  at this point I need  $Q$  at that point I need all okay. So, because we were in the situation we decided that we look at the physical problem there actually because so far we have just been dealing with the chalk dust working only with the theory.

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So, we said the physical problem just to recollect because it is a 1 dimensional flow If I remember right I took a pipe of length  $L$  right and there was a value on one end. Where you place the value will only determine the initial conditions right I should not say only will determine it will determine the initial conditions. If you have the value here, then whatever this is open it on the say to the atmosphere it is typically would have  $P$  ambient prescribed on that side.

So, the initial condition would be whatever the ambient conditions right since the value is on this side values on the another side it would be whatever is in the chamber right and here you have total pressure and total temperature. These are typical conditions that you would use right if you

were performing an experiment am I making sense in a gas dynamics lab okay and it is not a very exciting even that experiment is not a very exciting experiment.

But still further for the for our purposes it is simple enough that we can play around with it. So, the physics of the problem requires as I pointed out in the last class right the physics of the problem requires 3 conditions prescribing  $T_{\text{ambient}}$  here does not really do anything for you. Unless of course you needed the  $T_{\text{ambient}}$  here you require in order to determine the initial condition it is a boundary condition it does not do anything for you okay you understand.

It is a pressure difference that is going to drive see this is just like normally when you say that you have let me think of a situation so if you think of all the sources of power that we have the most. So, sources of power that we have every compress a fuel air mixture and we add energy right we add we ignited combustion and then there is expansion the process always involves compression expansion you understand what I am saying.

If I were just to create a fireball in the open and raise the temperature of all the air we are ignoring gravity here all that will happen is the temperature of all the air we go up right in order in this room if I were to raise the temperature the temperature in the room will go up there is no change in pressure nothing of that sort you actually need. So, I need a low pressure at one end and a higher pressure here.

And the pressure drives the air and we extract the energy from the fluid that is why always thermodynamics deal with enthalpy and not directly with your directly the pressure or temperature right because you need the pressure gradient in order to drive to start the motion so that you can extract the energy having the temperature alone right through mechanical processes is not enough and we are making sense okay.

So, in that sense it is the pressure difference that is going to drive the flow it is the pressure difference that is driving the flow is that fine right. So, in this side we can measure total temperature and this total temperature will basically do what what does the total temperature do? Does it determine anything do we need it you go back to your gas dynamics and see the way I

need to prescribe the total temperature there.

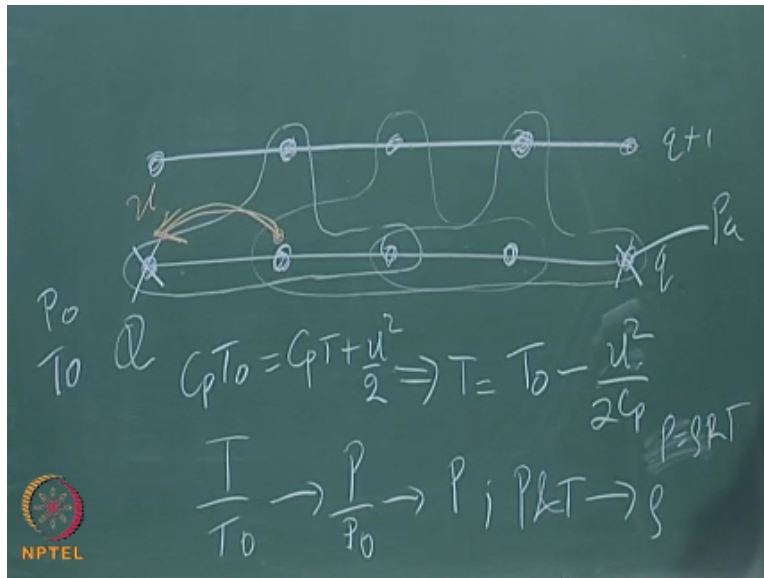
So,  $P_{\text{ambient}}$   $P_0$   $T_0$  and this problem can actually work using gas dynamics fine okay from the mathematical point of view we have already said that because we have a time derivative we have a  $\frac{dQ}{dT}$  term because of the  $\frac{dQ}{dt}$  term at  $t$  for some given  $T$  we will take  $T=0$  at  $T=0$  I need a condition so I need to have an initial condition for all spatial locations and I have  $\frac{dE}{dx}$  term right for all I have a  $Q$  at  $T=0$  and I have  $\frac{de}{dx}$  as first derivative.

So, integration in my mind will give you 3 constants of integration which have to be determined and  $P_0$   $T_0$  and  $P_{\text{ambient}}$  are given. So, both physically and mathematically we will determine okay it is just that our FTCs requires that I need a  $Q$  at this point and a  $Q$  at that point right and consequently I need to determine something here and something there is that okay right the experiment thought experiment we are running yesterday was if  $P_{\text{ambient}}$  was  $P_0$ .

As I just indicated there would be no flow and if I started to lower there are some mechanism by which I could lower  $P_{\text{ambient}}$  right if I have another value is open so if I lower the  $P_0$   $P_{\text{Ambient}}$  the effect will be there there will be a  $U$  set up here because of the lower lower  $T_{\text{ambient}}$  right so presumably this boundary condition communicates itself here to an increase in  $U$  right I am just giving you a hand waving argument.

So, you want the 3 quantities you want 3 quantities here you are having  $P_0$   $T_0$  I am saying here is a way by which we can do it so what we do is from the first interior grid point maybe I go back here from the first interior grid point.

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So, whatever I have here a copy the  $U$  to this point extrapolate the  $U$  to that point that okay so this is a very naive way of doing this but this is the first cut part is that there is a there is a we will see how we can improve we can improve. Right I copy the  $U$  here you could also copy  $\rho$   $U$  if you wanted but I copied the  $U$  there. So, what does that give me at this grid point I have prescribed  $P_0$  and  $T_0$ .

And now what we need to do is little gas dynamics right given given  $U$  that I will extrapolated  $U$  I use energy equation what does it tell me 1 dimensional energy equation not the differential equation that we are using  $c_p T + U^2/2$  and if you give me the  $U$  and you give me the  $T_0$ . This tells me that  $T = T_0 - U^2/2c_p$  is that fine and once I have  $T/T_0$  I can find  $P/P_0$  so from  $T/T_0$  I can find  $P/P_0$ .

And consequently I can find  $P$  and given  $P$  and  $T$  I can use equation of state I can use the equation of state or they can I guess I might as well write  $P = \rho RT$  and find  $\rho$  you are set you have everything you have a  $\rho$  you have  $\rho U^2$  you can write you can calculate everything. You have  $\tilde{Q}$  and therefore you can find  $Q$  right is that okay fine what about the right hand side I have only I have only prescribed at this point.

I have only  $P$  ambient prescribed at this point right I only have  $P$  ambient being prescribed at this point at time level  $Q$  remember this is not initial condition initial condition we knew everything

this is at time level  $Q$  just in case you are wondering what is the deal that is why I said it is at the time level  $Q$  right. So, we have only  $P$  ambient at that point clearly not enough I need 3 quantities right.

So, I need to extrapolate 2 fine what can I extrapolate the different variations in all that you could do it there are so many possibilities even upstream there are so many possibilities you could have extrapolated  $P$  instead of  $U$  am I making sense. So, these are the things that one can try out there is lots of scope for you to play around here okay. So, one possibility is that for example you say that there are no sources.

And keep sources energy sources in between it maybe extrapolate  $T_0$  possibly I could extrapolate  $T_0$  in  $U$ . I could extrapolate  $T_0$  and  $U$  possibly. I am just making a suggestion right the point that I am trying to make is that you need to extrapolate 2 quantities I have to see these are these are boundary conditions that I have to generate they do not exist. The physics does not provide them mathematics is not right now giving me a clue.

We will look at it to see whether we can use the differential equation somehow to generate these is that fine okay. So, if extrapolate the  $T_0$  if extrapolate the  $T_0$  in  $u$  what do we get again if I extrapolate the  $T_0$  in  $u$  I can find the static temperature and given the pressure and static temperature I can find the density it is the same game and you can go through right is that fine everyone okay.

It seems like a rather arbitrary thing to do it seems like a rather arbitrary thing to do. So, this is something that you can try out as I said it did leave scope so you can try let us say to extrapolate density and  $\rho$  and  $\rho u$   $\rho$  and  $u$  right the various quantities that is the thing that you want to make sure you do not do these are these are things that you would be careful you already have  $P$  ambient here you do not want to really extrapolate density and temperature for instance.

Because potentially you could violate equation of state at that point am I making sense if the pressure here is different from the pressure here by extrapolating the density in temperature you could potentially violate equation is so there you have to be very careful what quantities you

extrapolate but you could try you could try out various quantities some of them may be totally unstable not work at all right try out quantities.

But these are these are some thing that I thought okay you can just start fine is that okay can I justify that is it possible for me to justify that maybe I will stick with this is it possible for me to justify that justify extrapolating these quantities okay. So, the question is when can I do this what are the conditions is there what is there any an inherent assumption that I have made and saying there are these quantities can I always extrapolate with it always work okay.

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$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} \rightarrow \frac{\partial Q}{\partial t} + A \frac{\partial Q}{\partial x} = 0$$

$$\frac{\partial \hat{q}_i}{\partial t} + \lambda_i \frac{\partial \hat{q}_i}{\partial x} = 0 ; \lambda_1 = u$$

$$\lambda_2 = u+c$$

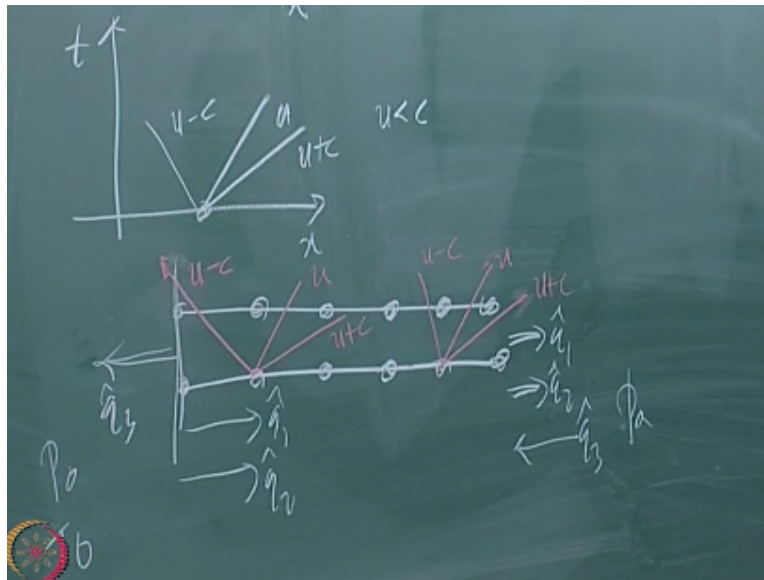
$$\lambda_3 = u-c$$

$$\frac{\partial \hat{q}_1}{\partial t} + u \frac{\partial \hat{q}_1}{\partial x} = 0$$

So, let us go back let us go back remember that this could be written as  $\frac{dQ}{dt} + A \frac{dQ}{dx} = 0$  and this could be further decoupled into 3 equations I am going to write now component form  $\frac{d\hat{q}_i}{dt} + \lambda_i \frac{d\hat{q}_i}{dx} = 0$  where  $\hat{q}_1, \hat{q}_2, \hat{q}_3$  are characteristic variables we are going to solve the equation in this form because multiple dimensions it is not going to help us right so that it does not make sense going there.

But it is useful to look at and what are  $\lambda_1, \lambda_2, \lambda_3$   $\lambda_1$  is  $u$   $\lambda_2$  is  $u+c$  and  $\lambda_3$  was  $u-c$  this is what we found is that fine okay now we will actually investigate these equations so if you look at 1 equation if you look at 1 equation you have  $\frac{d\hat{q}_1}{dt} + u \frac{d\hat{q}_1}{dx} = 0$   $\hat{q}_1$  is being propagated along characteristic.

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I will just draw a small figure here so at this point that is being propagated along the characteristic whose slope is  $1/u$ . The slope is  $1/u$  what about  $Q_2$  hat,  $Q_2$  hat is being propagated along that and this is this this corresponds to I would not write  $1/u$  so I will just write out what characteristic this corresponds so this corresponds to  $u$  this corresponds to  $u+c$  and  $Q_3$  hat  $u+c$  or  $u-c$   $u-c$  this is  $u-c$  this is  $u+c$  remember this is  $x$  so it is travelling faster this is  $u+c$  okay.

So, if this is  $u-c$   $u$   $u+c$  and all these slopes are positive what does that tell you about the slope  $u+c$   $u$  is  $>c$  so this actually corresponds to supersonic flow so this actually corresponds to supersonic flow that is  $u>c$  right if you are talking about subsonic flow if you are talking about subsonic flow in the  $xt$  plane right you would have  $u+c$  there and you would have  $u$  here and you will have  $u-c$  headed out in the other direction.

Am I making sense okay so  $u-c$  will be negative this is for  $u<c$  this is for  $u<c$  everyone okay at a subsonic inlet at a boundary condition back to those boundary conditions. So. This happens to be subsonic if that boundary happens to be a subsonic okay that boundary happens to be subsonic or if you look at what happens to the first interior grid points. If the flow is subsonic these are the grid points if the flow subsonic if the flow if the flow subsonic.

What happens at this point? So you know one characteristic which is  $u-c$  maybe use a different color here  $u-c$  you have one characteristic which is  $u$  and one characteristic which is  $u+c$  is that



fine. Actually I sort of deliberately grew this characteristic this way this if you go up think about your stability condition or the grid size or stool the  $\Delta t$  is too large all of these characteristics should be above that right just go back think about your stability condition.

But were going to look at the stability issues in this seriously anyway but I thought I would just sort of stick there that I could resist the temptation to have that right you could have  $u+c$  that is going that way is that fine okay what this basically says is there is one characteristic that does propagating  $Q1$  hat or  $Q3$  hat in this case it is propagating  $Q3$  hat in that direction outward directions right.

And if I look at it if I look at the boundary itself it is at the boundary itself at the boundary itself since the flow subsonic I have one characteristic that is propagating  $Q3$  hat out and I have 2 characteristics that are propagating  $Q1$  hat and  $Q2$  hat in is that fine and if it is a subsonic exit I would have the same situation. So, I draw at the last but one point so I would have  $u+c$  here I would have  $u$  there I do not know why I am putting arrow heads.

And I have  $u-c$  here I guess I am looking at it I am putting arrow heads because instinctively I am looking at it as being propagated along that direction right okay so in which case then you have  $Q1$  hat going out  $Q2$  hat going out  $Q3$  hat coming in okay. Now when I look at the problem when I look at the physical problem that we prescribe we prescribe  $T_0$   $T_0$  2 incoming characteristics 2 quantities prescribed right that is nice.

Then I prescribed  $T$  ambient one incoming characteristic coming into their domain 1 quantity prescribed right. So, the mathematics and the physical experiment that they they are the same I am happy with that the way it worked out and we are extra plating 2 quantities We should be extrapolating  $Q1$  hat  $Q2$  hat obviously that would be the way to do it right and we will see how we go about doing that.

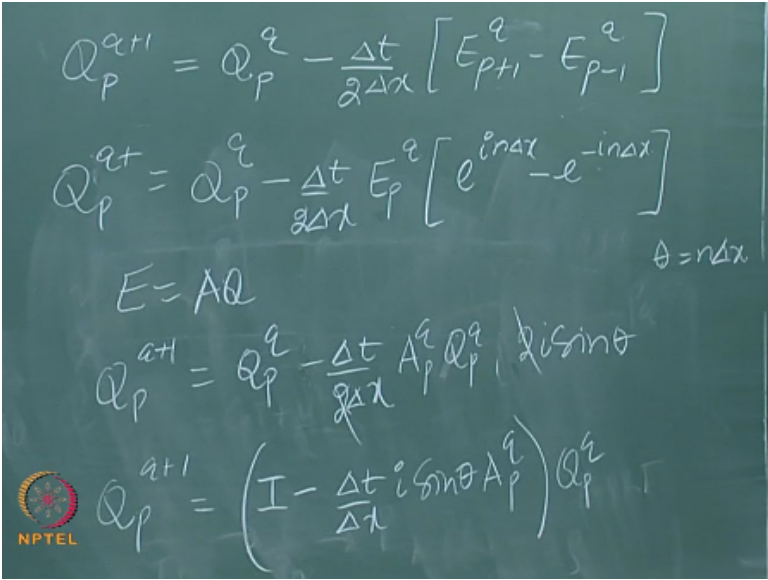
So, there are 2 quantities that you are extrapolating that correspond to in your mind the characteristics  $u$  and  $u+c$  right and 1 quantity say I am giving you a rational it is a hand-waving argument I should what is being propagated on  $Q1$  hat  $Q2$  hat  $Q3$  hat that is what are being

propagated but I am I sort of say to extrapolate it is easier to extrapolate the  $u$  instead of going to all these characteristically variable fine if it works well I am willing to live with that right.

But as I said we would also look at the legal way to legally the proper way to do it and find out any questions okay. So, this is this is one one mechanism by which we can apply boundary conditions as I said we come back to we will revisit boundary conditions again it is not it does not end with this they but this is something that you can definitely I implement and try out but before you do that the question that we have this is the is the going to be is it going to be stable.

Is it going to work right or do we need to do something in FTCS. So, this is this going to work or do we need to do something so we do not come back to our original equation.

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Handwritten mathematical derivations on a chalkboard:

$$Q_p^{n+1} = Q_p^n - \frac{\Delta t}{2\Delta x} [E_{p+1}^n - E_{p-1}^n]$$

$$Q_p^{n+1} = Q_p^n - \frac{\Delta t}{2\Delta x} E_p^n [e^{i\theta} - e^{-i\theta}]$$

$\theta = \kappa \Delta x$

$$E = A Q$$

$$Q_p^{n+1} = Q_p^n - \frac{\Delta t}{2\Delta x} A_p^n Q_p^n 2i \sin \theta$$

$$Q_p^{n+1} = \left( I - \frac{\Delta t}{\Delta x} i \sin \theta A_p^n \right) Q_p^n$$

NPTEL logo is visible in the bottom left corner of the chalkboard image.

How can we what is the or do we do the analysis how can we do the stability analysis now we have a system of equations earlier we had only one static situation. Now a system of equations what do we do nice additions. Well there are different ways to do this maybe we cheat a little bit different ways to do this on you could write it in terms of delta  $Q$ s and so on. But let me just say just a little cheat here.

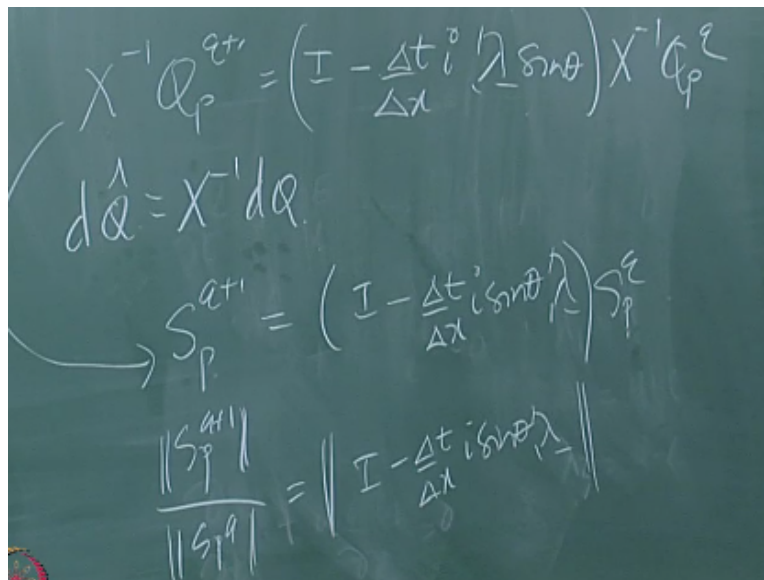
So, first we use the shift operator and get this out  $Q_p$   $Q_{p+1}$   $Q_{p-1}$   $-\frac{\Delta t}{\Delta x} E_p q^n e^{i\theta}$  and  $-\frac{\Delta t}{\Delta x} e^{-i\theta}$  obviously there has to be a  $2\pi$  and  $1$  and so on I am acting as

though I am going to wave my hands a little and say they are okay I lose  $2\pi$  okay. And I do not know whether you check this out right the reason why I said I will do a little cheat is because in this particular case it turns out  $E$  is  $E_u$  can be actually written as a  $q$  right.

You now you tried it out but  $E$  can actually be written as  $aq$  it so happens  $E$  can be written as  $aq$  okay. So, consequently this is going to turn out to be  $Q_{p,q+1} = Q_{pq}$  okay so what do you do as I know I am going to replace an  $E_q$  here I am going to have an  $AQ$  here okay. We do not have an  $Aq$  here oh okay  $-\Delta t / 2 \Delta x$  already see that  $A_{pq} Q_{pq}$  what is  $E$  power  $I$  and  $\Delta x$  — power  $-\ln \Delta x$  why  $2i \sin I$  will call it  $\theta$   $\theta$  and  $\Delta x$  okay.

As earlier the  $2\Delta x$  go away. So, I can write this as  $Q_{p,q+1} = I - \Delta t / \Delta x i \sin \theta A_{pq} Q_{pq}$  which of course is a very familiar looking the only difference is that they are all matrices it looks familiar but they are all matrices. What now there is one way to decouple it we could pre multiply the equation by  $X$  inverse right okay that  $X$  is eigenvectors I pre multiply the equation of  $X$  inverse.

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The image shows handwritten mathematical derivations on a chalkboard. The first equation is  $X^{-1} Q_p^{q+1} = \left( I - \frac{\Delta t}{\Delta x} i \sin \theta A \right) X^{-1} Q_p^q$ . Below it, the transformation  $d\hat{Q} = X^{-1} dQ$  is written. Then, the equation is rearranged to  $S_p^{q+1} = \left( I - \frac{\Delta t}{\Delta x} i \sin \theta A \right) S_p^q$ . Finally, the ratio of the norms is given as  $\frac{\|S_p^{q+1}\|}{\|S_p^q\|} = \left\| I - \frac{\Delta t}{\Delta x} i \sin \theta A \right\|$ .

And of course I can choose  $X$  inverse so that it is normalized in some sense it does no stretching only rotations it can take unit vectors right  $X$  inverse  $Q_{p,q+1} = X$  inverse  $-\Delta t / \Delta x I$  have that right  $X$  inverse  $A X$  inverse  $Q$  this is  $I - \Delta t / \Delta x i \sin \theta A * X$  inverse  $Q_{pq} \sin \theta$  oh  $\sin \theta$  very important okay how can I forget my  $\sin \theta$  thank you just remember  $X$  inverse  $q$  is

not they are not the characters variables right.

$X$  inverse  $q$  is not they are not the characteristic variable characteristic variables are related through the idea of a derivative  $Dq$  character is  $x$  inverse  $dq$  characteristic variables are related this fashion right this is some strange map. We are just doing some is doing some kind or transformation am I making sense so in a sense this gives me some and I just give it a name  $s$  so that you do not confuse it with  $Q$  caret right I will just give it some some name  $upq+1$ .

So, want you to be careful right okay now I just take the it takes the norm I take the norm I want the norm of this divided by the norm of that norm of  $PQ+1$  /norm of  $SPq$ =norm of this is a we have seen as an iteration matrix before and we did Laplace equation right system of equations think back to the example that I did with Laplace equation I had a scalar equation performed a rotation right I had 2 scalar equations.

I performed a rotation and showed that they became coupled right 2 scalar equations I made them into a couple system by performing a rotation. I am doing the opposite here right I have a couple system of equations and I am trying to undo the coupling. So, we could might as well take this spectral norm. What does the largest eigenvalues is what determined right the largest eigenvalue is what determined whether that sequence of iterations converged or not.

I could have used just used their argument directly but I just wanted to tie it with this we have actually done this before right we have actually done this before. So, what is the largest Eigen value  $u+c$  it looks like the largest eigenvalue but you do not know the sin of  $u$  you understand what I am saying so the largest Eigen value or  $u+c$  there you will see through it it does not matter  $c$ .

And they are same okay thank you fine  $u+a$  it looks like or  $u+c$  or whatever mod  $u$  or  $c$  or mod  $u+c$  using it in general mod  $u+a$  would be the largest taken but okay in this case it is clearly going to be unconditionally unstable really going to be unconditionally unstable just like it was for FTCS apply to any one of the equations scalar equations.

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$$\begin{array}{c|c}
 u+a & c=a \\
 |u|+a & (|u|+c) \\
 \mu_2 \frac{\partial^2 Q}{\partial x^2} - \mu_4 \frac{\partial^4 Q}{\partial x^4} & \sigma = \frac{(|u|+a)\Delta t}{\Delta x} \\
 \mu_2 \Delta x^2 \frac{\partial^2 Q}{\partial x^2} - \mu_4 \Delta x^4 \frac{\partial^4 Q}{\partial x^4} & \\
 u, u+a, u-a & \sigma < 1 \\
 & \sigma_{u+a}, \sigma_{u-a}, \sigma_u < 1
 \end{array}$$

So, I could add I can add what do I want to add I can add artificial dissipation that will make it up winding right. What was the term that I wanted to add what is the term that I wanted to add I will add a  $\mu_2 \frac{\partial^2 Q}{\partial x^2}$  and subtract  $\mu_4 \frac{\partial^4 Q}{\partial x^4}$  to the 4th is that fine okay fine I usually prefer to write this in terms of because if you think about the term that we added to eliminate.

I prefer to add a  $\Delta x^2$  but it does not matter and you will see as you go along here rather make it appear to  $\mu_2 \Delta x^2 \frac{\partial^2 Q}{\partial x^2}$ . Or you can try this for  $\Delta x$  to the 4th right see if you think you can find any difference and of course the minute I give you 2 options like this one of the one thing one possibility that catches the eye. I can see a few smiles are there obviously you can try  $\mu_2 \Delta x^2$  see what happens right.

See what happens what are the right know I am not going to give you any helpful hints on what possible values of  $\mu_2$  may work right. You try it out you see what it is you already have remembered what we have done for the 1 dimensional flow right. You already have enough you already have enough of a background to figure out what you need to add fine the only difference there is there you had only 1 propagation speed here you have 3 propagation speeds right.

That is the problem  $u, u+a$  and  $u-a$  right and after adding these this dissipation term. If you do get a stability condition that corresponds to the  $CFL < 1$  this CFL condition that the  $\sigma < 1$  which

sigma do you use right because there are 3 sigmas actually so for the wave equation you only had sigma right 1 sigma but here, you have sigma u+c sigma u-c or u-a or sigma u there are 3 possible sigma because there are 3 propagation right and all of them have to be <1.

Even if you do this even if you do this right and If you go back up winding. You basically add the right amount that is you add the right amount it goes back up winding which one do you add which 1 do you see which is the most constraining which of these is the most constraining that could would seem to be u+a right it seems to be u+a. So, along these lines I will write again that you want sigma one that constraints to be mod u+a delta t /delta x is that fine.

But remember that you can actually define 3 of them. So, there are 3 propagation speeds and actually define 3 of them are there any questions it is fine okay. So, you can ask the question why are we bothered with FTCS when we know that it is unconditionally unstable right why are you wasting our time when you know it unconditionally unstable fine I know that we can add this right.

We can do something with it let us go to let us go to a more sensible scheme that we had. But the only problem there was what was the problem or the problem BTCS.

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The image shows handwritten mathematical derivations on a chalkboard. The first equation is:

$$\left. \frac{\Delta Q}{\Delta t} \right|_p^{q+1} + \left. \frac{\partial E}{\partial x} \right|_p^{q+1} = 0$$

An arrow points from this equation to the second equation, which is a Taylor expansion of  $E^{q+1}$ :

$$E^{q+1} = E^q + \left. \frac{\partial E}{\partial t} \right|_p^q \Delta t + \frac{\Delta t^2}{2} \left. \frac{\partial^2 E}{\partial t^2} \right|_p^q$$

Below this, the relationship between  $E$  and  $Q$  is given:

$$E(Q) \rightarrow \frac{\partial E}{\partial t} = \frac{\partial E}{\partial Q} \frac{\partial Q}{\partial t}$$

Finally, the BTCS scheme is derived by substituting the Taylor expansion into the first equation and neglecting higher-order terms:

$$\Delta Q_p^q + \Delta t \frac{\partial}{\partial x} \left( A \Delta Q \right)_p^q = -\Delta t \left. \frac{\partial E}{\partial x} \right|_p^q$$

BTCS the advantage was BTCS you may also see as backward Euler centered space whatever

BTCS unconditionally stable it gave you a system of equation right and here we already have a system. So, now we have to be a bit careful and see what does that mean so what is the we did a semi discretization. If you remember when we did this  $\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = 0$  I am going to write this in the delta form  $E$  at  $\Delta t$  at time level  $PQ$  and this is a time level  $PQ+1$ .

I want to write  $\frac{\partial Q}{\partial t}$  at time level  $3q+$  you want and want to read book your duty time level  $q+1=0$  just like we did last time  $E_{q+1} = E_q + \frac{\partial E}{\partial t} \Delta t$  at  $q$  times  $\Delta t$  higher order terms okay and we need to truncate the series here and am planning to only backward right backward Euler or backward time step and again I will use chain rule  $E$  is a function of  $Q$  so  $\frac{\partial E}{\partial t}$  I will write as  $\frac{\partial E}{\partial Q} \frac{\partial Q}{\partial t}$  okay.

So, that the see equation becomes  $\Delta Q + \Delta t \frac{\partial E}{\partial Q} \frac{\partial Q}{\partial t} = 0$  of  $A \Delta Q = -\Delta t R$   $\frac{\partial E}{\partial x}$  this is at  $Q$  all at  $P$  is that fine everyone.

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$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = 0 \quad \text{Seeking steady state}$$

$$\rightarrow \frac{\partial E}{\partial x} = 0$$

$$\left( I + \Delta t \frac{\partial A}{\partial x} \right) \Delta Q = -\Delta t R$$

$$\Delta Q_p + \frac{\Delta t}{2\Delta x} \left( A_{p+1} \Delta Q_{p+1} - A_{p-1} \Delta Q_{p-1} \right) = -\Delta t R_p$$

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So,  $\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = 0$  if we are looking only for steady state seeking steady state we will tell you what you do when if you are going to look for a transient right now we will look for only the steady state seeking steady state so this tells me the  $\frac{\partial E}{\partial x} = 0$  is the equation that I am actually going to solve okay so that the equation can be written as  $I + \Delta t \frac{\partial A}{\partial x} \Delta Q = -\Delta t R$  fine.

Just like we got for the wave equation the generalized wave equation with a little difference here what is the difference each of the delta Qs each of the delta Qs is a matrix vector. So, each of these entries is a vector is a matrix right. So, what does this equation look like if you write it for an arbitrary PQ? If you write it for an arbitrary point P and what does this and if you central the differences to discretize that use central differences to discretize that.

This is going to give me delta q pq I would not write the q it is the time q right delta p at q q at p+delta T times Ap+1 delta Q p+1-Ap+1 Ap-1 delta Q p-1=RP okay. So, you will get a tridiagonal system you get a tridiagonal system. The point is that each entry in the tri diagonal system is a block a matrix block. So, the general equation will turn out so you will get a tridiagonal system what I mean by that is you will get something.

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$$\begin{bmatrix} & & \\ -\frac{A_{p-1} \Delta t}{2\Delta x} & I & \frac{A_{p+1} \Delta t}{2\Delta x} \\ & & \end{bmatrix} \begin{bmatrix} \Delta Q_p \end{bmatrix} = \begin{bmatrix} R_p \end{bmatrix}$$

$$-\frac{A_{p-1} \Delta t}{2\Delta x} \quad I \quad \frac{A_{p+1} \Delta t}{2\Delta x}$$

So, this is delta Qp right this is the equation p going through and what you will get here is what is on the diagonal there is an identity matrix on the diagonal right there is a Ap 1 delta T/2 delta X on the sub diagonal and there is a Ap+1 n delta T/2 delta x on the super diagonal you understand what I am saying that is the diagonal that is the sub diagonal that is the super diagonal everyone is that fine.

And each of these entries each of these entries is a matrix each of these entries if a matrix Ap-1 delta T/2 delta x just make it clear I Ap+1 delta T/2 delta x those are the entries in a matrix. So, if



you decide for example to do Gaussian elimination if you decide for example that you want to do Gaussian elimination right. So, where in Gaussian elimination you would say divide through by the pivot element.

Here you have to pre multiply the inverse am I making sense pre multiply by the inverse fine if you are planning to do this if you are planning to do that it is better that you factor this  $A^{-1}$  using your LU decomposition right you will always have this question. When you do numerical linear algebra you know that LU decomposition and goes in elimination are essentially the same right Lu decomposition and Gaussian elimination operation for operation.

You can actually show that they are essentially the same am I making sense okay you do the elimination part in Gaussian elimination you end up with a upper triangular matrix you do LU decomposition you have a lower triangular upper triangular you do a forward substitution of the L and you end up with an upper triangular matrix. You would expect that you can map the 2 operations together you actually can.

You can show that they are equal you can show that there are operation for operation you can actually identify that they are equal fine. So, then the question always crops up crops up why would you choose 1 or the other well if you do LU decomposition if you factor it as  $L$  nu then you only have back substitutions and forward substitutions if you pre factor as L and U if you are going to multiply through by right.

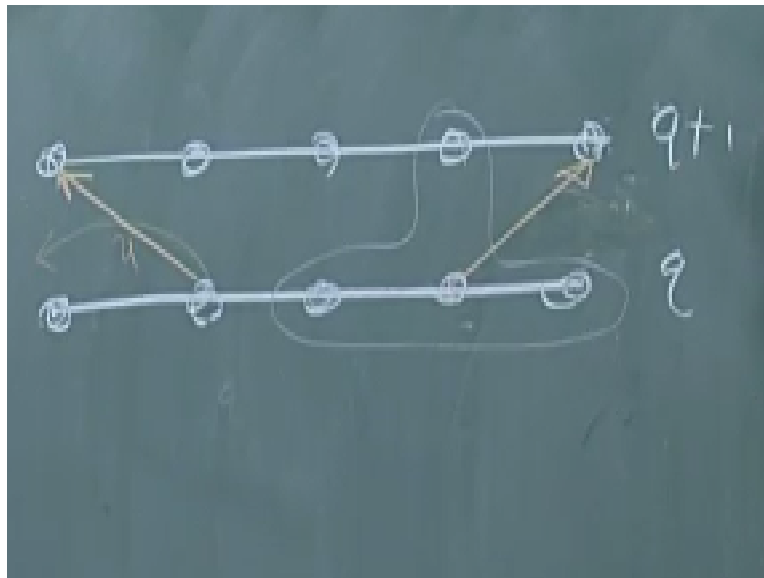
The inverse you could actually do back substitutions forward substations and back substitution that is the idea that if you are using if you are using Gaussian elimination even if you want to do Gauss Seidel you are doing iterative method you may still have to do something of that sort but of course Gauss Seidel the biggest advantage the diagonal matrix here is I right so Gauss Siedel is not that bad Gauss Seidel is not that bad am I making sense.

If you are using an iterative matrix it is not that bad there may be consequences but it is not that bad okay is that fine everyone okay. So, then we have only 1 question left how do we apply boundary conditions here we have a system of equations how do we apply boundary conditions

here in this problem how do we apply boundary conditions. Let me see where is the best place for us to discuss that maybe I will do it here.

How do we apply boundary conditions here so in the application of boundary conditions we need to maybe look at it a little more carefully?

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Again I will draw 2 timelines  $q$  and  $q+1$  like I did the first time around I am not going to choose I am going to choose few grid points I would not choose a lot of grid points. So, that it does not become too messy right and what I had indicated at that time is or you need a value here that you can extrapolate some quantity from the interior say for instance  $T_0$  or  $u$  or whatever from here you can extrapolate  $u$  fine.

It need not be from the current lead time level to the current time level it right now what we want as we want the value here. So, actually you could instead of instead of extrapolating from the current time level to the current time level you could actually propagate now that looks more like it looks more like our characteristics. You can actually propagate from the previous time level to the current time level. You can extrapolate both in space and in time.

So, many different ways by which you can do this you can extrapolate both in space and in time interior value goes from the last but 1 grid point at the previous time level to the time level to

which you are stepping fine okay that is possible I mean if you are going to do it handwavingly that is possible. So, in a similar fashion even here you could do the same thing by the way this does this I am suggesting this.

The suggestion comes to mind only because we are looking at an Implicit scheme but even in the case of FTCS we could have done that even in the case of FTCS even in the case of FTCS you are going to use this stencil in the case of FTCS you are going to use the stencil. You assume at initial condition this is known you will find all the interior points then you can ask the question what do I do at the boundary.

And you can actually extrapolate from the current time level to the next time level even at FTCS am I making sense right even an FTCS you could do that and then when you go  $Q$  to  $Q+1$  when you go to the next step  $Q$  becomes  $Q+1$  you have everything at the current time. So, the minute you start this so it is this we needed the conditions the conditions were required because of the scheme that we came up with we have no physical basis by which we can justify this.

Right now the only mathematical basis that we have done is though the characteristics are propagating in that direction so that rationalizes why I am extrapolating that is all we have done the question that we can ask is cant we just directly use the characteristics themselves why not just use the characteristic equations am I making sense why not just use the characteristic equations directly.

And say that at all of these interior point we use the regular equations and at the boundaries we will do something with the current with the character with the equations say in characteristic forms we will allow we will use the equation in characteristic form to determine what is the boundary condition what is the equation that is solved at this point am I making sense. We will admit that only  $p_0$  and  $T_0$  is provided here.

Which means that from the differential equation we need we need one differential equation for 1 parameter to be solved at the boundary am I making sense okay and it is the same thing at the right hand end that is you have  $p$  ambient  $p$  ambient being provided the only 1 condition given so

2 of them should come from the interior is it okay right. In the next class what we will try to do we will try to determine.

How do we apply these boundary conditions using the equations governing the equations themselves is that fine okay? Thank you.