Introduction to Computational Fluid Dynamics
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Lecture - 25
One-Dimensional Euler equations - Attempts to decouple

So, last class we started looking at one dimensional flow, I think we have derived the equations right.

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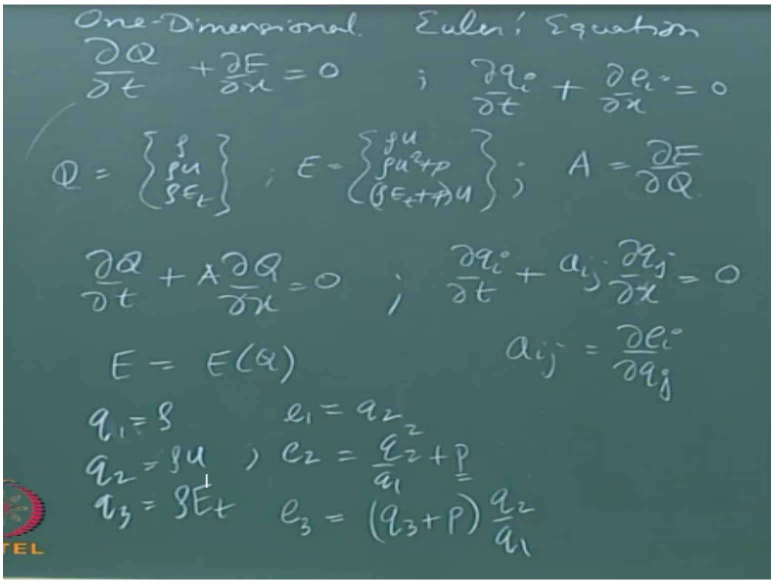

So the equation that we got, this is one dimensional. We just write one dimensional Euler's equation dou Q/dou t + dou E/dou x = 0. If you want that in index notation it is just for, so that would be dou Qi/dou t + dou Ei/dou x = 0. We saw that Q had components rho, rho u, rho Et and E had components rho u squared, rho u square + p, rho Et + p*u, is that fine, this is what we have got.

We basically said we do not like the fact that this equation looks a little different from the wave equation that we were looking at. So we wrote it in terms of the flux Jacobian. So this is a conservative form. We wrote the non-conservative form, which is dou Q/dou t + A dou Q/dou x = 0. You will see it is one of the many possible non-conservative forms. It is not the typical non-conservative form because the Q is still in the so called conservative variables.

The equivalent of that in component notation, so what is A, A is dou E/dou Q, the equivalent of that in component notation would be dou Qi/dou T + Aij dou Qj/dou x = 0, is that fine. It is a matrix equation. So there is a summation over j, is that fine everyone, Aij clearly is dou Ei/dou Qj, that is basically Aij. I am just writing it in this component form because we need to find out, is it possible for us to evaluate to find what is A.

So implicit in this definition of the flux Jacobian, it is sort of an assertion that E is a function of Q, that I can write E as a function of Q. So I should be able to write this E as a function of Q. So e1 is obvious, look at e2 and e3 right but before I start writing E as a function of Q, I want to give you motivation as to why you should always make sure that you write this properly in terms of q1, q2, q3 and so on.

So this is like a gradient right, it is derivative, it has derivatives of the first component with the first component, second and so on. So what does dou rho u/dou rho, and that is why you have to be very careful how you write this out. I do not want you to hesitate to answer because I set you up, but you should all do. So dou rho u/dou rho, the con the reason why that question is not properly posed.

That is I am differentiating with respect to these dou rho u/dou rho these 2 kept constant dou rho u/dou rho, rho u kept constant, so rho u is a constant. So dou rho u/dou rho is 0. You see what I am saying, you have to be very careful. If you try to do it directly here you will make mistakes, I also make mistakes I want to protect myself from myself. So I am very careful. So I will write e1 as, what is e1, let me write. So q1 therefore is rho, q2 is rho u, q3 is rho Et.

What I am suggesting to you is always write it in terms of q1, q2, q3 that way you will not make mistakes, e1 is q2, e2 is q2 square/q1 + the P, we have to figure out. So then we have to figure out this P, let us pay attention to that P and e3 is q3 + P*q2/q1, do you understand. So the only issue is how do we find P. For that we have to go back to the original definition of e total, because that is the relationship that we had.

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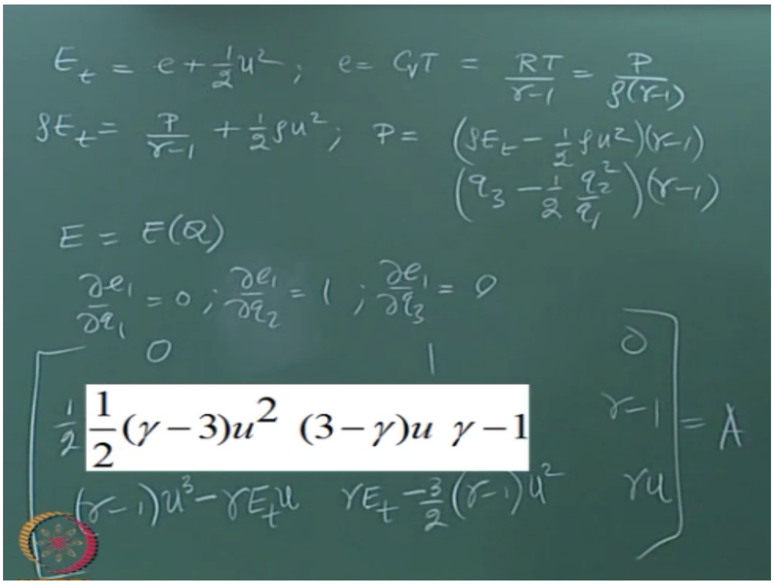

We had E total is e + 1/2 u square, or rho E total, whichever way and e was, this is CvT which equals the equation of state or RT/gamma - 1, I substitute it for Cv = P/rho * gamma-1, is that fine. So you go back here E total, rho E total is P/gamma - 1 + 1/2 rho u square. This tells me that P is rho E total – 1/2 rho u square * gamma - 1, which tells me this is q3 – 1/2 q2 square/q1 into gamma – 1, am I making sense.

So now we managed to write E as a function of Q and it is actually possible for you to find dou E/dou Q. So the first term for example, therefore dou e1/dou q1 = 0, dou e2, I will do the easy ones I will leave the others for you, dou e2, I will write it out but I would suggest that you verify you check that I have not made any mistakes dou e2/dou q1, what I want to do? That is not what I want to do.

As I said I am going to do the easy ones dou e1/dou q2 = 1, dou e1/dou q3. I am keeping the row a constant and I am changing the column yeah, dou e1/dou q3 = 0. You can do the same thing now. So you say dou e2/dou q2, there is no sense to me doing it. I have a little cheat sheet again. So I am going to take my sheet out, I am going to write it out. So that I do not want to make mistakes here.

So I will start with the bottom row because that is the largest. That is a bottom row first element, that is the second element, that is last element of the bottom row and I have 1/2 gamma - 3 * u 3 - gamma, I wrote the bottom row first and, gamma – 1 and the first entry of course is 0, 1, 0. That is the matrix A, is that fine. You should make sure that you are able to do at least a few of these, you should check.

And I assure you that if you do not write it in terms of q1, q2, q3, you will make mistakes. I have made mistakes every time I try that is why, little difficult to keep track of what is happening right okay.

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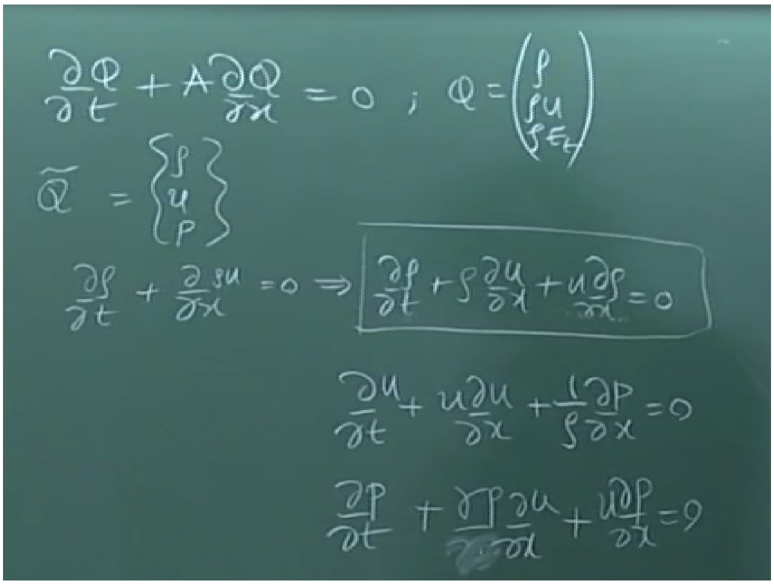

So we are back to the original question that though I have managed to write the equation as dou Q/dou t + A dou Q/dou x = 0, A is a very, the first row is okay but it is a pretty complicated expression and the system of equations are still coupled. So though it resembles our linear wave equation, it is nothing like our linear wave equation. In fact, I would think that if you have done linear wave equation, I get 3 decoupled wave equations.

Something of that sort that is what I want. So maybe it was just a poor choice of variables. We wrote the conservation equations and as a consequence of our densities of the quantities that were conserved. As a consequence of our densities, we picked Q which is rho/rho u rho Et. It was a choice forced upon us in a sense by the physics of the problem, maybe if you choose a different set of variables we will get an equation that is more amenable to solution.

After all I am writing it in a non-conservative form, and if you think back, then standard non conservative variables form either you use rho u and temperature or you use rho u and pressure or something of that sort right, you are interested in pressure. You are not really interested very often in rho Et. You would like to know the pressure distribution. See I am trying to motivate why I want to do a change of variables.

So you are interested in the density, you may be interested in the speed, you may not be really interested in the momentum density at a point, you may be interested in the speed of flow at a point. So it is possible that we should therefore choose a Q tilde, which is rho u P instead of rho/rho u rho Et. So I will try to derive this now from our governing equations. I will try to derive the equations in terms of this, which means that I am going to do a little arduous but not too arduous calculus.

Expand out the terms so that I can get equations, isolate equations that are derivatives, isolate the terms in the equation, so that the derivatives in terms of rho u and P is that okay. So just bear with me, we will try to go through this quickly. So conservation of mass, first equation here dou rho/dou t, it is not bad start + dou/dou x of rho u = 0. So I can write that as dou rho/dou T + rho dou u/dou x + u dou rho/dou x = 0 is that fine.

So we have one equation. You need to get the other 2 equations. For this I go back maybe to the front board. So we will retain this part to write all the equations. We will go back to the beginning.

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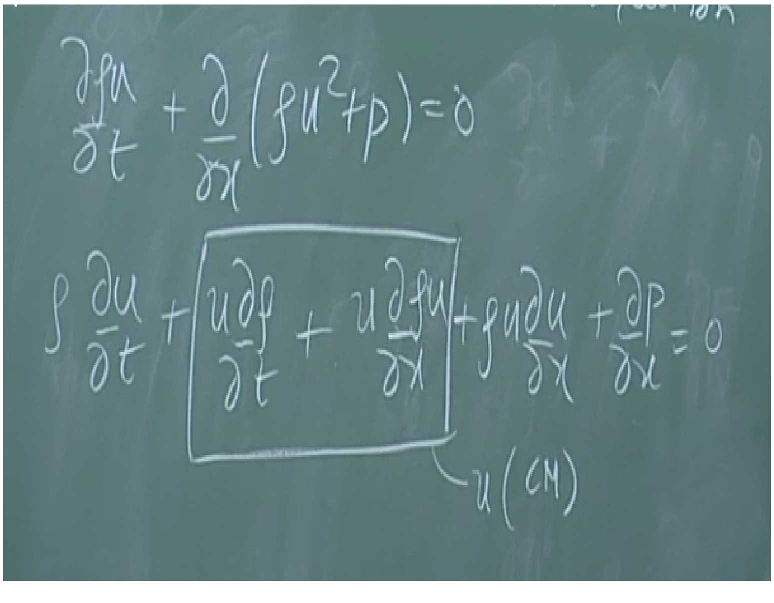

What is the momentum equation dou rho u/dou t + dou/dou x of rho u square + P = 0 expand that out, rho dou u/dou t + u dou rho/dou t + u dou rho u/dou x + rho u dou u/dou x, you know why I did it that way? I did it that way because I see I have a u dou rho u/dou t here and in my mind I am already thinking u dou rho u/dou x, so conservation of mass. An anticipation of that I have split it in that fashion + dou P/dou x = 0 is that fine.

Indeed, these 2 terms do correspond to u times conservation of mass, so that term will be 0. So this gives me rho dou u/dou t, rho u dou u/dou x, dou P/dou x. So there are only terms that have only the parameters that we want. There is a dou u/dou t, there is a dou u/dou x and a dou P/dou x. So I am going to take this over. I will divide through by row when I transfer it to make sure I do not make any mistakes.

So I have a dou u/dou t + u dou u/dou x + 1/rho dou P/dou x = 0 is that fine right. So let us get back to the energy equation now.

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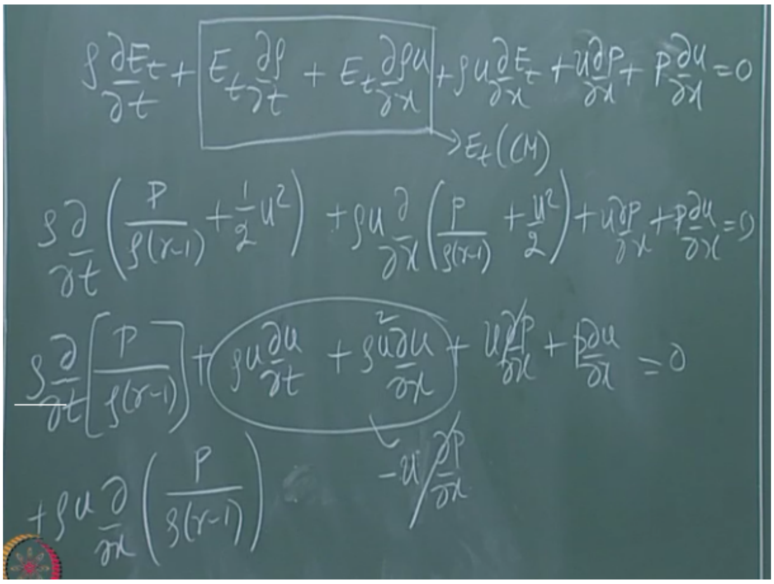

So I have dou rho Et/dou t + dou/dou x of rho Et + P * u = 0. So this becomes rho dou Et/dou t + Et * dou rho/dou t. Now we suspect we can play the same game, because there is a rho u in here. So that becomes + Et * dou rho u/dou x + rho u * dou Et/dou x + dou/dou x, you want dou P/dou x * u + P * dou u/dou x, it is the expression it is = 0.

Again we have a simplification due to conservation of mass. This is Et times conservation of mass, left hand side of conservation of mass and therefore it = 0. Is there anything else we can do. It looks like we have no choice we have to expand the Et. We have to expand that Et. What was Et, I had that somewhere here, what was Et. So there is a 1/rho, but Et is here, so rho dou/dou t of Et was P by rho * gamma - 1 + 1/2 u square + rho u dou/dou x of P/rho * gamma - 1 + u square/22 + u dP/dx + P du/dx = 0.

We will try it before, see the P looks a little complicated expression. We will get to that. We will try to get rid of these because we have done conservation of mass, it is a momentum equation helps us out here. I will get a u dou u/dou t from here and I will get a rho u dou u/dou x from here does that make sense. So there will be a rho u dou u/dou t from there and the rho dou u/dou x from here, and we will see from the momentum equation whether something can be knocked over.

So there is a hope that, we will follow that path, so I leave the first term untouched for now. So I get dou/dou t of P rho gamma - 1 that is the first term, the second term is rho u dou u/dou t, then this term gives me a + rho u dou/dou x P/rho * gamma - 1 + rho u/rho u square dou u/dou x +, I can add that anywhere + u dou P/dou x + P dou u/dou x and this sum of all of these = 0 is that fine. Can we look at momentum equation and see what we have?

Our dou u/dou t + dou u/dou x. So rho u dou u/dou t, rho u square dou u/dou x and that is I can substitute by – 1/rho dou P/dou x. So these 2 terms I can replace by -1/rho dou P/dou x and pardon me into rho u - u dou P/dou x. There is a u dou P/dou x here that will go away, so that also goes away, get rid of the stuff.

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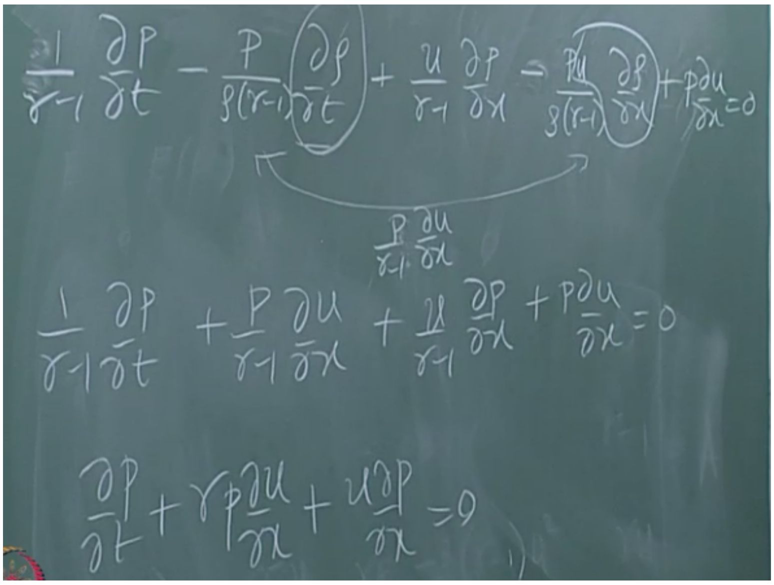

What we have? what are the terms that we are left with, we will just collect them before we simplify it further rho dou/dou t of P/rho gamma - 1 + rho u dou/dou x P/rho gamma - 1 + P dou u/dou x = 0, just make sure I have not made any silly mistakes. So we expanded it does not look like we can do anything. We will have to face it. So this is rho if I take dou P/dou t so this comes out, so I get 1/gamma - 1 dou P/dou t because the rho will cancel -.

If I differentiate this I get a 1/rho square but one rho will cancel, so I get a P/rho gamma - 1 dou rho/dou t, is that fine the derivative of 1/rho is -1/ rho square dou rho/dou t, but one of the rho will cancel the 1 in the numerator. Fortunately, these are the same. So you just replace the t's with the x's. So it is not too bad, it is not as bad as. So this gives me u/gamma - 1 is that right, u/gamma - 1 dou P/dou x – P u rho gamma - 1 dou rho/dou x + P dou u/dou x = 0.

I have a dou rho/dou t here, dou rho/dou t here, so if I take out maybe I should have left this as, so there is a rho u somewhere here. Let us come back to conservation of mass u dou rho/dou x dou rho/dou t.

So I see a dou rho/dou t and I see a u dou rho/dou x and they both multiplied by P/rho * gamma - 1. So these terms can be combined, they have a - sign they both have a - sign in front, which means I am taking them over to this side that leave me a rho dou u/dou x, a + rho dou u/dou x. So these terms can be replaced by from conservation of mass rho dou u/dou x. It will multiply through by gamma – 1.

So I get a dou P/dou t + rho * gamma - 1 dou u/dou x +, what do you get here you get a P/gamma - 1. So if I multiply P times dou u/dou x but I have another P * dou u/dou x here, a gamma - 1, let me I am not going to jump this I am skipping a step I will make a mistake 1/gamma - 1 dou P/dou t + P/gamma - 1 dou u/dou x + u/gamma - 1 dou P/dou x + P dou u/dou x = 0. So if I add this to that, I will get a gamma/gamma - 1.

So I will multiply through by gamma – 1. Now I will skip the step. I get dou P/dou t + gamma P dou u/dou x + u dou P/dou x = 0 is that fine everyone. We will find out if I made a mistake we are going to find out dou P/dou t + gamma P dou u/dou x + u dou P/dou x = 0. How is that, so if this is Q tilde, I have rho u P, I can write this as a matrix rho u P and I have dou u/dou x, I have rho u P right, I have a dou rho/dou x, I can actually rearrange these terms and write it in the matrix form. So let us see what we have here as a matrix form.

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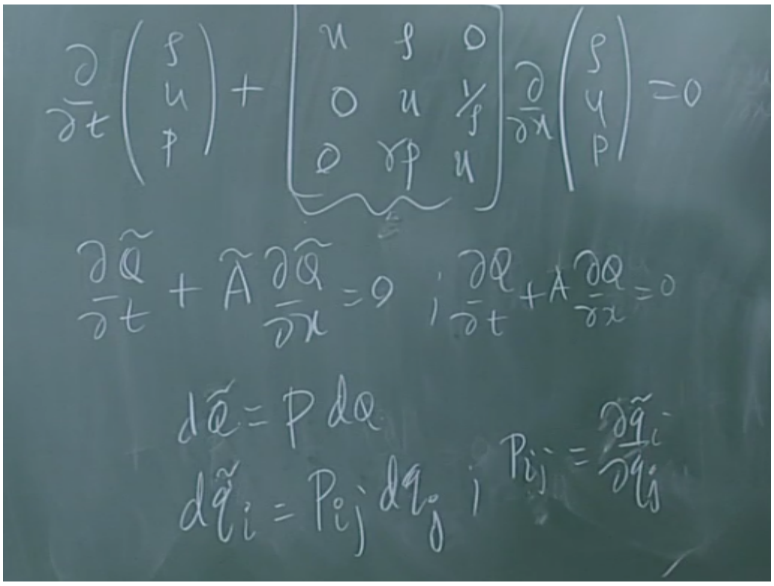

What do we have, so Q tilde is rho u P, so the first term obviously is dou/dou t of rho u P + can you tell me, what the rest of it is, is there a dou rho/dou x here, u dou rho/dou x + rho dou u/dou x. There is no dou P/dou x term there + 0 dou P/dou x I am writing the matrix now. The second one is there a dou rho/dou x no, 0 dou rho/dou x, u dou u/dou x, 1/rho dou P/dou x. The last one there is no dou rho/dou x, 0 dou rho/dou x, gamma P dou u/dou x + u dou P/dou x * dou/dou x of rho u P = 0.

So I am now able to write this as dou Q tilde dou t + I will identify this matrix as A tilde and identify it on the top of it, a different non conservative form. In this case the variables are not the variables we got in the conservation equations as densities. So you can call them non conservative variables if you want right as opposed to those variables, which are conservative variables. So you have another different conservative form. Expressions are easier simpler.

So clearly we are sort of going in the right direction but this is a pain. Even if I have to sit down now write the next one I can try this instead of rho u P try out rho u t. How do I know, if you are going to do this trying to hit the thing blindly, how do I know I am going to get there right? How many variables can I try and I may not even chance upon it. I may not just, we tried once we did not luck out, we try to rho u P and we still got a system of equations that is coupled.

So we have to obviously get a more systematic way by which we do this. So the question that we ask ourselves is, how did I go from Q to Q tilde, can I is it possible for me to transform from Q to Q tilde. Am I making sense, if I have the equation and if I do that, if I figure out how that happens, how these 2 equations are related, if it is possible for me to do something to this, then I am set. So what is the relationship between these 2.

So I have 2 sets of variables I can write these in terms of q1, q2, q3, I can write Q tilde 1, Q tilde 2, Q tilde 3 in terms of q1, q2, q3. Therefore, I can write dQ tilde is or dQ whichever way. I do not know which way you want to do. So if I want to transform from here to there, then possibly want it in terms of Q tilde dQ tilde is some P * dQ. This is like classic definition of a derivative. It is a linear transformation there is a direction.

Remember what I said earlier. This is a vector. This is a linear transformation. So we do not know that it is a derivative. I am writing it out as a derivative. You do not know that it is an exact differential, but I can write out this relationship. What is P, that is an index notation. This could say dQ tilde i is Pij dQj, you understand the matrix components, component wise that is how it would be written dQ tilde.

Component wise that is what you would write, or we are basically saying that Pij is dQ or dou Q tilde i/dou Q tilde dou Qj. If I have this, there are different ways to look at this. One way to look at it, as I said is, you are looking at it as a derivative. Another way to look at it is that this P transforms dQ to the dQ tilde coordinates. The dQ variables get transformed to the dQ tilde coordinate. That is another way to look at it.

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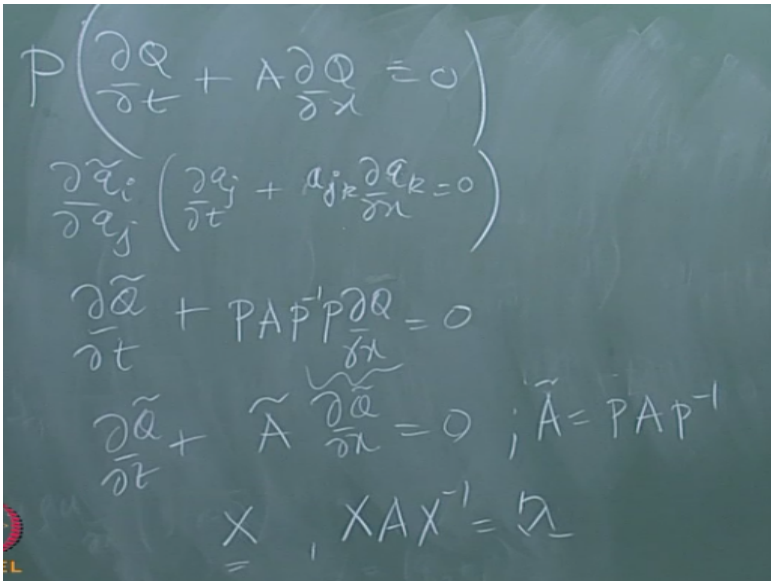

So if I pre multiply this equation by P, it should transform from the Q coordinates to the Q tilde coordinates is that fine. Another way to look at it is that, I am basically just using the chain rule, different ways by which you can look at this. If you look at what this is, this is the dQ tilde if you look at this equation just I want to make sure that, there is no confusion here dQj and what does the equation itself, because I have already used i, I use k.

So you can do dou Qj/dou t + ajk dou Qk/dou x = 0. You can look at it as just using chain rule, am I making sense, different ways by which you can look at this. So what will this give me dou Q tilde/dou t + what we do here P A P inverse P dou Q/dou x = 0. This of course gives me dou Q tilde/dou x. So there should be dou Q tilde/dou t + A tilde dou Q tilde/dou x = 0, A tilde is P A P inverse is that fine everyone.

So if you have not seen this before, I will suggest that you just go take a quick look at matrix algebra. This is a similarity transformation, and what does this similarity transformation give us, what does it tell us, what is the point of similarity transformations. The Eigen values are the same and if you are lucky in this case we are lucky, we have distinct Eigen values they are real, I mean they have a lot of nice properties.

So we can actually find some matrix you can actually find some matrix, some transformation which will diagonalize the matrix A. What I am saying is, yes there is a matrix X, so that X A X inverse or X inverse A X depending on which way we have written it is a diagonal matrix lambda. Whether we call it A inverse or A is something to our, I mean X inverse or X is something.

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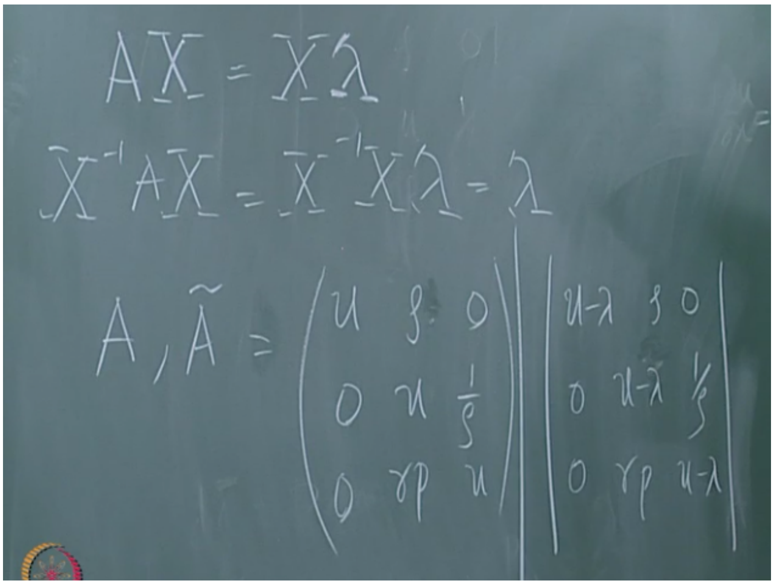

So your usual Eigen value problem is AX = lambda X, I usually prefer to write it as X lambda because otherwise when you make it a vector you can run into trouble. So if this is a 3 x 3 system you have a full set of Eigen values, then you have 3 of these K = 1, K = 2, K = 3. So I can put those in column matrix. So I get AX, X corresponds to a column matrix = X lambda. This is a matrix, lambda is a diagonal matrix, the matrix is lambda X and X lambda are not the same you have to be very careful.

That is why I say somehow do not I prefer to say instead of AX = lambda X when we are talking about lambda in a scale of one Eigen value I really prefer this is lambda K. I really prefer to say AX equals X lambda, because when you write it as the matrix that is how you would write it. So if I were to multiply through by X inverse, which = lambda. So such a transformation exists.

You can actually diagonalize is that fine. Such a transformation exists can actually diagonalize. Now as it turns out, we can either use the A to find the Eigen values and Eigen vectors or we can use A tilde to find Eigen values and Eigen vectors. So now I will use some prior knowledge that I have, that using A to find the Eigen values and Eigen vectors is difficult, so I will use A tilde.

As I said, already I raised my A tilde so tell me what my A tilde is. What is A tilde u rho 0, 0 u 1/rho, 0 gamma P u what it is. I will do that, so you want the determinant of u - lambda rho 0, 0 u - lambda 1/rho, 0 gamma P u – lambda, you want the determinant of that, that is relatively easy, that gives me. What is the characteristic equation u - lambda * u - lambda square - gamma P/rho = 0.

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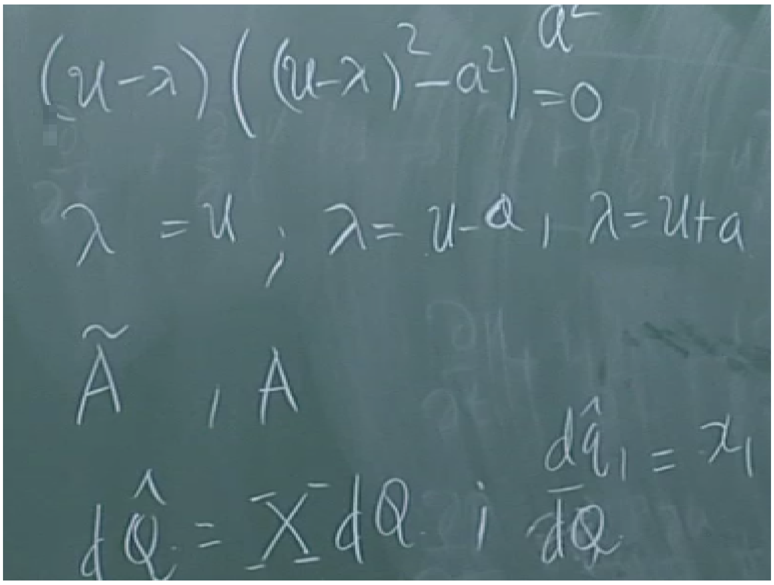

And here little gas dynamics, what is gamma P/rho that is this square of the acoustic speed. So that is a square, so you get u - lambda * u - lambda square - a square = 0, so lambda = u is one solution and the other 2 solutions are lambda = u - a and lambda = u + a, bit of a strain you have been going through a little intense derivation here. So we are almost there, so it looks like if we find what I will do is right now.

You can try to find make sure that you are able to find the Eigen vectors. I want you to try to do 2 things right, find the corresponding Eigen vectors, do it for A tilde first and then try it out for A. What do you think, do you think the A tilde Eigen vectors for A tilde and Eigen vectors that for A will be the same, do you think they will be the same? Actually we performed this doing the similarity transformation is like doing a coordinate.

You are doing a coordinate transformation; you are performing some kind of a rotation doing a coordinate transformation. So it is unlikely that the Eigen vectors will be the same, try it out. I would suggest that you start with the A tilde one first because that matrix looks easy and everything was easy with respect to it. So try out a tilde one first right and then see if you can find the Eigen vectors for A.

What we know as a consequence is that if I have the matrix X, so then I can say that I can find the dQ caret, which is X * dQ. So you have to decide whether you are going to put them as columns or rows, otherwise it has to be X inverse. So you can decide which one is going to, whether it is going to be row vectors or column vectors. So dQ caret is x * dQ, and then you can do the transformation.

So of course I am not saying anything about dQ caret right now, about Q caret is locally this differential, this relates in that tangent plane that is why I said think of it derivative. Locally in that tangent plane, it relates dQ to dQ caret right, in order to find Q caret, I have to be able to integrate, I have an envelope of tangents, I have to be able to integrate.

I have to find that envelope I have a whole bunch of tangents the issue is can I find an envelope that is what the integral is okay. So if you think about you go back to your calculus and think about. So whether I can integrate this or not is a different issue to find Q hat there may be circumstances under which I can actually find this Q caret or Q hat. Think of it as a you have seen this kind of a thing before dQ hat.

This is a smaller side I did not want to get into this, dQ hat 1 by I will say I will write it as dQ indicating this is a gradient with respect to Q and this is the first Eigen vector x1. You have seen something like this before. The gradient of a scalar field is a vector field. When can this be integrated, when can you find the q1 hat, in the curl of the vector field is 0. It is like a velocity potential or some just like a potential the q1 hat exists, it is integrable.

There is an issue of intent whether all those bunch of tangents they have to have a certain property that they have to be able to get that envelope, so that requires that the curl of x1. So the curl of this has to be 0 in order to be able to find Q. We do not know in a special case it turns out we can actually do it. So if I pre multiply by that X or X inverse or whatever I call it, if I call it X as I said X or X inverses matter what we make it.

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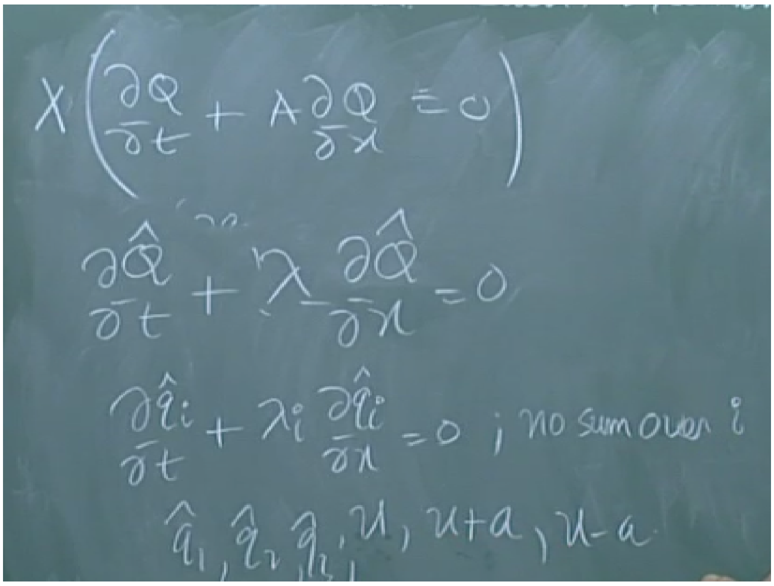

This will result in an equation dou Q caret/dou t + the diagonal matrix dou Q caret/dou x = 0, or in component form dou qi caret/dou t + lambda i dou qi carrot/dou x = 0, and there is no sum over i, there is no summation over i. You will not get carried away too much with index notation no sum over i. So there you have it decoupled we have 3 as promised non-linear because if you have a u you know there is only so much we can deliver.

So the lambda 1, lambda 2, lambda 3 are u, or u + a, u - a. So what are the 3 propagation speeds u, u + a, u - a. So you will find what are q1 hat, q2 hat, q3 hat. You will find the q1 hat, q2 hat, q3 hat corresponding through these 3 Eigen values, to figure out what is it that is being propagated. Something is being propagated at the speed u from gas dynamics, you may already be aware that is a contact surface.

Go back, look at it and u + a, u - a very clearly acoustic speeds. So if it is a stationary, there is no motion u is 0 and there is a disturbance is going to propagate in 2 directions + a and - a at the speed of sound. On the other hand, if there is an underlying motion in a certain direction then whatever you have is propagating it at u the medium is propagating at u, you have a u + a and u - a is that clear.

So we are sort of actually picked up. We have actually picked up at least we feel right now from our gas dynamics background, we have picked up the necessary physics, what we need to do is we have to ask the question can I apply FTCS to this, can I apply some scheme to this and see if we can get a solution is that fine. So I will see you guys next class. Thank you.