Introduction to Computational Fluid Dynamics
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Lecture - 23
Quasi-linear One-Dimensional wave equation

Good morning right so we have been looking at linear wave equation so far right. So yesterday in fact we finished with the demos.

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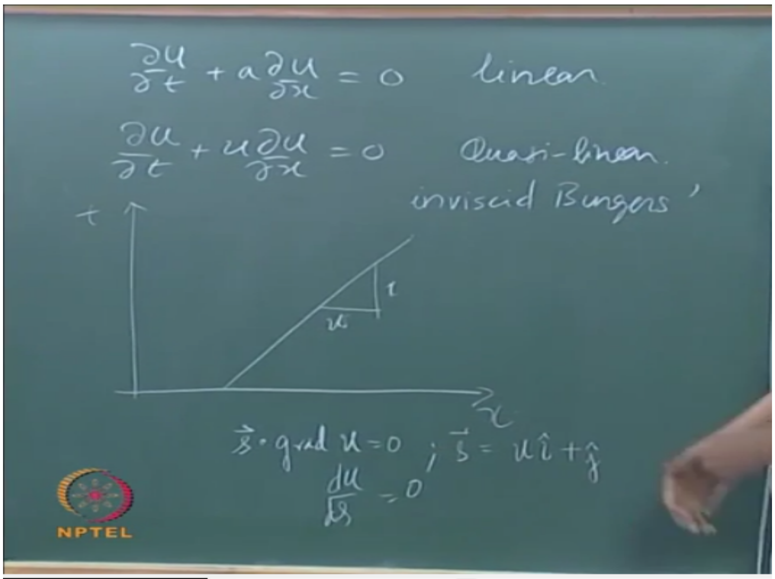

So to be precise we have been looking at linear first order one-dimensional wave equation. Of course as I noted somewhere in between it looks similar to but it is not quite identical to the class of equations that we are looking at in the fluid mechanics as such right though I can give an interpretation as of some stream traveling at a constant speed a, carrying the property u with it.

Another typical equation that we run into is this equation. We will spend a little while. We will try to spend like a part of today or whole of this class talking about it. I could of course spend the rest of the semester talking about it but I do not think I am going to do it. We have spent enough time on the linear one okay. So this is linear, this is said to be quasi-linear okay. So if you had dou u/dou x whole squared then it will become nonlinear or dou u/dou t whole squared then it will become nonlinear okay so this is quasi-linear.

And look at this equation and see whether we can figure out any properties of this equation from an analytic solution point of view just like we did earlier right and see where we can take it from there and after that I will leave this alone and then we will go on right looking at other equations of interest to us like Euler's equation and so on okay. So you will hear this being called either the quasi-linear one-dimensional first order wave equation right.

You may even see it sometimes in literature or in books as the inviscid Berger's equation okay anyway so it could also be called inviscid Berger's equation but you may see it, you may even see it being called inviscid Berger's equation. Now just to recollect in the beginning when we were talking about this equation we realize the importance of characteristics and in the xt plane right in the xt plane.

We saw that we had characteristics that had slope a that basically means of slope 1/a in this coordinate system that basically means that whatever it is we were doing was propagating at the speed a in a given amount of time right in a unit time. It is basically propagating at distance a in unit time so that propagation speed was a, that is what we had. In this case, the propagation speed is u. Is that fine?

So the first question is the argument that I used for this does this argument still hold here? That is can I still say that some s dot gradient of u=0 where s= I do not remember what basis vectors I used last time but anyway it is okay ui+j where i and j are unit vectors along x and t. Am I making sense? So this notion of the directional derivative still works here. So you still have du/ds=0 where this s is measured along this direction s okay.

And that argument that we use that time saying that along this line because the right hand side is 0, the right hand side is 0 then the story is different because the right hand side is 0 along this line use a constant. Is that okay? And along this line if you use a constant along that line the propagation speed is a constant fine. Is that work for you? So now all we are going to do is we are going to say oh this has slope u, 1/u.

You travel a distance u in unit time. Remember it is still quasi-linear right. It just so happens that along this line u is a constant that is very important. So I sort of think that to make sure that there is no issue. So just like we did last time, we will use this to see whether we can come up with solutions to this equation just like we did for this case we will see if we can come up for solutions for this case okay.

Clearly, we cannot do the exponentials like we did last time because the exponential function had the a in the exponent and then you will be writing u in terms of itself that does not make sense okay. It does not help us mean it is sort of an implicit form that does not really help us. I do not say it does not make sense but it does not really help us right.

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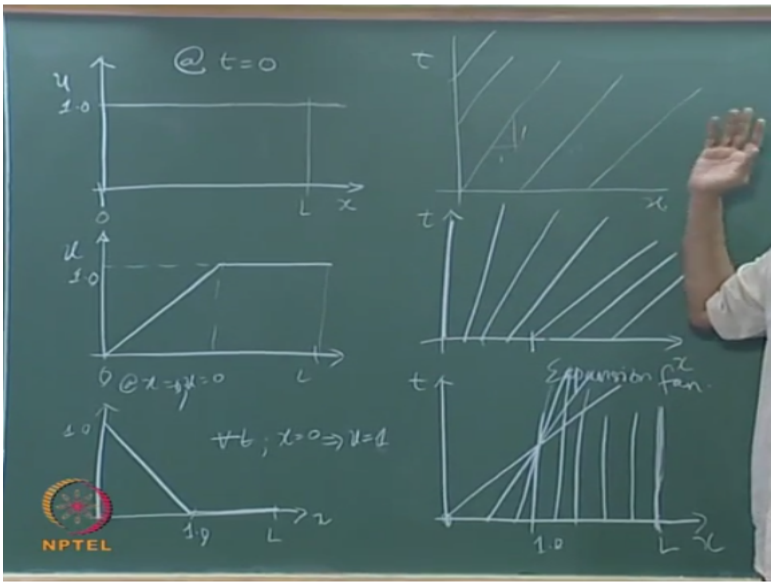

Let us consider some initial condition so you consider initial condition. This is u, that is x okay. We will consider a series of initial conditions and see what we get. You consider an initial condition where u equals and we are going to only look at the just like we did before either a unit length or a length L or whatever it does not matter okay. All of this is at t=0. If the initial condition is 1.0 everywhere what is the solution?

On the xt plane what do you expect? Actually, it is rather boring problem, it is the same as dou u/dou t+dou u/dou x=0 that is what it degenerates to so the initial condition then basically says that all of these characteristics are 45 degree lines right with propagation speed 1 and if my boundary condition also continues to be 1 then you will characteristics coming out of that. Is that fine? Okay.

So this in that sense, it sorts of degenerated just because u is a constant in this case it happens. So if u had been a, it would be the same as that equation. Is that fine? Okay. Okay let us consider another one. So if you have something that starts off from 0 and goes to 1 and then is a constant and the boundary condition on the left hand side is 0. It is going to continue to be 0.

The boundary condition that I am going to apply for all time at x=0, u=0 is the boundary condition that I am going to apply okay. What do we get for the corresponding characteristics? You do the easy one first. So this is a 45 degree line, it goes to 1 so clearly this goes to 1 right. So let us do that first. So this is 1 right let us do the easy part which is to the right hand side.

They are all 45 degree lines likes this same deal, so they are all 45 degree lines likes this. So you get okay. Is that fine? What happens in between? What is it here at the left most point? The slope is infinite basically, the slope is indeterminate, the slope is infinite. So the characteristic here is a nice line there and what happens in between? For instance, what happens at 0.5?

The slope is 2, it is a steeper line right so the slope is 2 at 0.5 you would basically get steeper line and as you go towards this, the lines get steeper. As you go towards this, the lines get shallower till you get to 45 degrees okay right. This is called an expansion fan so this is an expansion fan okay that you would have seen in gas dynamics right so this is an expansion fan. So this behaviour is very different from this behaviour.

So now we are starting to see the u du/dx sort of kick in right, the effect of the u du/dx stuff. Is that okay? My third example of course is the more interesting example. Third example is going to start at 1, it is a 45 degree line going down to 0 here so this is also 1.0 x=1 that is x=L and then it is 0 afterwards right and for all t x=0 tells you that u=1, Is that fine? Okay. So as a consequence what are you going to get?

Again we will do the easy one first so between 1.0 and L you are basically going to get vertical lines because stagnation right u is 0 if this propagation speed u is 0, the flow there is stagnated. So you will get vertical lines, well if I can draw vertical straight lines right up to that and of course at the characteristic at 0 if we look at that again we will get a 45 degree line so now something bizarre happens, the characteristic intersect right.

So I will continue this through and then what happens to the in between points? The slope is going to increase and in this particular case it will turn out that they will all pass through this point okay. So I would not draw anymore because it will become a mess. Is that okay? So how do we interpret these two right? This is very clear. This is just pure simple translation. The first one is very clear.

How do you interpret the other two? So as the second one for instance as t evolves what do you expect will happen? How does this graph, how does this graph evolve from looking at this? How does it evolve? See you know at least you know the propagation speed of this right. You can actually work out the propagation speed of each of these individual quantities okay.

So effectively what you are going to get? As time progresses, we will get a series of functions, so this is 1, this is what you started off with. Let me draw it with a white line like this or shall I draw it there. May be I can draw it back there. I will draw it back here.

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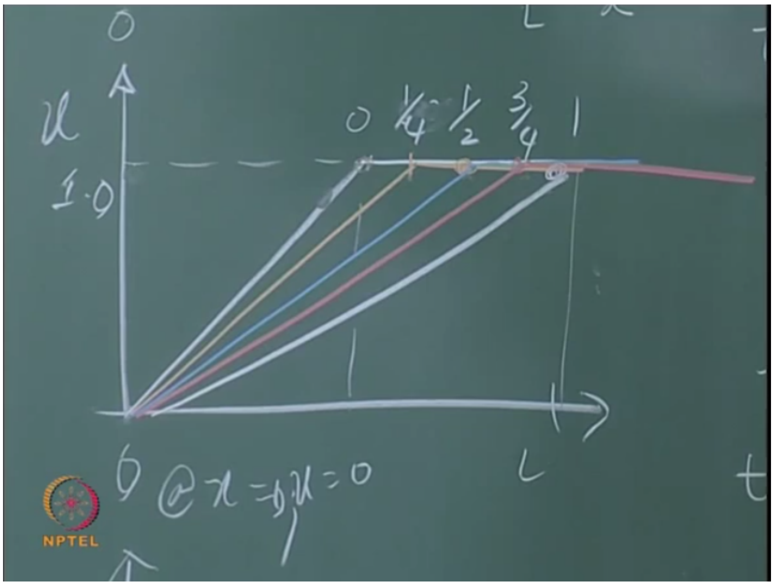

So as time progresses at different times this point is going to move to the right, right so this point is going to move to the right and you will get a function that looks like this. Am I making sense? Okay that is why it is an expansion fan so if you waited for second time unit that is twice the first time unit, it is translating at a constant speed of 1 okay. Then you will end up with maybe I choose a different color.

You will end up with a function like that okay. Is that fine? And you can see that yes indeed so at different time intervals so if I got this right just say this is at 1-time unit if you want if you use seconds one second if you use meters per second and second. So this is at t=0, 0.25 or 1/3, 2/3 whatever and by the time you get to 1-time unit here right equivalent roles, they are equivalent roles, it will propagate equal distances here.

On this end it does not propagate at all. In between it propagates at the proportional speed okay. Is that fine? And indeed this line is getting stretched. Is that okay? So this is at different time intervals okay maybe I will add one more just for the fun of it. I think from that I add so this is at t=0, this is at 0.25, 1/4, 1/2, 3/4, 1-time unit right. By the time you would propagate 1-time unit, it has traveled a distance of 1 fine everybody.

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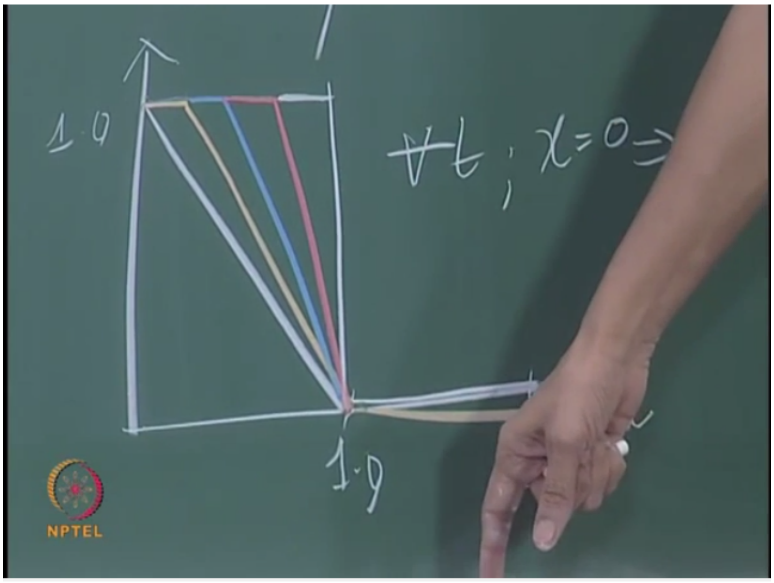

What happens in this case? If I were to just follow through so after a small time interval right, the left most point is propagating at one unit per second right and I would get a solution that is like this. If I waited a little more time then I would get a solution like this, as you going to go through 0 because these are all stagnation point, they are not moving. Basically, those points to the right are not moving.

And if you waited a little more time you would get that and what is the final one? By the time you came to 1 unit, you have now come to a point where you have traveled a full distance right and in fact at this point all of these characteristics meet. Can we go beyond that? Does it make sense? Can we go beyond that? Well see that really depends on what is the actual application.

Whether we are able to go beyond that really depends on what application. For example, if you were to look at these as trains traveling on a railroad track. If you were to look at these as trains traveling on a railroad track, there is possibly railway station here, ignore all of the stuff. There is a slow passenger train right, reasonably fast train and then an express train and if they make it at the same time to the railway station, at the railway station you have something called sidings right.

So they can pull a train over to and they can actually cross each other. At the station it is actually possible that they cross, it depends on the application. Am I making sense? All this basically says that this train is traveling faster than the other trains and if there is a way for you to shunt the trains from one track to another track so that they can actually overtake each other, see that is the key that they can actually overtake each other.

It is actually possible that they do propagate this way. Am I making sense? Is that fine? Right so it depends on what is the interpretation that we give to this, what governs these equations, what is interpretation that we give to these characteristics right? The other possibility of course is and these kinds of things happen, the solution can become multivalued, so the solution can become multivalued here.

You look at these characteristics and say wait a minute, u is constant along this, u is constant along this, each train has its own speed, each u is constant along and yes at the railway station the trains can be multivalued. You are there, your train, a stationary and you have seen other trains on other tracks going by right at a reasonable clip fine. So the u can be multivalued at a given point right.

They are so used to thinking of this possibility that they are multivalued of being right I mean you do not encounter it very often at least we do not encounter it very often. The other possibility of course is that you do have a function even in a regular physical setting which is multivalued. So if you think of I think the standard classic example that has given as a wave breaking.

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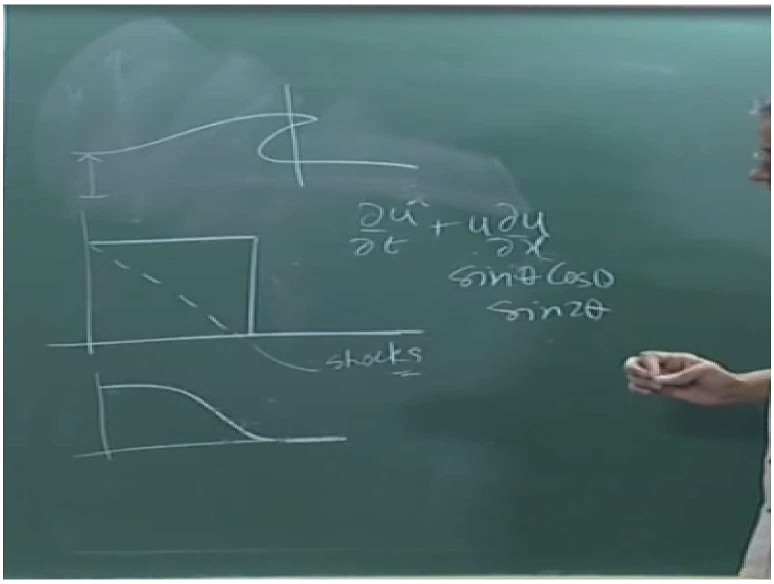

So you can have if you go the beach I have removed all the froth and all of that stuff but you could actually have a wave breaking right and the wave it can become multivalued. If your u indicates the depth right is the measure of the depth okay or function of the square root of the depth because it is tie to the propagation speed right. Then it is possible that it actually becomes multivalued.

It depends on the context that you are talking about right. If you are on the other hand, there is a traffic light here and all of these cars are stopped then if you do not stop in time yes this is going to happen right so you can have a pile up okay. So there can be situations where something like this actually happens fine, right and in gas dynamics what we are used to which is why you have the response and no, no it is not possible.

So if you have a pipe right which is why I gave you, if you have a single lane traffic and you cannot get around, then there is a problem, then you are going to end up with the situation that I have shown here where you start off with something that looks like you can end up with a function that ends up like this. You start off with a function that looks like that and you end up with a function that looks like this okay.

So what is the big deal? Why am I harping about this? I could have just said oh dou u/dou t a dou u/dou x=0, just try it out for different things but why am I harping on it? Why am I making a big deal about it? If you think about yesterday's demo right when we had dissipation or when we used heat equation, if you have a step, the step tends to smoothen out okay.

When we had oscillations, everything was smooth right everything that we had was actually relatively smooth. This is a strange situation right and it is also possible you will hear people say oh nature does not like discontinuities and so on right I mean you can make this broad statement nature does not like discontinuities but here you have a situation right where this governing equation actually generates at discontinuity where none existed okay right.

And for us it is important because we saw that the dissipation term on the right hand side of the linear wave equation was smoothing, that resembles viscous term in the Navier-Stokes equations okay. That very closely resembles the viscous term and it is smoothing right but the left hand side that is the dou u/dou t+u dou u/dou x term, this kind of a term it is also there in the Navier-Stokes equation on the left hand side.

It has a tendency to make it steeper okay. Is that fine? You understand what I mean by making it steeper, what is happening to the high wave numbers here when you go from here to here what is happening to the high wave numbers? They are decaying, their amplitudes are decaying that means that if you are going from something like this to this so imagine you draw the characteristic for this, you expect that this is going to become a shock like this.

Why? Because if this is a propagation speed, this is traveling faster than that right, so if you just have a smooth initial condition the demo we went from here to here but instead of going from if you have started off with this and you are solving this equation what we see now is this is going to become that, it is a competing opposing effect. Am I making sense which is interesting right?

And where does it come from? What is the source? That means I am adding from this low frequency I am generating high frequency terms. Where does that come from? Okay the term is here. Just imagine that u is like sin theta sin n delta x right, u is like sin theta what happens to this? This becomes sin theta cos theta which is sin 2 theta and that is where the frequency doubling up is.

You are adding to the sin 2 theta, you add to the sin, you understand you are going to add to the sin 2 theta term if you were to do a decomposition of some kind you are going to add to the sin 2 theta term, you are generating a sin 2 theta term where you only started off with the sin theta term okay.

So even a smooth function like this is going to get steeper, so the actual equations where you have both viscous term and this quasi-linear term both these effects are competing and therefore what you have studied in gas dynamics that is the thickness of any shock that forms so in gas dynamics of course these are called shocks right okay. Occasionally in mathematical literature you may hear them referred to it as an internal boundary layer which is sort of an oxymoron.

But if you know boundary layer there you know what they are talking about right okay right. So there are shocks right these are called shocks and you know that the thickness for the shock depends on the viscosity. Am I making sense? Viscosity is trying to do this. The shock is the quasi-linear term is trying to push it back here and they come to some kind of an equilibrium right.

Left to its own devices the quasi-linear term will actually give you a discontinuity. So viscosity is a sort of has the opposite effect which is why that balance is what gives you the thickness of that shock right and it changes, it depends on what is viscosity of the fluid is. Is that fine? Okay. Are there any questions?

This is as far as what should I say as far as this quasi-linear equation is concerned we will see whether let me what happens can you say anything beyond the shock? So obviously there are different possibilities from the one hand you can have this effect, on the one hand you can have this effect, on the other hand I will redraw this with the shock there on the other hand.

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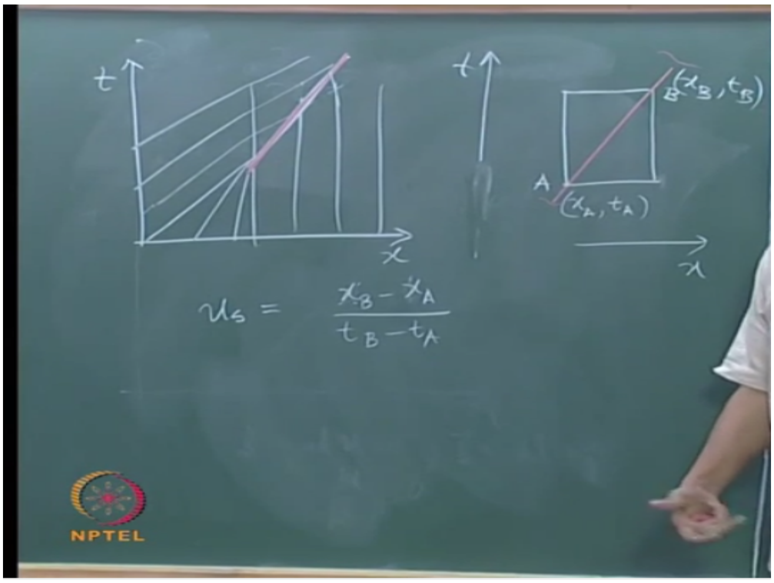

So you would expect to get this. I should have drawn these vertical lines first but it is okay I will change to right and then you have these lines and of course you have conditions coming from the left which I have not drawn there right or you have all of these lines so all at 45 degrees. They are all parallel to this right and they form a shock. So I need a different color for the shock.

These characteristics merge and they form a shock and basically the shock sort of consumes the characteristics, it eats the characteristics. Characteristics disappear into the shock right okay. Is there a way for us to find the propagation speed of the shock? Right I mean to say is the shock just located there, is it going to travel, is it going to be stationary? The way I have drawn it, it is going to propagate.

The shock is going to travel that is the way I have drawn it. Is there a way for us to find the propagation speed of the shock? Okay. Is there a way for us to find the propagation speed of the shock? So we will zoom in on this, we will zoom in on this area. We will choose a little more so, now I am not going to, will just say t, I will not show you the origin. I will say x, you do not know where this is.

Somewhere here there is a shock; our objective is to find so of course the shock. I do not care where it originates, I do not care what it is going, somewhere here there is the shock and I will choose a convenient control volume, it points A and B okay. It is going to be a rectangle. I will tell you what I am proposing to do okay.

I choose this rectangle, I will rewrite my equation in a fashion that I am able to relate the conditions on the left hand side of the shock to the right hand side of the shock and I propose to take the limit as A and B approach each other right and from that I am hoping that I will be able to infer something about how fast the shock is propagating. Is that fine? For instance, if this point is xA and this point is xB right, so this is xA, tA in the xt plane, xB, tB.

Then, we already know that the propagation speed of the u shock, here u shock is xB-xA/tB-tA fine. I seem to have implicitly assumed that the shock is propagating at a constant speed okay. Let us see where this takes us.