Introduction to Computational Fluid Dynamics
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Lecture – 19
Effect of Higher Derivative Terms on Wave equation

We have been looking at the modified equation. Just remind you the modified equation is the actual equation that we think we are solving when we get our approximate solution to the original equation. Now I am trying to modify the equation. We have defined consistency and well auxiliary on the side, I defined what is convergence? We also saw that the modified equation to our original wave equation.

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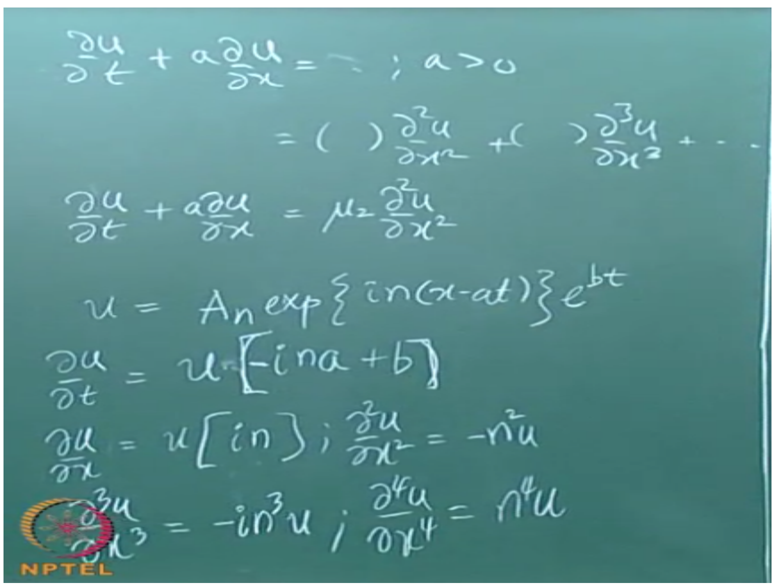

The original wave equation, the way we are looking at it had a 0 right hand side, but the modified equation ended up having terms that were of the form instead of being 0, had some coefficient second derivative of x. We had things with third derivative of x and so on. We wrote it up to and including the fourth derivative. So we then asked the question. Just take 1 such term. The equations are linear.

Why do not I deliberately add a term on the left hand side and right hand side which has dou square u/dou x square. Mu 2 is a constant. I am very suggestively I called it mu 2. Clearly I am going to have a mu 3 and a mu 4. Now for this equation, I am just repeating what we did at the end of the class. We said that a solution could be written in terms of Fourier series and I will write 1 term. It is still a solution so that u could be written as An exponent in.

I think I decided to get rid of the 2 pi and n x - at. This was for the original equation. We decide to go the semi-inverse route which means that we are going to try out come up with the candidate solution. We will guess that the solution to this equation is in this form and that gives out one disposable constant b that we need to determine. All the others are fixed. Now since I know I am going to add extra derivative. Let me just write out all the derivatives that I require.

So dou u/dou t and then we can just substitute. We can keep tract and then substitute whatever we want. Dou u/dou t is what we get. So if we differentiate this and we use product rules and we will get the sum of 2 terms which ever you want to do it. So you will get the u times I * n that comes from differentiating the first one with a - sign because it is a - at + b. What is dou u/dou x? u * u multiplied by, u times it is not u of maybe I use a square bracket. u of u * in.

What is dou u square u/dou x square? So you will get another in, every differentiation with respect to x we will get an in so there will become a - n square u. dou cube u/dou x cube gives me in times that - n cubed u and finally do to the fourth u/dou x to the fourth is n to the fourth *u. Everybody is with me? Now you can just work it out. So we will work it out first for this equation as only second derivative term.

So, clearly all of them have u as a coefficient that will go away. So if I substitute the first term is going to give me a.

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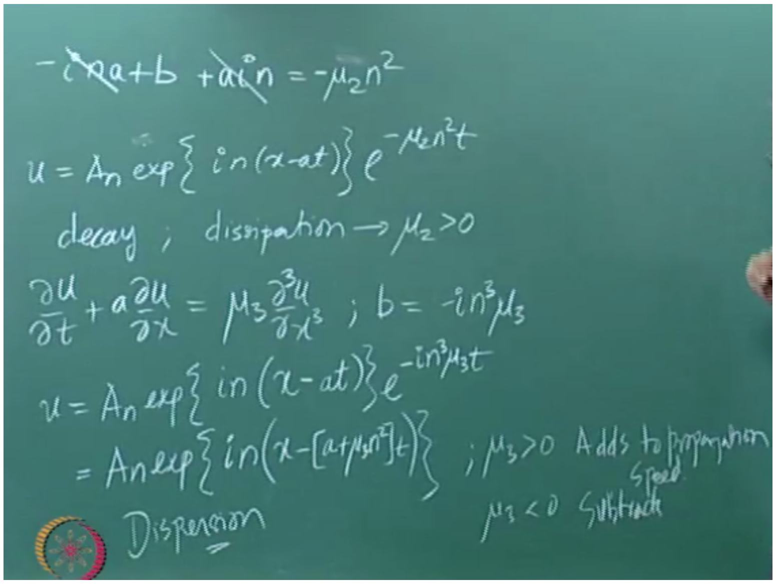

- ina + b and the dou u/dou x has an a multiplying it so that will give me a + a in which is a whole point which is why our original equation was satisfied = - mu 2 * n square that is b. This tells me that for this equation that we have for this equation this tells me that the b here should be - mu 2 * n. So what is that mean. What is the implication of that? u I will just write it out is An exponent I will just write it out again in x - at e power - mu n square t. So what does this tell us about u.

This is just bunch of sines and cosines. the amplitude depending on the n, sin next sine 2x, sine 3x depending on the n and the amplitude is An, the An does not change with time. It only changes with wave number, it does not change with time whereas, what happened now is because is being multiplied by this mu 2 n square t the solution is going to decay. I will prefer the word decay in literature very often we call it a dissipative term.

It is called the dissipative term but dissipation. The scheme is said to be dissipative so the solution decays. The addition of the secondary derivative term the solution decays given that when does it decay mu 2 has to be positive. mu 2 has to be > 0. So you get this decay if mu 2 is > 0 is that fine. So far it looks okay. There is something else that is interesting. It does not gets decay. There is an n square term here.

So basically says that higher frequencies decay faster than lower frequencies so higher frequency will decay faster than lower frequencies. If mu 2 is negative the exponential is positive the solution will diverge. So what we are saying is see remember I added this term to the right hand side because such a term appeared in a modified equation and what is the modified equation.

The modified equation is equation to which by approximate solution the solution that my program is generated. The solution that my program is generating is a solution to the modified equation. So the modified equation has this extra term has the second derivative term, the modified equation has the second derivative term and the mu 2 is negative by scheme is going to diverge and mu 2 is positive my scheme will converge, my scheme will be stable.

We have the added information about what are algorithm or the solver (DES). I want to understand by writing out this equation we are saying something about how our program is actually going to behave that we can look at the modified equation and make an observation that is what we are trying to do. This basically says that the program that you write using a scheme which has a second derivative term will cause the high frequencies to decay faster than low frequencies given that mu 2 is positive.

And mu 2 is negative high frequencies will diverge faster than low frequencies. So this is the property that we are able to get. So what happens if you have a third derivative now? Let us consider the situation where you have dou u/dou t + a dou u/dou x = and we add a mu 3 dou u/dou x cube. And in this case b happens to be from the third derivative b is - I n cubed mu 3. So the u for this equation = An exponent in x - the b also is going to have an in.

And either we can take it in the one step or maybe we will do it 2 steps let me not hurry to it. At exponent e power - I n cubed mu 3 t and we can in fact combine these terms. This gives me An exponent in x - a + mu 3 n square * t. So what was the coefficient that was multiplying the t. The coefficient that was multiplying is the physical speed with which it was the speed with which our equation is propagating the signal whatever you are modeling it was u.

Whatever it was that u are modeling, this was the speed with which it was been propagated that is p there is no changed. So the addition of the third derivative term we have a purely complex term here. It is not going to decay. There is no real component it is not going to decay, it is going to keep on oscillating added to it, it turns out that our mu 3 term actually adds to the speed of propagation that is worse.

What does it have. So of course I will say I am saying added because my mind is saying mu 3 is > 0. It adds if mu > 0 it adds, if mu3 < 0 it subtracts. So it adds to the speed, propagation speed and mu 3 < 0 it will subtract. There is reverse. The speed that it adds depends on the frequency. So in this case if mu 3 > 0 high frequency you will travel faster than low frequencies that is basically what it says not only is the function that we are monitoring, not only it is propagating at the wrong speed.

But if you decompose it using Fourier series into its component frequencies the high frequency will travel faster than lower frequency. This is called dispersion. In a technical context most of all you have first time encountered dispersion was possibly when you are looking at Newton's experiment with optics. That is most probably the first time where you saw dispersion of light and light pass through a prism at an angle.

Dispersion the fact that different name numbers are traveling at different speeds causes it enables you to see that white light may actually have a spectrum associated with it. So that is dispersion that is most probably. The first time you most probably encountered it is when you are sharing a bag of potato chips or something of that sort where you know that if you take a standard bag of potato chips the larger wave numbers.

The most interesting wave number or I should say a smaller wave numbers, the larger wave lengths, the more interesting wave lengths tend to be on the top and the fine dust tends to be in the bottom so the person who grabs the bag first gets the biggest pieces and invariably so any container if you look at you will see that the dust tends to settle if you have aggregate material. You will see that it gets there is a process of dispersion as you disturb it where the higher frequencies end up at the bottom.

And the lower frequencies end up at the top, the larger pieces end up at the top and the final dust ends up at the bottom and you get a distribution. So it is always what you known intuitively that you should grab the potato chip back to the front is a fact because simply clearly because of dispersion that high frequencies in this case, there is a separation that takes place. You would like the signal to be in certain form, but there is separation that takes place. So what does that mean?

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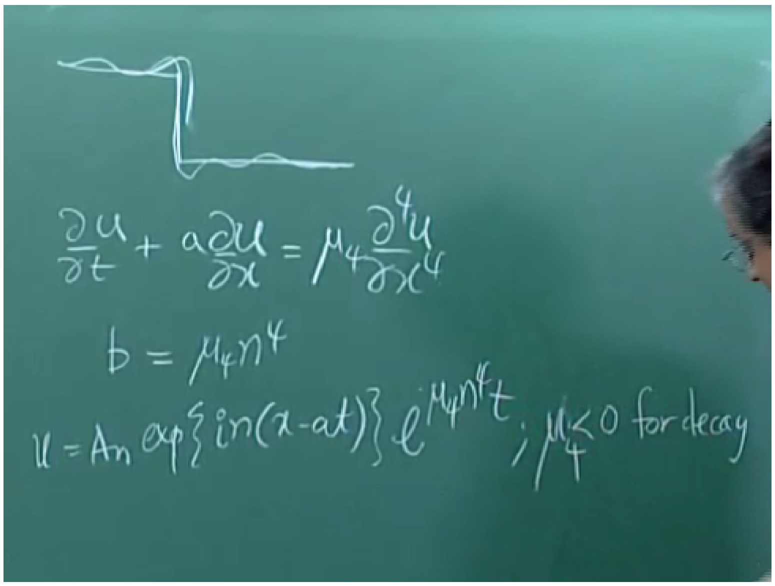

Well if you have what that basically means or if you have an initial condition which is like a step and you do a Fourier series expansion for this you will ignore ideas of Gibbs phenomenon or all that. You do a Fourier series expansion for it that as it propagates in time that we expect not to get a step, but to get something that is wavy simply because the way you do Fourier series all of the coefficients.

All the way numbers they combine together in such a fashion as to where there is a peak which you do not want it is cancelled by a higher frequency and if they are travelling at different speeds what gave you a sharp step in the beginning as it propagates because the different frequencies are travelling at different rates they will not quite add up exactly. So there is a in a sense if you look at the one that is propagating at the right speed.

There is a phase error that is introduced to the system, some of them are not in the right phase, they are not all adding up together is a phase error that you are accumulating that basically what happens. So that is dispersion, what about fourth derivative. dou u/dou t + a dou u/dou x = mu fourth * dou fourth u/dou x to the fourth. We do not have to go through the whole process b = mu 4 n power 4. So this is just like the second derivative.

There is a difference now. So u is An exponent in (x - at) e power mu 4 n power 4 * t. So for stability we require you will have stability is mu 4 is < 0. Mu 4 has to be < 0 for stability for decay. It is not < 0 it is going to diverge. I want to emphasize what we are doing here though we seemed to looking at these equations individually with these terms thrown in looking at this behaviour you can look at your modified equation and then determine.

What is the behaviour of your program that is the idea? This has an n power 4. This is a lot worse. You can just look at. If you consider wave number 2 versus wave number 1, 2 power 4 is 16. If you look at wave number 1 versus wave number 10, now you are talking about exponentials raise to the power 10,000 this is really high frequency you have just completely being eliminating extremely rapid.

But again we have the same story that high frequency is decay faster than low frequencies, but because you have an n power 4 involved here and the decay occurs then mu 4 < 0. If mu 4 > 0 it is going to cause a divergence and high frequencies will then diverge faster than low frequencies. So this seems to tell us that may be you have worried about the high frequency content. What about the modified equation?

Do you remember the modified equation? You do not remember it. Just to recollect. You remember that I said that in order to in the modified equation you have time derivative terms and to eliminate the time derivative terms you should really use the modified equation itself to eliminate the time derivative terms and all the mix derivatives that come because of that. What I did in class last time I just basically use the governing equation.

There is a reason why I did that because we need that equation also though it is not really the true modified equation. We need that equation also. What I will do now is I am not going to do FTBS originally I thought maybe I will do FTBS I am not going to do FTBS. I will save a little time. I got myself a chit sheet from my book which I have got a little table of all the terms second derivatives so you have seen second derivative, third derivative, fourth derivative.

What do you expect to be the nature of the fifth derivative term if there is one, you expect it to be dispersive so for the linear wave equation if you add to the right hand side and there is somehow you add to the right hand side even derivative terms the sign may change, but it is likely to be decay if you pick the right coefficient and if you add odd derivative term it is not going to decay it is likely to be dispersive.

Whether high frequencies travel faster or travel slower will depend on the coefficient that term has that has to be determined. So just looking at second, third, fourth you can guess because we can see that we get I square, I cube, I to the fourth and so on so it is very clear that it is going to become real imaginary real, imaginary and the sign will keep flipping. So that is very clear. The process is obvious. So what is going to happen? Let me just write out the slip. Let me just write out this table for you. What I am going to do is?

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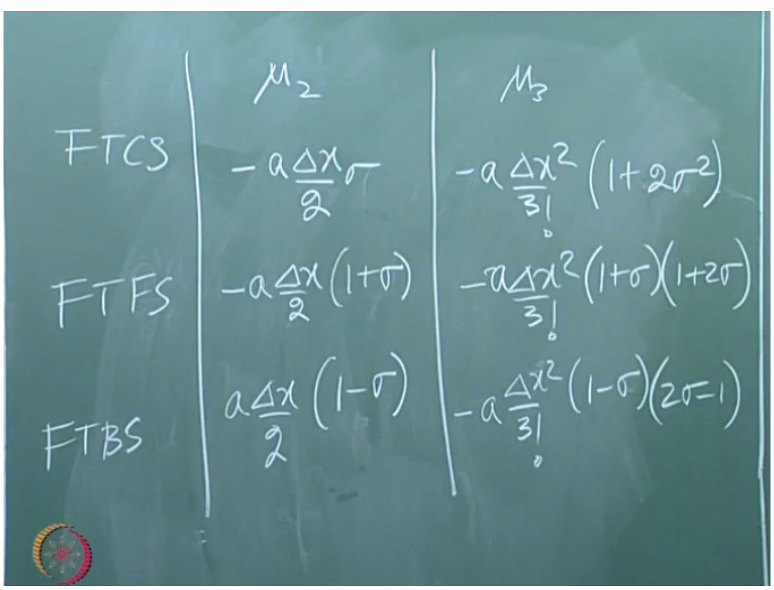

I am going to write forward time central space, forward time forward space, forward time backward space and this will look different from what I did last time, because in this case I am actually using the modified equation to eliminate the time derivative term. Is that clear? Okay. So the coefficients that I get for FTCS will look different. I will show you. When I do a demo I will show you easy way by which you can calculate, but otherwise it is quite, it is a pain, I mean you have to do it.

I have done all of these manually because even though I have done them using a symbolic manipulator I did not trust the symbolic manipulator. I actually did it manually, but I will do now is I will have here let me first write mu 2, so this is the mu 2 term for FTCS. So you get a - k delta x/2 * sigma. What is sigma = a delta t/delta x. So where possible I will write it in terms of sigma because sigma is something that seems to be significant for us.

It showed up in all our stability conditions and so on. So you have to write it in terms of sigma then it is possible. So then FTFS mu happens to be - a delta x/2 that is the same for all of them 1 + sigma and FTBS is a delta x/2 1 - sigma. So right here, you can see it is negative unless something happens to-- this is negative, that could be positive which is what we got, this is 0 < sigma < 1 we got a stability condition it is possible.

You may not always get the stability condition out of the modified equation or you may not always get the perfect stability condition of this modified equation, but in this case it actually worked. So it really shows that whether mu 2 is negative you get unstable situation. If mu 2 is positive you get decay and the solution of the algorithm is stable. So what we did with the stability condition we actually got here.

What about mu 3? mu3 is a bit of mess so that is going to - a delta x square/ 3 factorial 1 + sigma, 1 + 2 sigma square and a - a delta x square by 3 factorial I could have most probably saved a little space by keeping all the constant a delta x square. Those tend to be the same. 1 + sigma * 1 + 2 sigma + 1 or 1 + 2 sigma and the last one is - a delta x squared by 3 factorial 1 - sigma * 2 sigma - 1.

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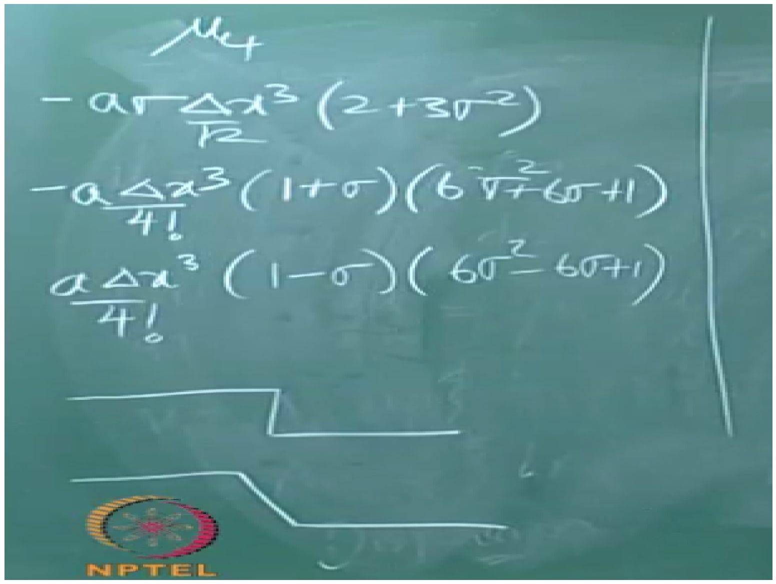

I am writing it a little big if you do not mind. I will write the third column here so that is mu 4. So you get a - a sigma * delta x cube/12 * 2 + 3 sigma square. So first thing just do not take this down you are happy with it. You should try to check it at least few of them. They are enough of you that you can take bits and pieces and check it out. The other thing is you should at least do the using the wave equation instead of modified equation to eliminate the time derivatives to see what is the difference between what I have given you.

And what you would get and you will get a little practice actually doing the modified equation. It is a good idea for you to do. So then you have - a delta x cube by 4 factorial * 1 + sigma * 6 sigma square + 6 sigma + 1 and FTBS are our star performer amongst the 3 of these is a delta x cube/4 factorial 1 - sigma * 6 sigma square - 6 sigma + 1. That is the full table. So you will look at this.

This basically says that for all of them you have dispersion and depending on what happens so depending on what values that you take the sign can actually change it is possible for the sign to change. In some case is not and in this case what happens you have to see whether what whether sometime travel what whether it is high frequencies travel faster so I will allow you to work through this to see what happens.

So you have FTFS, FTBS, all of them can be dispersive and you have mu 4 these 2 have negative quantities so you say wait a minute if it is negative they should be stable, but apparently it is not enough to stabilize apparently it is enough to stabilize and that is likely because of the n power 4 which is more stabilized than the wave number is close to 1 and what we have here finally. Here we have positive.

It is supposed to be destabilizing, but apparently it is not dominant enough to actually create problem for us. So we have got the second derivative, third derivative, fourth derivative looking at this I would suspect that may be the higher derivative is not really that important as the first order conclusion that I come to if you are going to write it out I have got actually run programs yet.

That if FTBS actually does deliver, we can conclude that the higher derivatives may be on that important. We may fine use for them later, right now that looks they are not that important. Any other observations that we can make if you said sigma = 1 for FTBS it looks like all the terms in the modified equation is 1 - sigma. If you said sigma = 1 it turns out by the way, it is a fact. If you write out the term the general terms in the modified equation all of the terms will have a 1 - sigma.

So in the case of FTBS if you said sigma = 1 your approximate solution is a solution to the exact equation, the original governing equation. That is nice, but remember what does that mean? See you can just say I am solving the original equation. We are solving the original equation, but our function is still represented only using a 11 grid points or 100 grid points or 1000 grid points. We still have a computer representation of the function.

That function is being propagated at the right speed our approximate function is being propagated at the correct speed. Our approximate function is being propagated the way it is supposed to be propagated by the wave equation. That is all it means. If sigma = 1 the modified equation for FTBS becomes identical to the original equation. So the approximate solution actually satisfies the equation it is just that it is an approximate solution still.

Am I making sense? Because, it is representing using a finite number of points. If you are to try to represent that step function that I talked about earlier using a finite set of points what you would actually get would be a ramp. What you would actually get would be a ramp, will be a value here and the next value will drop and it is somewhere in between there is a step. What you will actually get this is about the best that you can do.

If you are going to be using linear interpolates if you are going to use Hat function it is the best that you can do. So the actual representation would be a ramp. All that says is if your initial condition is a ramp anything is fine. If your initial condition is a step, then there is an approximation in the representation of that step. Is that okay everyone. So what we have seen is that we have the modified equation.

I have written it in a strange fashion, but we have the modified equation for all 3 of these. You can try out the modified equation what I have let out BTCS and as I said you can use the wave equation itself and little exercise just try it out make sure that you are able to do it, get the modified equation for the rest. Are there any questions? Questions everything is fine. So, then what we have now. I will go to this board here. What we have now. So the question is we have seen that because of mu 4 we have seen that because of FTCS having this term in the up front.

Because FTCS has this term of front that the coefficient being negative that is the reason why it is not working that is the conclusion we draw from this analysis and in this case for sigma between 0, and 1 this coefficient being positive that is the reason why this thing is working. So it seems to be a case for up winding that is one thing. In other thing we can ask the question then what actually is the difference between the 2 after all, all you are doing is in the forward time central space part you have.

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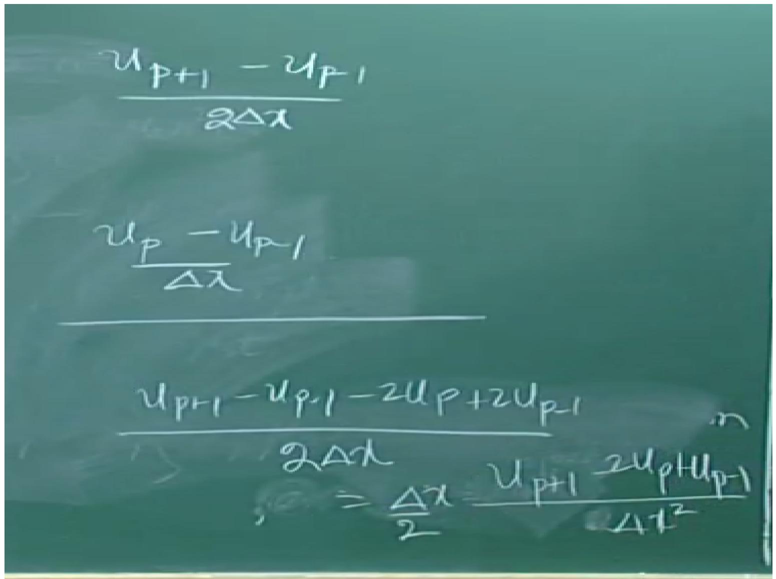

Up + 1 at time level q - up - 1 at time level q/2 delta x. I have left out the time level q and in the case of FTBS what you have is up time level q - up - up - 1-time level q/delta x. Really between the 2 scheme is actually the only difference between the 2 schemes that is only different and the modified equation changes in the sense that second derivative term the sign of the second derivative term changes.

So what happens if you subtract 1 from the other what is the difference between the 2 let me ask the question that way. What is the difference between the 2? So the LCM is 2 delta x. ut + 1 - up - 1 - 2up + 2 up - 1 and this gives me up + 1 - 2up + up - 1/2 delta x. I can multiply and divide by delta x. If I multiply and divide by delta x, it gives me delta x I take the 2 back here delta x square, that looks familiar. That is the second derivative. So this sort of brings a little closure here.

So the difference between the modified equations and the difference between what you are doing here between central differences and forward difference and backward difference if you are doing up winding what you are effectively doing is you have taken central differences and added some kind of dissipation. This is the exact amount of dissipation that you have added which, cause the change here modified equation is that fine? So there is a school of thought that basically says why not add the dissipation of ourselves the 2 ways that you can pose it.

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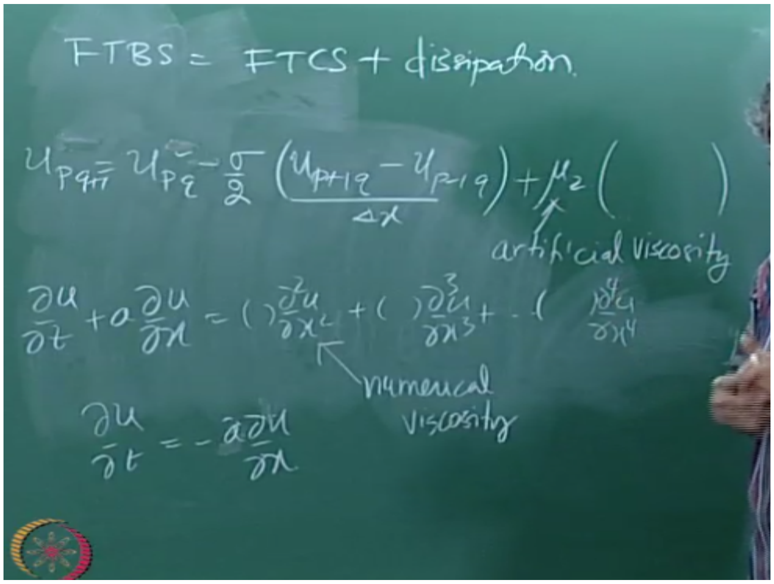

So basically what we are saying is FTBS = FTCS + dissipation. So it is almost as though by possibility that any scheme can be written as FTCS + add on terms. So that is one possibility, but right now we do not go that far. We basically say I can take FTCS and I can add any amount of dissipation that I want to do it. I can take FTBS and add any amount of dissipation that I want to it. That is what it seems to say, I can take FTCS and add any amount of dissipation I want to.

So you can just basically say up q + 1 = up q - sigma/2 up +1 and then you can decide to add yourself, you can add some kind of a term that looks like a mu 2 * a second derivative. Because after all I mean see you are not solving the original equation. You are only solving the modified equation. So why not pick the modified equation that you are going to be solving that would be the argument, why not pick the modified equation that you are going to solve?

So if you know the modified equation you can knock out whatever terms that you want in the modified equation that is 1 way to look at it or you know the modified equation it has some structure that you do not like you can specifically modify that structure to something that you like. Now you can ask the question is it possible for me to take FTCS and just eliminate the dissipation is it possible for me to take FTCS and eliminate a term?

Is it possible for me to take I should restate that properly? Is it possible for me to take the modified equation of the FTCS and do something to FTCS so that 1 term disappears? The second derivative term disappears the third derivative term disappears. In order to do that if you want to do it sequentially that is if you want to eliminate the second derivative term, you want to eliminate the third derivative term that term that you use.

The term that you add here in order to eliminate a specific component of the modified equation just so that you understand what I am saying dou u/dou t so it is not in the air I am not saying this term, that term and so on = and there is a dou square u/dou x square + dou cubed u/dou x cubed + dou the fourth u/dou x to the fourth and so on. Let us see we just decide that we want to get rid of that term. You want to get rid of this term or you want to get rid of these 3 terms.

But you are going to do it starting it the second derivative you are going to eliminate terms from the modified equation that means I have to add something to FTCS so that these terms disappear. There is something that I want to think about now. To determine what is the term that needs to be added? You have to use the modified equation derived from the original equation. You have to use the modified equation derived from using the original equation in eliminating the time derivatives and not the modified equation derived from using the modified equation.

Do you understand what I mean when I say modified equation, derived using the modified equation? Modified equation has time derivatives and spatial derivatives. They eliminate the time derivatives using the modified equation; they eliminate the time derivates using the original equation. If you want to eliminate any term in this, you should use the equation. I am not going to give you an answer here. You think about it.

You have to use the modified equation that comes from this and not the modified equation that comes from that. You just ponder on it and you will see why it is that I told you to derive the modified equation using this term. So you can actually systematically add terms here. To answer the question what term shall I add here in order to eliminate this? You should derive the modified equation using the original equation to eliminate the time derivative term.

Is that clear? You have to go back that statement maybe a little confusing. You will have to go back look at how you derived the modified equation and you will understand what I mean. So it is possible for us to eliminate it. It is possible for us to actually add it. If you were to deliberately add a second derivative term it is referred to as artificial viscosity, then Navier-Stokes equations has the second derivative term which is viscosity.

This looks like that so this is called artificial viscosity. The term that shows up in you modified equation naturally because you just did a discretization is called numerical viscosity. So this comes from your discretization u discretized it so you end up with a term a second derivative term which is not there in the original equation. So it looks like your equation has become viscous when it is actually not viscous so that is numerical viscosity.

The term that you add deliberately because you want to fiddle at the modified equation that is called artificial viscosity. Then we say we have said now see I just casually made a stage that any amount of artificial viscosity that you want is that true what is the consequence of that. What is the consequence while adding any amount of artificial viscosity that you want?

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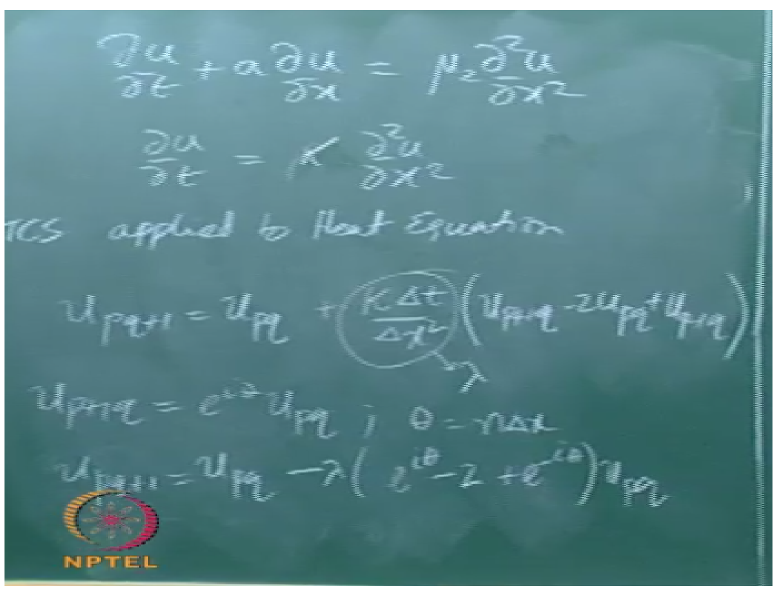

What is the consequence of adding? So if you add a large amount of artificial viscosity what happens? Say a propagation speed is 1, 1 meter per second if you want units 1 meter per second and since I have said that your artificial is caused it can be anything I add mu to 10 power 6. Is that a concern should you be concerned? See now you really totally change this equation. You no longer solving the linear wave equation.

You are solving something that looks closer to a heat equation. a is so small, this is almost like dou u/dou t = mu 2 dou square u/dou x square of course you do not normally say mu 2 there we use the term kappa for thermal conductivity, thermal diffusivity actually, but anyway I will use it some conductivity, thermal diffusivity. So it is essentially degenerate to this equation. What is the stability condition associated with this equation? If you have to apply FTCS to this, apply to heat equation, somewhat over rapid fires occurs here.

So upq + 1 is upq + kappa delta t/delta x square. I am not really doing this for memory I am sort of working out in my head you can work it out in your head. up + 1 q - 2 upq + up - 1q. There is a term here that looks similar to what we had earlier like sigma we called as lambda. So we will substitute our exponential. We will go through that same process. So up + 1 q is e power I theta * upq where theta was n delta x or n delta x/l n delta 2 pi n delta x however we define our interval.

See I have not even bothered. Now the fact that it is a local analysis is very clear I am not even saying how large may domain is. I am not even bothered. It shows up as theta and the theta will appropriately vary. So this basically becomes up q + 1 = upq + lambda * e power I theta - 2 + e power - I theta * upq and therefore from our earlier stability analysis.

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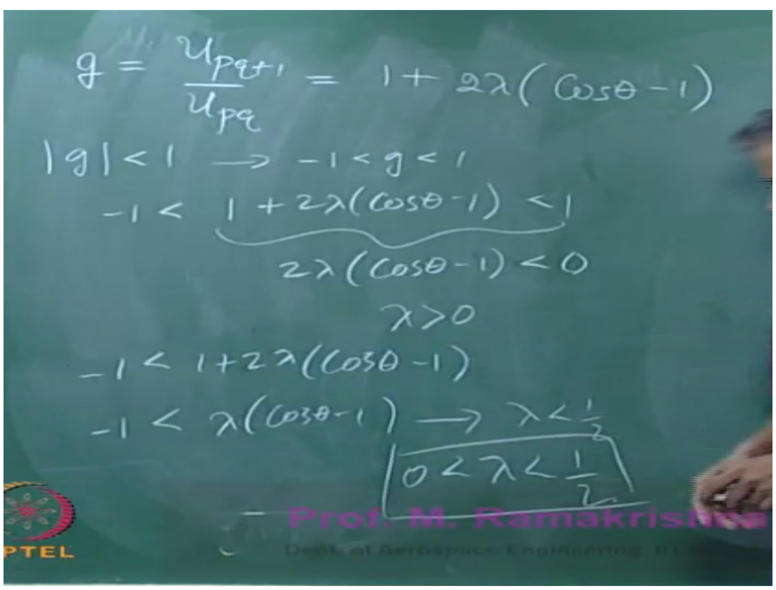

Therefore, g which is upq + 1/upq again across 1-time step, the amplification across 1-time step is 1 + what do I get e power I theta is e power - theta 2 cos theta so that gives me a 2 lambda * cosine theta - 1 and I want mod g to be < 1. What is that tell me. This is real. Mod g < 1 tells me - 1 < g < 1 or - 1 < 1 + 2 lambda cos theta - 1 < 1 which way do you want to do this. You have to take an inequality at a time and see what happens. So if I look at that one that tells me 2 lambda * cos theta - 1 < 1 or lambda should be > 0.

What is the other condition give me? - 1 < 1 + 2 lambda cos theta - 1 and do I mean there are different ways by which we can do this or - 1 < lambda cos theta - 1 I took this - 1 over there - 2 divide it through by 2 I skipped a step. Then lambda is so what does it tell us about cos theta - 1 what can happen what does this constraint that this gives me lambda is < 1/2 we can come back to this 0 < lambda < 1/2 we can constraint that we get.

We will come back to this. We will do this with a 2-D context. We will come back to this. So I will see you in the next class.