

**Introduction to Computational Fluid Dynamics**  
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**Lecture - 18**  
**Modified Equation**

So, in the last class we did FTCS, FTFS FTBS right, we saw that the first 2 are unconditionally unstable, we saw the third one was conditionally stable. And we sort of quickly did BTCS, which I will look at again one more time right, and when we saw that BTCS was unconditionally stable okay.

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BTCS :  $\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 ; a > 0$

Stencil diagram showing points  $P_{i,q+1}$ ,  $P_{i+1,q+1}$ ,  $P_{i-1,q+1}$  on a horizontal line and  $P_{i,q}$  below  $P_{i,q+1}$ .

$\frac{\partial u}{\partial t} : \frac{u_{P_{i,q+1}} - u_{P_{i,q}}}{\Delta t} ; \frac{\partial u}{\partial x} : \frac{u_{P_{i+1,q+1}} - u_{P_{i-1,q+1}}}{2\Delta x}$

$u_{P_{i,q+1}} + \frac{a\Delta t}{2\Delta x} (u_{P_{i+1,q+1}} - u_{P_{i-1,q+1}}) = u_{P_{i,q}}$

stable for all  $\sigma$

$|z|^2 = \frac{1}{1 + \sigma^2 \sin^2 \theta} < 1$

So it is basically where we are, so backward time central space BTCS, I will just so that you know all applied to  $a > 0$  okay, that is the equation that we are looking at. And this stencil just to be clear, the stencil looks like that okay, so this would be the  $\Delta t$ , that would be the  $\Delta x$ , and we decided to call the new time level  $q+1$  staying consistent with what we are doing before. So this is  $p_{q+1}$ , I will just go through it quickly okay that is fine  $p+1$   $q+1$ ,  $p-1$   $q-1$ ,  $p_q$  okay.

And the difference between what we did FTCS and so on, and this is that the equation is being now represented at the time level  $q+1$  that is the difference okay, please remember you have to keep that in mind, after a point once you are starting to do these different discretization and different schemes, it is easy to forget where you are representing the differential equation. So the differential equation is being represented at the point  $p_{q+1}$  okay.

So then it is straight forward, as I indicated  $\frac{du}{dt} u^{q+1} - u^q / \Delta t$ ,  $\frac{du}{dx}$  is  $u^{q+1} - u^{q-1} / 2 \Delta x$  okay, substituting back into our differential equation gives us  $u^{q+1} - u^{q-1}$  or maybe I do not skip a step here whole points. So substituting back and what we are going to do is we are going to keep all the new time level  $q+1$  on the left side of the equation, and everything else is going to go on to the right hand side.

So I get  $u^{q+1} + a \Delta t / \Delta x$  so  $2 \Delta x u^{q+1} - u^{q-1} = u^q$  fine, and again we identify this sorry wrong one that is  $\sigma$ . So we have gone through this stability analysis for this, and we have shown that it is unconditional stable for all  $\sigma$ , because that gain mod  $g$  or mod  $g$  squared, I am not going to go through the whole thing I just wanted to get to this point okay mod  $g$  squared was  $1 / (1 + \sigma^2 \sin^2 \theta)$  right, and we wanted that to be  $< 1$  is that fine okay,  $\theta = 0$  it is 1, but otherwise it is strictly  $< 1$ , is that fine right okay.

And as always I am not really concerned about the DC component right okay, are there any questions? **“Professor - Student conversation starts”** This is for one particular  $n$  right  $g$  or we can say this like from  $g$ , no this is like, it should have it should be  $g$ . You say that  $\sigma \text{ mod of } g n^2$  should be  $< 1$  right, because we cannot just take this one wave number and I mean we have to take, the system of equations is linear.

So essentially we are looking at the it is like saying that if you say  $\sigma = 0$ , it is like you are saying that the  $x$  component goes, the system of equation is linear right so you can just look at the individual components. Finite dimensional vector space so just having each component of being finite does not necessarily mean the norm of the whole vector is finite? Well no that you have seen there is an issue here, we are just basically saying that you know there are more serious issues right.

I am just basically if you are going now go to the fact that it is an infinite dimensional vector space, the original space fine because they are not gone to the limit right, this is definitely finite dimension. Because I cannot represent anything over a certain largest wave number, so in this case for the numerical scheme that question does not I mean it is not an issue right. I am going to take 10 grid points, I know 5 is the largest wave number, I take 10 intervals 5 is the largest wave number I can represent, so that is it, that finished with that.

However, if you are talking other issues if you talking about it from an analysis point of view, there even you know like if I say take the sin theta component, if you actually go through the process you will see that you will be trading summations infinite sums with there are 2 limit to infinite process that you will be exchanging. so then the issue of uniform convergence and so on all those also become an issues.

So even before you get to this there are other issues that you have to hand, am I making sense okay right. So if you want to look at it from a mathematical analysis point of view there is a problem there, which I am not looking I am not paying attention to any of that right, because I never going to go to that infinite process right. So really you will not hear me typically if I say uniform convergence yes then I am talking about SOR, I will say something about it is not uniform convergence it is not interesting right.

But in this case you want uniform convergence, if you are actually going to an infinite sum and you are going to say that I am going to exchange the differentiation right. If you substitute you expand in terms of Fourier Series, and you want to substitute it into the derivative. If you expand in terms of Fourier series, and you want to substitute it here, if I want to trade I want to exchange the derivative with the infinite sum.

Then there are certain properties that I need, uniform convergence being one critical property that I need okay right. But we are not going to do that, but it is an important consideration I am not saying it is not, it is an important consideration, but we have sort of in a finite dimensional space okay, so in that sense it does not matter fine okay. And this has a consequence to something that I am going to say later in class, this has a consequence, but we are in the finite dimension is that fine okay right. **“Professor - Student conversation ends.”**

In this case and like we always do in mathematics or in any of our mathematical physics are right deriving our governing equations. The wave number was chosen arbitrarily, that is I just basically saying for an arbitrary wavenumber  $n$  for an arbitrary wave number  $n$ , what does this do? And then you ask is there any particular wave number for which it is very bad right. So and it turns out that there is no sigma, see that is basically what it is.

Because  $n$  is what do you call it there is the  $n$  is embedded in here, there is no sigma for which this thing is going to blow up, there is no combination of parameters physical parameters that we have as a choice right, either  $\Delta x$ ,  $\Delta t$  or of course given the speed of propagation for which this is going to any of the wave numbers will diverge okay, is that fine, that is basically what we have.

So it is unconditionally stable okay, now we have got that you know in fact yesterday I said great we have finally got something that is unconditionally stable, because a bit disappointing you do you FTCS you expected to work and it did not work right. So it is very nice that it is unconditionally stable, but at what price? What is the price? If I look at FTCS and look at this equation it is sort of messed it up, so I can rewrite it.

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BTCS

$$-\frac{\Delta}{2} u_{p-1,q+1} + u_{pq+1} + \frac{\Delta}{2} u_{p+1,q+1} = u_{pq}$$

Implicit Scheme

FTCS

$$u_{pq+1} = u_{pq} - \frac{\Delta}{2} (u_{p+1,q} - u_{p-1,q})$$

Explicit Scheme

$$u_{pq+1} = u_{pq} + \frac{\partial u}{\partial t} \Delta t + \frac{\partial^2 u}{\partial t^2} \frac{\Delta t^2}{2!} + \frac{\partial^3 u}{\partial t^3} \frac{\Delta t^3}{3!} + \frac{\partial^4 u}{\partial t^4} \frac{\Delta t^4}{4!}$$

$$= u_{pq} - \frac{\Delta}{2} \left( u_{pq} + \frac{\partial u}{\partial x} \Delta x + \frac{\partial^2 u}{\partial x^2} \frac{\Delta x^2}{2!} + \frac{\partial^3 u}{\partial x^3} \frac{\Delta x^3}{3!} + \frac{\partial^4 u}{\partial x^4} \frac{\Delta x^4}{4!} \right) - u_{pq} + \frac{\partial u}{\partial t} \Delta t + \frac{\partial^2 u}{\partial t^2} \frac{\Delta t^2}{2!} + \frac{\partial^3 u}{\partial t^3} \frac{\Delta t^3}{3!} + \frac{\partial^4 u}{\partial t^4} \frac{\Delta t^4}{4!}$$

So if I say that now I am going to write it in a slightly different form, so that we get it okay. I have written it in this form, so that you can see that this actually forms entries in a tridiagonal matrix, the diagonal term is  $u_{pq+1}$ , that sub-diagonals are  $p-1$ , the super diagonals are  $p+1$  okay. And when you come to the, again if you take the right, this is a time level  $q+1$ , so when you come to this point clearly you need the boundary condition there okay, that will go over to the right hand side.

And when you come to that point when you come to computing the value peak  $p$  is this value, then you need  $p+1$  okay, so this is a boundary condition that is required because of our numerical scheme choice of numerical scheme, I chose central difference because I wanted more accuracy. So we need to do something here, there is a boundary condition that needs to

be specified at the right hand side, which is not required as part of the either physics or the mathematics okay.

More we get a system of equations okay, so you may say what is the big deal so you get a system of equations. But if you solve the system of equations what do you get? You have taken one-time step, I want you to understand this, you solve the system of equations you have taken one-time step. Now you get ready to take the second time step, and you have a whole system of equations again okay, as it turns out because these coefficients are constant.

It is not a nonlinear equation right, this is the set up where something like Lu decomposition will do well right, so you are pre-decompose then now you are left with only forward substitution and back substitution okay. So this is the situation where a scheme like Lu decomposition works well okay. You do some work any work that you do before hand on the matrix, this work that you do not have to do later right, because the matrix has not changed as long as it does not depend on these variables, that is one way to do it, it is a direct method.

Direct methods have their own problems and the matrices grow very large okay, because if you do look at the process of forward substitution and backward substitution, you will see that the last entity that you determine is based on the series of computation that you have made on the previous  $n-1$  quantities okay. And therefore, it is prone to error if you take if you are solving  $1000 \times 1000$  system, and you have 1 million unknowns.

Or in 3 dimensions you are solving right  $100 \times 100 \times 100$  system which is still 1 million unknowns, then the last quantity that you calculate the 1 millionth quantity that you calculate comes as a consequence of calculating all the previous 99 lakhs whatever 999999 am I making sense right. So it is too much right, so the error the round off error tends to accumulate. So for that reason I typically do not use direct methods this is a personal opinion.

I typically do not use direct methods for matrix, that are if there are banded matrix like this maybe have to about  $1000 \times 1000$  right. And if they are not, if the dense matrix maybe about  $200 \times 200$ , this is the personal opinion you understand what I am saying. If you give me a matrix when I say dense matrix all the entries essentially all the entries are non-zero right.

You are doing panel method or something of that sort okay  $200 \times 200$  that is about the largest, otherwise I immediately switch to iterative schemes, which is why I started you off from Gauss Seidel okay. If it is a sparse matrix like this maybe I go up to about  $1000 \times 1000$ , 1000 unknowns but once it goes beyond that I will always go to an iterative scheme. I may use this Lu decomposition as an initial guess fine.

In this case however, if you are wondering why the heck is this guy being so much, in this case however, with iterative schemes there is an advantage. What is the good guess for  $u^{p,q+1}$ ?  $u^{p,q}$  right, you have a good guess, if your  $\Delta t$  is reasonable size right you have a good guess already, you have  $u^{p,q}$  as an initial guess am I making sense. So you already have something with which you can start right.

So it may not be that bad, but the fact of the matter is that you have to solve the system of equations and when you are done solving it, you have taken only one-time step, I want you to remember that. So there may be some effort that has to be put into the solve this other than just the schemes that I have told you fine, there are ways by which you can accelerate these things, I am not going to spend time on that, there are ways by which.

So this really gives you a system of equation that is the key, point that I want to make is this gives you system of equations, and because this equation appears together this expression these terms appear, the quantities that you want appear together act together is called an implicit method fine it is called an implicit scheme. I will say a little more about implicit schemes, explicit schemes as we go along you will see that I will illustrate various kinds of situations that occur.

So this is called an implicit scheme because you get a system of equations where the unknowns are tied together in a linear fashion, and you cannot you have to solve system of equations to get it okay. It is the reason why right in the beginning I said some of the mathematics that you want to learn implicit function there is one, so it will tell you when you can. In this case of linear system of course the system just has to be invertible right.

But implicit function theorem will tell you in general, if you have an implicit expression when you can invert okay. Implicit scheme as supposed to I just write it just for the sake of this is FTCS, this is BTCS, this is an explicit scheme. Explicit, because I have the expression

that I want explicitly occurring on the left hand side, no further work need to be then okay fine, are there questions okay. So now we get to this point that we are talking about earlier.

So I have 10 intervals we have already shown that the largest wave number that we can represent is basically 5 right okay, so bear that in mind that is so if I have something that as if I have  $\sin 10x$ , we have already seen in the demo of that  $\sin 10x$  right  $\sin 20x$ , all of these  $\sin 15x$  all of them are basically going to look the same. You have only 11 grid points on which you are sampling the function okay, so bearing that point in mind.

I asked the question now, these equations approximate the original linear wave equation, I go through the effort of solving the system of equations, and I end up with a solution, the solution is an approximation to the solution to that equation that we want to solve okay. So I am going to ask an ill-posed question basically, to what equation is it an exact solution? The solution that I am calculating here is an approximate solution to the problem that I am trying to solve.

To what problem is it the exact solution? Put another way, what is the problem that we are solving? I want to solve the linear wave equation but I am making an approximation, I am solving some other equation actually. When I say I am making an approximation, so I am sort of turning around and then saying, well I have this  $u$ , I have this function right and if this function is if this solution this approximate solution is close to the exact solution, then I must be solving a problem that is close to the exact problem.

And we solving a differential equation that is close to the exact differential equation what is that equation? Does that make sense? Does that the question make sense? I say it is ill-posed because I just told you if I have only 11 grid points, I have these values at 11 grid points, we already know that there are so many an infinity of functions that it represents okay. but though it is ill-posed still the worry questions to ask, because the answer is interesting okay, am I making sense is that clear right.

So the questions we are asking is I have an approximate solution to the equation that I want to solve, to what equation is it the exact solution? We want that equation okay, is that fine right. So what we do is we do a very simple thing maybe I will start off since I have FTCS here I will start off with FTCS right, and we will go back to our friend Taylor series, we will use

Taylor series expand this about the point FTCS we were representing the equation at the point  $p_q$  that is important okay.

This is what we say you have to constantly remind yourself, so in FTCS we are representing the equation at the point  $p_q$ , so we will expand Taylor series is always this where are we are presenting the function about which point are we expanding right. So we are going to expand about the point  $p_q$  okay. So let us do  $u_{p,q+1}$ , so  $u_{p,q+1}$  is  $u_{p,q} + \frac{du}{dt} \Delta t + \frac{d^2u}{dt^2} \frac{\Delta t^2}{2} + \frac{d^3u}{dt^3} \frac{\Delta t^3}{6}$ , so for do you want to go maybe we will add one more, is that fine.

And this supposedly equals  $u_{p,q}$  which is the first term  $\frac{\Delta t^2}{2}$ , this is going to get a little messy  $u_{p,q+1}$ , so is this  $u_{p,q} + \frac{du}{dx} \Delta x + \frac{d^2u}{dx^2} \frac{\Delta x^2}{2} + \frac{d^3u}{dx^3} \frac{\Delta x^3}{6} + \text{the 4th derivative}$ , this part is okay,  $-u_{p,q}$  -or+ be careful the -sign there,  $p-1$  so it is  $-\Delta x$  okay,  $+\frac{du}{dx} \Delta x - \frac{d^2u}{dx^2} \frac{\Delta x^2}{2} + \frac{d^3u}{dx^3} \frac{\Delta x^3}{6} - \text{the 4th derivative}$ , it is a mess but lots of things will cancel well enough things will cancel, not lots of things will cancel but enough things will cancel.

First of all, the  $u_{p,q}$  goes away, the second derivative goes away, you may ask why you are doing this we did it for right, we did it when we are doing the derivation of the different scheme, but anyway humor me, 4th derivation goes away, is that fine anything else,  $u_{p,q}$  here goes away okay. I think that is it we are stuck, we cannot do anything else here, so I will rewrite that that tells me that right, I will rewrite that.

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Modified Equation

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} \frac{\Delta t}{2} + \frac{\partial^3 u}{\partial t^3} \frac{\Delta t^2}{3!} + \frac{\partial^4 u}{\partial t^4} \frac{\Delta t^3}{4!} + \dots$$

$$= -a \frac{\partial u}{\partial x} - \frac{\sigma \Delta x^2}{3! \Delta t} \frac{\partial^3 u}{\partial x^3} + \dots$$

$$\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x}; \quad \frac{\partial^2 u}{\partial t^2} = -a \frac{\partial^2 u}{\partial t \partial x}$$

$$\frac{\partial^3 u}{\partial t^3} = -a^2 \frac{\partial^3 u}{\partial x^2} = +a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^4 u}{\partial t^4} = a^3 \frac{\partial^4 u}{\partial x^4}$$


And if I divide through by delta t that gives me  $\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} \frac{\Delta t}{2} + \frac{\partial^3 u}{\partial t^3} \frac{\Delta t^2}{3!} + \frac{\partial^4 u}{\partial t^4} \frac{\Delta t^3}{4!} + \dots$   $= -a \frac{\partial u}{\partial x}$  that is the first term, there are 2 of them so  $-a \frac{\partial u}{\partial x}$  okay, then what else do we have  $-\frac{\sigma \Delta x^2}{3! \Delta t} \frac{\partial^3 u}{\partial x^3}$  will go away because 2 of these  $3!$  factorial times delta t  $\frac{\partial^3 u}{\partial x^3}$  cubed, please check to make sure there are no issues.

And I do not think we have right the 4th derivative goes away, and no other there are higher order terms, so I should say + dot, dot, dot right, everyone. So I can easily see that I have  $\frac{\partial u}{\partial t} - a \frac{\partial u}{\partial x}$ , I mean you should expect that I am going to get that. What do I do with the time derivatives? You know I would really like to have this as because you have already seen I mean we have gone through forward time or backward time right.

Because this higher derivative in time where clearly not that easy to handle okay, so as far as possible you want to retain this as a first derivative in time, I do not like this higher derivative in time for that reason. Spatial derivatives at a given time level I have right, I know I am comfortable with that at a given time level I can evaluate the spatial derivative right, whatever order derivative that you want, that is of what is right second derivative, third derivative 4th no problem.

Temporal derivatives, it is a bit shaky because when I start off I only have one-time level at which the initial data right, so somehow I do not want this. So what I proposed to do is I proposed to replace this derivative time derivative by a space derivative right, so the game

that we play is this, so if you say the  $\frac{du}{dt} = -a \frac{u}{dx}$  this is our differential equation okay, then  $\frac{d^2 u}{dt^2} = -a \frac{d^2 u}{dt dx} = +a \frac{d^2 u}{dx^2}$ , is that okay.

Yes, at this point someone should have a complaint, does anybody have a complaint? I am proposing to replace this second derivative with this okay, I am proposing to replace the second time derivative with this second spatial derivative fine okay. So if I go to  $\frac{d^3 u}{dt^3}$  which is the time derivative of this that should give me fine okay, see if you ignore that  $u$ , this basically says that  $\frac{du}{dt} = -a \frac{u}{dx}$ , it is essentially what it is saying right  $\frac{du}{dt} = -a \frac{u}{dx}$  fine okay.

So 4th derivative would be something similar, so 4th derivative would be okay, so you end up actually end up something like this. So I am proposing to substitute this here, see okay I will tell you the complaint that you could have, the actual complaint that you could have is this equation what is this equation? This is the equation that we are actually solving, when you do FTCS right when you use FTCS, FTCS in this case happens to be unstable.

But if you were to solve or FTBS or whatever, when you were trying to solve using FTCS right you are trying to get an approximate solution to the original equation, you are getting an exact solution to this equation, this is the equation that you are actually solving okay. So this is since this equation  $\frac{du}{dt} = -a \frac{u}{dx}$ , it looks very similar to the original equation right, this is called the modified equation, this equation is called the modified equation.

But there is a point here, the point is if I am going to substitute for the derivative time derivative, the second derivative to eliminate the terms there, should I not use the modified equation instead of the original equation, you understand what I am saying. If this is a modified equation, this the equation that I am actually solving, if I am going to make the substitution should I not use the modified equation to do it instead of the original equation? The answer is yes you should right.

I am going to use the original equation for 2 reasons: one doing this in eliminating the terms is not more difficult right, so that if you want you can look it up, when my book comes out you look at the book whatever, you can work through it. Because the trouble is you will get a

lot of mixed derivatives, it is a very messy equation right, you can make the decision that you will only keep terms up to the 4th derivative.

And every time you differentiate this you know that for instance this term will go away, once you differentiate with respect to time, this will become a 5th derivative you can drop it right it is you can there are things that you can do to simplify. But still using this equation is a messy process okay. And there is a second issue that will become clear later, that using this actually by itself has a certain advantage okay.

You will see what I mean when I will explain that when I come to the end, using this by itself has an advantage. So instead of using the modified equation to eliminate the time derivative terms, which is what you should technically do, I am going to use the original equation to eliminate that right those terms, am I making sense? Is that okay. Simply because, a: on the blackboard it is easier to do right.

B: because that equation that you get is still the useful equation, are there any questions, **“Professor - Student conversation starts”** Is it Taylor series expansion only if the function is analytic, but here we? No, we have an issue that is how many terms of the Taylor series see you get to that you cannot in a sense get to even any of our fine differences, you understand for the differential original differential equation **“Professor - Student conversation ends.”**

We have to ask the question right if the Euler equation supports shocks discontinuities, then how did you right, so that goes into a different realm right, that goes slightly different. So maybe we do not, we will assume that we can do this and get away with, in fact even for the Euler equations if you do quasi 1D flow also you will get shocks right, we will look at this you will get shocks.

And we will use finite difference methods and you will use the differential equation representation that actually do through and solved okay right. So then issue is why does it work or there is okay, so that theory is something that that is sort of outside the scope on this course fine right. Because if you get look at it from a point of view theory of differential, the way I usually handle it as I always go with integral equations.

I do not usually solve them in the differential equation right, but even otherwise there is a whole theory of weak solutions and on right. So when we say a solution we are saying the solution is something that you can substitute into original differential equation to verify whether it satisfies that equation or not. But if the solution has discontinuity, how do you substitute it into a differential equation okay.

So you have to do something weaken that differential equation, so that is a there is a whole mathematical machinery that goes with it. So we are not really going to maybe at the end of the semester if we have time we will get to it okay right. So at this point we will just leave it at this, and basically say that you know I will sort of repeat something I should right in the beginning that I can take as many as derivatives as I want right okay.

I can take as many derivatives as I want, and typically I will stick to 4th right okay. So I will substitute this back there, and we will see what we get, let me.

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Modified Equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = -\frac{a^2 \Delta t}{2} \frac{\partial^2 u}{\partial x^2}$$

as  $\Delta x \rightarrow 0$   
 $\Delta t \rightarrow 0$   
 $\Delta x, \Delta t \sim h$

$$\left( \frac{a^3 \Delta t^2}{3!} - \frac{a \Delta x^3}{3! \Delta t} \right) \frac{\partial^3 u}{\partial x^3} - \frac{a^4 \Delta t}{4!} \frac{\partial^4 u}{\partial x^4} + \dots$$

$\Rightarrow \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$  Consistency

$\hookrightarrow u^h \rightarrow u$   
 $R \rightarrow 0$

So I get my original differential equation  $\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} =$  and what does with equal is there a second derivative term  $-\frac{a^2 \Delta t}{2} \frac{\partial^2 u}{\partial x^2}$  okay, then what else do we get, do I have a third derivative term -and that is a + so I get a cubed delta t squared/3 factorial, I think I got that right -sigma delta x cubed/delta t and then there is a 4th derivative terms -a to the 4th delta t/4 factorial, is that fine okay, you can check whether I have.

In fact, in this case, so this is something that I preferred to do my modified equation, in this case you can actually factor out of the  $\Delta x$  cubed right, this will turn out to be this, you will get you can write this expression in terms of  $\sigma$  in the parenthesis, did I miss the sign somewhere? The first one is  $-a$  squared, a squared  $\Delta x$ , okay that is fine. And there are infinite in such number of terms, we are not going to look at as I said I am not really bothered right.

We may be even able to write a general expression of looking at this for the even terms and odd terms, if you write this in terms of  $\sigma$  that pattern may become more obvious, but it does not matter, it is material for me right now. What matters to me right now is that we have this modified equation okay, modified equation I have it in terms of the original expression=all the extra terms, and we ask the question what happens when  $\Delta t$  and  $\Delta x$  go to 0? Okay.

And again when you have 2 such things going to 0, there is always an issue as to whether something is going faster, so we want  $\Delta t$  and  $\Delta x$  going to 0 in a similar fashion okay,  $\Delta x$  and  $\Delta t$  go to 0, all of these individual terms go to 0. The right hand side all the terms go to 0 is that fine everyone, as  $\Delta x$  and  $\Delta t$  go to 0 in comparable fashion right, basically that I am saying that the ratio of  $\Delta x$  and  $\Delta t$  does not change for instance.

In a comparable fashion as  $\Delta x$  and  $\Delta t$  go to 0, this equation the modified equation goes to the original equation, as this goes to this becomes  $\frac{du}{dt} + a \frac{du}{dx} = 0$ , the modified equation goes to the original equation okay. So as the number of points that you are using to represent the solution increases, which is what  $\Delta x$  going to 0 means and as the time steps that you take in evaluating it go to 0 right.

So as your representation gets more and more accurate, the modified equation goes to the original equation which is the good thing, we may be happy if that happens. This property is called consistency, this scheme is said to be consistent, so you say I took a differential equation I discretize the equation, I used Taylor series in this case I discretize the equation I end up with a modified equation.

If in the limit my  $\Delta x$ , when I did the finite difference scheme I said I am not going with the infinite process, if in the limit of doing the infinite process I actually let  $\Delta x$  and  $\Delta t$

go to 0, yes my discrete equation goes to the original equation the scheme is said to be consistent okay. You may be under the impression, you can say wait a minute we substituted finite difference method, individual derivatives the definitions work, how can you come up with the scheme that is inconsistent? Right.

It is possible as someone who has invented schemes that are inconsistent, it is actually possible, terms can cancel put them together, there are terms that can cancel and you can be left with it, you can be left with the term that just refuses to go to 0, do you understand right. So it is possible for you to invent the scheme that is inconsistent right, and you should suspect that I mean we invented a scheme which will not work.

FTCS which is modified which is consistent, it is consistent and it will not work right, it is possible actually for you to invent the scheme which is inconsistent it is possible. So this consistency is very important okay, so as  $\Delta x$  and  $\Delta t$  go to 0 modified equation goes to the original equation, the scheme is said to be consistent, as  $\Delta x$  and  $\Delta t$  go to 0. If  $u$  will write  $u^h$ , where  $\Delta x$   $\Delta t$  are like  $h$ , they are like  $h$ .

And going back to that remember that notation where I put the superscript  $h$  means that it is the discrete solution, if in the limit  $h$  going to 0  $u^h$  goes to  $u$ , then we have convergence am I making sense, as  $h$  goes to 0 our candidate solution approximate solution converges to the exact solution, I do not want you to get this confused with our iterations converging that is the different convergence that is a sequence of solutions that are being generated which converge.

This is the sequence of solutions with different  $h$ 's with different grid sizes, and as I change the grid sizes it converges, am I making sense. So you have consistency, you have convergence, we have stability of the scheme, there are 3 things that are there okay. And there is a famous theorem you can go look it up call the Lax equivalence theorem, which basically says if you have 2 of them you have the third okay, consistency, convergence, stability.

There are 3 properties for the scheme that we have seen so far okay, and they tell us how is our scheme going to behave, they tell us if you generate a solution are you generating does the solution converge the original right, does that converge to a solution, does the

approximate solution converge to the solution. There is the equation that you are solving for the scheme does the equation converge the actual equation okay.

We are to a large extent we are satisfied with this okay, but if you are doing mathematics formulae you would want to know not just that we use converge, you also wanted to know that the derivatives converge the  $\frac{du}{dx}$  converges and  $\frac{du}{dt}$  also converges okay right. We are as engineers a little more course we say okay you converges we are very happy right.

And when you are doing your actual calculations, once you have a scheme you say well somebody has looked at consistency convergence stability and all that, and you tend to just take the  $u$  for what it is right, but this is an issue okay, is that fine consistency, convergence, please remember in this context convergence what it means? Okay as  $h$  goes to 0  $u_h$  goes to the actual solution, and of course we have seen stability fine, that is all very nice.

Something else come out of this, we had a form of a solution for this equation, it looks like if you do this you get these extra terms. If you look at the modified equation you get these extra terms on the right hand side, is there a way for us to infer an analytics solution? We had an analytic solution proposed analytic solution for this in terms of Fourier series and so on. Is there a way for us to get an analytic solution with something on the right hand side? Am I making sense.

Is it possible for us somehow to figure out the way that we say I have this, I know the solution to this if I had something to the right hand side is it possible for me to get a solution? Is that fine okay. So we can I will just basically start us off and on that. What we are going to do is these are called semi inverse techniques, in the sense that we are going to guess the form of a solution, and try to find some disposable coefficient that we have thrown in, so that it becomes the solution right.

**(Refer Slide Time: 45:24)**

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = \mu_2 \frac{\partial^2 u}{\partial x^2}$$

$$u = A_n \exp\{in(x-at)\} e^{bt}$$

$$\frac{\partial u}{\partial t} = A_n \exp\{in(x-at)\} e^{bt} (-ina + b)$$

$$\frac{\partial u}{\partial x} = u in; \quad \frac{\partial^2 u}{\partial x^2} = -u n^2$$

$$u(-ina + b) + a u in = -\mu_2 n^2 u$$

$$b = -\mu_2 n^2$$

Because we already know that for  $\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$ , we already know solution of the form  $A_n \exp\{in(x-at)\}$ . I am writing only one term, I am writing only  $u_n$ ,  $i n 2 \pi$ , we will drop the  $2 \pi$  and  $l$  okay, we will just for now if  $l$  is  $2 \pi$  it takes care of it right, we will drop the  $i n x - a t$  right, just to keep the chalk dust to a minimum we will drop the  $2 \pi$  and  $l$ , we will just assume  $l$  is  $2 \pi$ . We know that if we substitute this in there it works okay.

So if instead of being 0, what if it is pick one of those terms? What if it is I chose  $\mu_2$  as a coefficient  $\frac{\partial^2 u}{\partial x^2}$ , what is it that form okay fine. Can you guess a solution for it, what did Laplace this looks like second derivative is like you know, so Laplace was smoothing, I would expect some kind of and from our physical intuition which is why to trigger that intuition I called it  $\mu$  right like viscosity, and expect that it is a right?

So anyway we have oscillation, we will see something can happen, see there are different ways by which we can argue this, remember we are only trying this out right, that is basically if it does not work it does not work, we are only trying this out. So I would say that I throw an extra parameter  $b$ , and look at the candidate solution  $u$ , I mean I should write  $u_n$  but I am not going to write  $u_n$  right of that form.

And ask the question if I substitute this into this equation, I say considered a solution see this is how you do it now know that we have guessed this, is a consider a solution of this form substitute that in there, and see if you can actually get a value  $b$ , and does that value mean anything, is that fine okay. So what is  $\frac{\partial u}{\partial t} = A_n \exp\{in(x-at)\} e^{bt}$ , you want to do it in one shot or you want to split it?  $e^{bt}$  times the derivative of the thing right.



I am basically using chain rule -i n a+ b is that okay, this is the derivative of term -i n a+ b. What is  $\frac{du}{dx}$ ? Instead of writing out the whole thing I will just write as u now that we know it is u times i n okay, substituted back there, so you get u times -i n a+ b. So sorry I need one more what is  $\frac{d^2u}{dx^2}$  = right. The u of course cancels, the i n a cancels with the i n a, I have a, it is not a b, so b is, is that fine everyone okay.

This is called a semi inverse method, you sort of guess the form, you have a few disposable coefficients you substitute it, and see if you can fit the coefficient, so that it turns out to be the answer okay, is that fine. We will come back to this, now we know that it is possible. What does this mean? What does it mean for FTCS? What does that mean for FTBS? Can be explained what we got, is what we are going to look at okay in the next class, fine, okay thank you.