Introduction to Computational Fluid Dynamics
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Lecture - 11
Laplace Equation - Convergence Rate

What we will do is we will look at, let me explain what we are trying to do now right, so that we get the context right. We know that the error in the solution en+1 satisfies this is determined by this iteration equation is that right. Now what we have is we are going to, this error corresponds to a function exy, which is error in the original solution okay.

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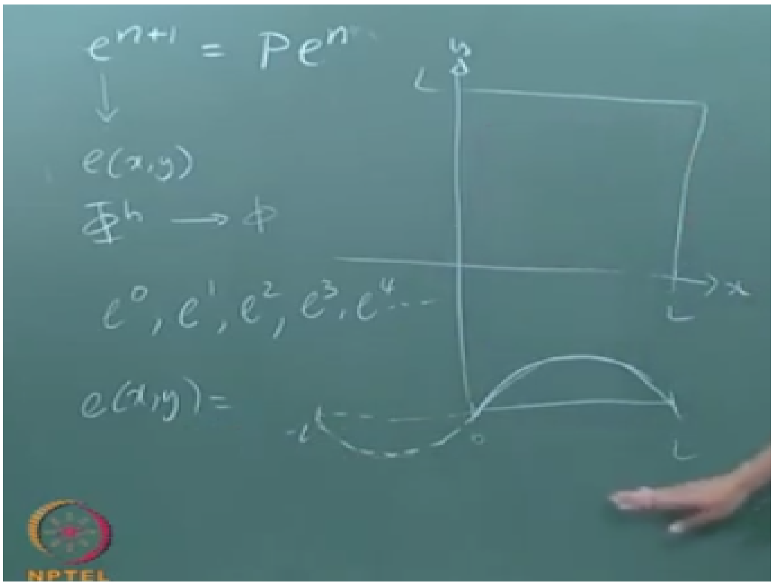

So remember that every time that you have phi h, there is a corresponding phi, right, there is a corresponding I would say, though they are set of points, it actually corresponds to a function, it is a function, because we can use linear interpolants to find the value at any given point, am I making sense, okay. See as I said before in an earlier class you are supposed to be giving me a solution to Laplace equation.

And if I say that I have a square and in this case we have actually gone from, we have taken an L/L square, for the sake of this discussion we have gone to an L/L square. If you say that I have gone to an L/L, if you are saying that you give me a solution to Laplace equation in the square, right, that means I have a right to pick any point and say what is a function value at that point and you have to give me a value.

You cannot say no it is not one of my grid points, I cannot give you a value that is not an acceptable answer, then you have not given me a solution, anywhere in this L/L square if you say that you have solved Laplace equation satisfying the boundary conditions that I have prescribed, then you must necessarily give me a value, right, at any given point, you understand what I am saying.

Which is why we talked about the HAT functions and all those ways by which given values at nodes you are actually able to give me intermediate points that was the objective, okay, that is the reason why that we know that there is an underlying function representation, though I am giving you nodal values, is that fine, so exy corresponds to that. Now what we are asking, the question that we are asking is, if I go through this sequence en+1 is phi times en.

If I keep repeating this, I will generate a sequence, which is e0, which is our first error that we make when we assume the solution, then you have e1, e2, e3, these are not e as an 2 points on one, this is e as an error, right, so e4 and so on, so the question that we have is does the sequence converge? What we are proposing to do now in today's class what we are proposing to do is we are going to expand this using Fourier series, okay.

The original equation, this is the linear equation. Fourier series is a linear combination of science and co-science, okay, right, so if I would substitute the Fourier series, any mode, any single mode to the Fourier series, if I ask the question what does this equation do to any single mode to the Fourier series, right, if I pick a general mode and ask the question what does this do the amplitude of that mode?

Is first wave number, or the second wave number, or the third wave number, right, I can address that question individually because the equation is linear, is that fine, okay, because this cropped up, so, essentially what I am going to do is, I am going to write e of x, y as the Fourier's, but it is in 2 dimension, the Fourier series in 2 dimensions and I know that the boundary conditions are 0, because this is the error.

So the value of the error on the boundaries is 0, because the boundary conditions are applied exactly, so it is nonzero only on the interior, okay, so if you do Fourier series normally you have to do something called periodic extension and all that, because your initial mode if you think about it just in one dimension if this is x, that is y, just in one dimension your initial mode sin x will be something like this, okay.

The initial mode sin x will be something like this, it is 0 here, it is 0 there, your error can contain for example your error could just be this, okay, so you have to able to represent this, so this goes from 0 to L, what we normally do is we extend it to –L and there is a function on that side that we are not going to worry about because I do not care it, it is not in my problem domain.

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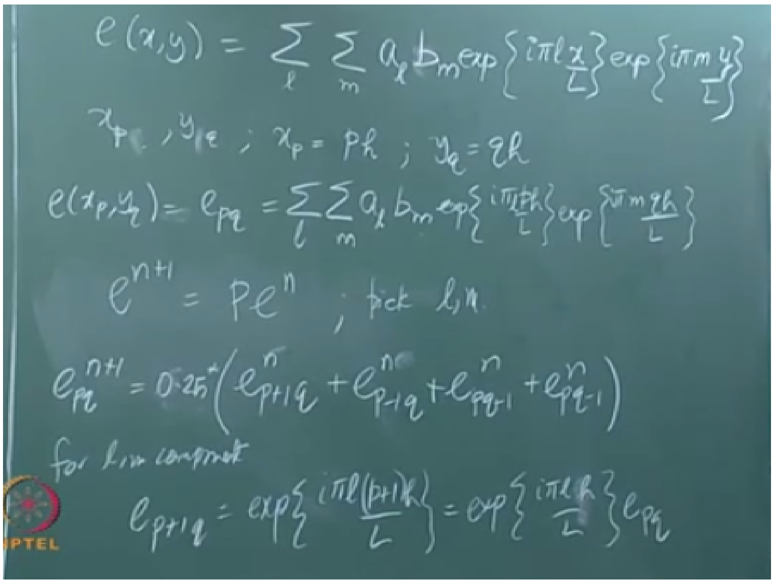

So as a consequence this e of x,y, I am just saying this so that you can go back and look up your Fourier series, right, and figure out where it is, okay. There is a consequence, I can write e of x,y as double summation because it is in 2 dimensions al bm exponent i pi lx/L and exponent i pi my/L, is that fine, everyone and the summation right now, this summation I will just write l and m.

The summation if you go look up your Fourier series, the summation actually goes from –n to +n or something of that sort, okay I will just write l and m, you go check this out, okay because there are grid points that go because I have extended it to –l, there are grid points that go the other side if I started numbering this at 1 this will be -1, -2 and so on, but we do not go and get into that, I do not want to get into that, okay.

So I just leave it as m and l, okay, fine. Now where do I want to, at what points, but I do not though this is a continuous function, I planned to substitute it into that iteration equation, right, which is a discrete equation and that is going to be evaluated at my grid points, which are xpq and ypq, so because it is a uniform grid, I do not need the q here, nor do I need the p here.

So xp is p times h, h is the grid size and yq is q times h, right, so this turns out to be exp, so I will write e of x, p, y, q, if I write that as epq turns out to be summation over l, summation over m, al bm exponent i pi x/l, x is ph/L and i pi m, there is an L here qh/L, fine, now we have already seen, so ep+1q what is the relationship between ep+1q and epq? So the difference will only be a e power i pi h, right.

You can work that out, so ep+1, so it will be turned out to be an e power i pi h/L times let me write exponent, I do not want to write e power because I am already using e for exponent, i pi h/L * epq is that fine, right, now because they are equivalent rules h/L is known, so I have to have the summation, okay, I do not want to do this, okay, I am doing this a bit early okay, let me do this, I have to take component by component first, okay.

Now this e I want to substitute into en+1 = p times en, okay because this equation is linear, this is linear, I can swop out the sums and I can swop out the double summation and so on and component by component, I can actually extract out the a's and L's component by component by appropriately dotting into the appropriate e power appropriate sign and co-sign, right.

There are all orthogonal to each other and so on, so in fact it is enough if I can take any one wave number and ask the question what is the effect that this iteration equation has on that wave number, okay, I have an orthogonal set essentially, I can ask what happens to any one of them, is that fine okay, so the objective of course is we want to find out which wave number is decaying the slowest.

You want to find out which of these wave numbers is decaying the slowest, so I can pick an l and an m, so for an arbitrary l and m, so I pick l and m arbitrarily and I ask what happens to that component, so if all the others was 0, right and that the error was in that form. There are different ways by which we can argue this, all the other coefficients was 0 and only the l and m components were left.

How would that l, m component grows, okay that is another way that we can look at it, okay, how would the l, m, if I had only this, how would this component grow, is that fine. There are different ways that you can look at this, so that is one possibility that you can ask the question how would that l, m component grow, so I should in theory also I add a subscript right, I should in theory also I add right.

So exy, I should also add a subscript saying that I am going to do the l, m component, but just so that I do not make it too complicated, I will just leave the, we know that we are dealing only with the, I have picked an l, m component, okay. So what does that do, let substitute it into our equation and see what that does, so epq at n+1 is 0.25 times ep+1 q at n + ep-1 q at n+1, we will leave this at n + epq+1 at n +epq-1 at n, is that fine.

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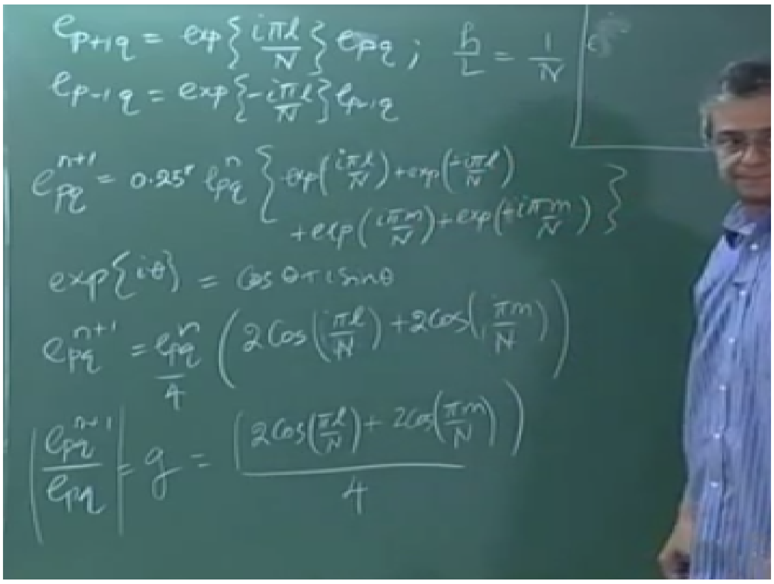

So for the l, m component, for l, m component what do I have? ep +1q in fact is exponent i pi lp+1 h/L = exponent i pi l h/L * e power pq, is that fine, okay, so we get these expressions ep+1q = this exponent i pi lh/LS N and epq, ep-1q = exponent –i pi l/n ep-1q and similarly you get the other ones with +m and –m, the wave numbers will change, so ep+1q going back to our iteration equation epq n+1 is 0.25 times, is that fine.

epq at n * exponent i pi l/N + exponent – i pi l/N + exponent i pi m/N + exponent i pi – i pi m/N, is that okay, is that fine, I have replaced h/L by N here okay, h/L is 1/N is that fine, h/L is 1/N. Now we will use the fact that e power i theta exponent of i theta, since I have called the error e, I pay the price for that right, so exponent of i theta = cos theta + i sin theta substitute back here and you will get epq n+1 remember this is for wave numbers l, m.

We are just asking the question what happens to the wave numbers l, m, okay, = epq n/4 of course * 2 cos pi l/N + 2 cos pi m/N. So the gain going from one iteration to another iteration the gain is basically the ratio of these amplitudes, okay, so it is epq at n+1/epq, which is g, this happens to be real, so we do not have to worry about, right, normally if g were complex, then we would have to do gg bar or something of that sort, it happens to be real.

So we want only the gain and we want the modulus, right, the absolute value and this is mod of 2 cos pi l/N + 2 cos pi m/N/4, is that okay, everybody with me. Now we ask the questions what is the value, right, when is this maximum, I want the maximum possible value. Now it turns out the error, see the error is 0 at the boundaries, we do not have electrical engineers, we would say we do not have a DC component, right.

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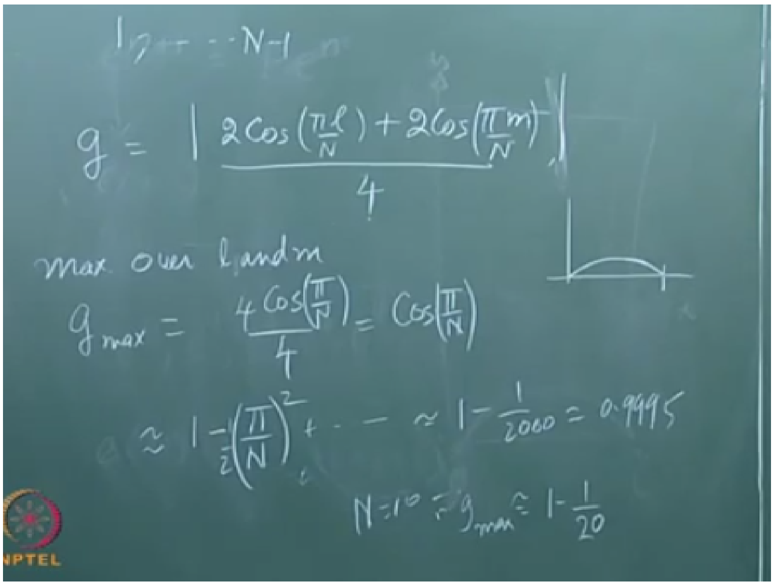

So, the summation need not start at 0, A0, we would not have an A0, or a b0 right, so the summation starts the wave numbers that we get go from one basically through n-1 that are of interest to us, okay, as I said the DC component is not there, the error is 0 on all the boundaries, so fortunately for us the expression that we have g as mod 2 cos pi l/n + 2 cos pi m, I just repeat it here /N fortunately the expression that we have mod/4.

We ask the question when does this become maximum, when does it take it is maximum value and that occurs fortunately for us it occurs when at the 2 extremes 1 and n-1, we do not really have to hunt anywhere in between, am I making sense, because if you go to 0, if at 0 was possible then the cos would be 1, if you went to pi cos would be -1, right, which are the extreme values that it can take.

We have a modulus so the sign does not matter, so I substitute 1 and this gives me the max over l and m, okay, so you see the game that we have played, we have used the fact that the equation is linear, you have considered an arbitrary wave number l and n, we have got the gain that we can get for that l and m, and now we are asking the question for which l and m is it maximum, is that gain maximum okay that is the key.

So the max/l and m gives me g max, which is 4 cos pi/N/4, which is cos pi/N, is that fine, okay, which of course I can expand using MacLaurin series, so MacLaurin series would give me and I will just use the first 2 terms, MacLaurin series, this will give me approximately 1-pi/N square 1/2, other terms I will ignore those pi/N square, okay, so this is our largest Eigen value, this is basically the spectral radius, you understand.

When I put it through the crunch once I iterated once, right, if I go from an n titrate to an n+ first titrate, n was chosen arbitrarily from an n titrate to an n+ first titrate the gain that I get is of this order, this is the largest gain corresponds to the largest Eigen value, right, it corresponds to the largest Eigen value, what is this value, so if you take say for example n is 100, what are you going to get?

We can just estimate it, if you take n as 100 pi squared is like 10, we will do an engineering approximation, pi squared is like 10, right, so the denominator gives me 1- it is of the order of 1-1/2000 that is like 0.999 is there one more 9 enough, fine something of that sort, do you understand what I am saying, okay, so for the first mode if you have an error I am drawing it only in one dimension now, we will forget the other.

If you have an error, if your initial guess is such that the error is like that, that is the first mode that is going to decay, this amplitude is going to decay at this rate, every iteration that you do that amplitude will be multiplied by this number, it is going to take forever to converge, am I making sense, okay, right, so as I said we will do, in the next class I will do a demo and you will actually see how bad it can be.

Fine, are there any questions, yes please, **"Professor - student conversation starts"** because exponential is orthogonal (()) (24:16) exponentials are orthogonal to each other and equation is linear, right, it is like the equivalent of saying that sigma f = 0 then you do statics or dynamics, you say sigma f = 0 then you can say the x component of force is 0, y component of force is 0, z component of force is 0, there you are using orthogonality, okay,

But you also need that the equation into which I am substituting, say sigma f = 0 fortunately happens to be a linear equation, right, so ijk are orthogonal, so you can do it component wise, but the equation that you are going to decompose has to be linear, if the equation is not linear then you run into difficulty, right because you can get coupling terms and all of that kinds, is that right, is that make sense.

You have to a bit careful because the ijk argument you have to be a bit careful, right, that analogy only go so far, right, be a bit careful, okay so if the equations were nonlinear in this case, right if you had u, du, dx term right which we are all familiar with from our fluid mechanics, if you had u, du, dx term, if you substitute Fourier series into it, let us forget exponentials, let us just stick with signs and co-signs.

If the u you are looking at the sin x term, du, dx would give you a cos x term right and suddenly u du dx gave you a sign 2x term, do you understand because this is the sin x cos x, okay the u du dx term contributes to the sin 2x component, so you cannot, when you say I am going to decompose it, you have to be a bit careful, okay, if you are going to decompose it component wise, you have to be a bit careful, do you understand, right.

So it is important that the iteration equation is linear, that is very important, fine, okay. **"Professor - student conversation ends"** So you can see that if you try to get any sensible, if you try to get even n=10 right it is going to give you like 1-1/20, even for n=10 you are going to end up even for n=10 g max you are going to be like 1-1/20, am I making sense, so the convergence rate, I mean it is quite, so you may be happy with how fast it runs with for n=10.

But the minute you want to say if you want to try something larger now you are talking about let me try at n = 1000 then it gets really bad, right, if for whatever reason you want to use higher resolution, then it gets really bad, okay, there are no other questions, the other thing that we looked at was writing A using the fact that A transpose = A, so we use the fact that A is symmetric, you remember when I say A what I am talking about, A is symmetric.

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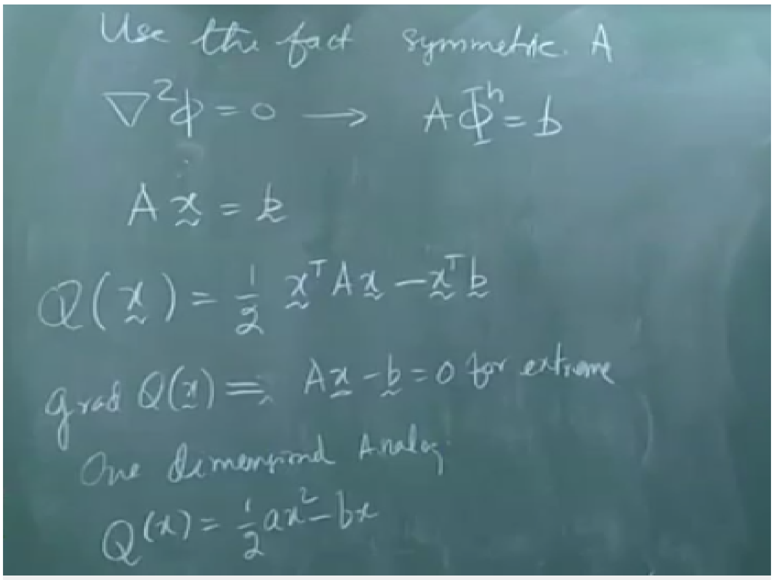

Symmetric A, right, that is this equation could be transformed to a discrete version, which was A phi = b, okay, this is what I meant, so this could be transformed into to stick to our consistent notation, this could be transformed into this and I had basically said that for consistency.

I mean not for consistency, but to make it look like a traditional standard problem that you are used to, if you just write it as ax = ax tilde = b then corresponding to this we can come up with the function q, which is the function of x tilde, which is 1/2 x tilde transpose A x tilde – x tilde transpose b tilde. I am sticking the tilde underneath just to indicate that they are vectors.

And I made it a b tilde because I have called it x tilde that is only reason why we are doing what we are doing, so that the equation is consistent and we already saw that the gradient of Q x tilde gives us because A is symmetric this gives us Ax tilde –b = 0 for extremer, I think I had a sign error there earlier, so the one dimensional equivalent of this or one dimensional analog if I just consider one coordinate, so I can have Q of x, this is just x = 1/2 Ax squared – bx.

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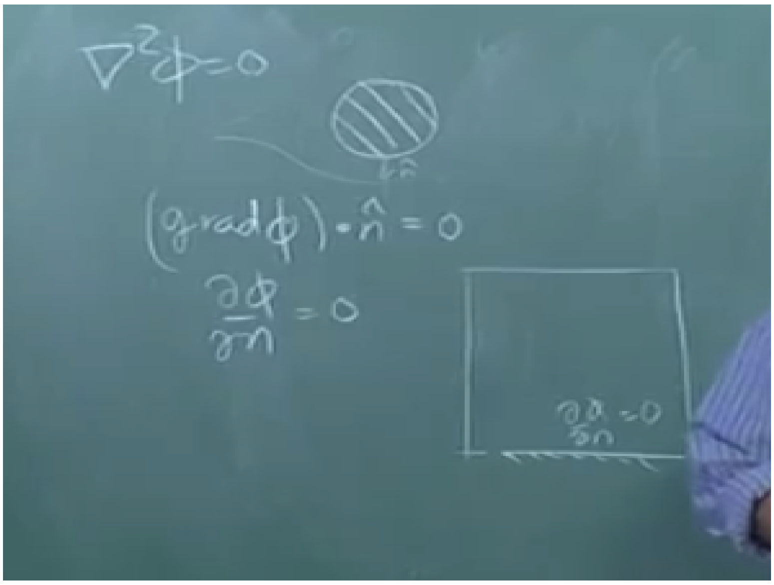

So normally when we do flow past anything, flow past cylinder or something of that sort, right, I just set up the boundary conditions and then we will talk about it later, so and we have Laplace equation for so if you say the flow is rotational, so I would not go through the fluid mechanics of all of this, right, so if this is the solid cylinder and there is a fluid flow past the cylinder, there is a flow past the cylinder, okay.

So what is the boundary condition that you have on the cylinder? They has no penetration boundary, it is called the no penetration boundary condition or the solid wall condition, right, which has the normal velocity is 0, the normal component of velocity is 0, so there is no normal component of the velocity, there is only a tangential component, okay, that is grad phi doted with m = 0, right.

And from our understanding of directional derivatives, which will need in a little while now, this tells us that dou phi/ dou n= 0. We are not going to do flow past cylinder here, this is just for motivation, we are not planning to do flow past cylinder, okay, so we will restrict ourselves to a box, but the fact that matters that anywhere on the box it is possible that your given dou phi/dou n =0 or dou phi/dou n = something, okay.

If this represents a room and there is some air coming in through a air condition or whatever it is then dou phi/dou n may give you some velocity at that point, which is a speed at which the air conditioner is injecting air into, so you may be given the dou phi/dou n value, what we have done so far is we have prescribed the phi values on the boundary, but the question is what happens if we are not given phi, but you are given dou phi/dou n, okay.

Fine, are we given these on all the boundary, are we given it only on one boundary, you might have studied this is in PDE, I am pretty sure in partial differential equation course, you would have seen Dirichlet problems, Neumann problems, and Robin problems. So Dirichlet problems you are given the function value on the boundary, Neumann problem you are given derivatives on the boundary, and Robin problem is a mixed problem.

You are given combinations of directives then function value, okay, so we have to basically see how we will handle the problems that have derivative conditions given on the boundary, right, so that is a very small segment that we will do that in the next class and we will also do a few demos, okay. Thank you.