Introduction to Computational Fluid Dynamics
Prof. M. Ramakrishna
Department of Aerospace Engineering
Indian Institute of Technology – Madras

Lecture - 10
Laplace equation - Iteration matrices

So in the last class we saw that Laplace equation discretized could be written as a system of equations. In fact, I just wrote that as A phi = b, to keep the conversation general, right, I will switch to the standard notation that you are used to which is Ax = b, I will put a tilde under the x to indicate that it is a vector, fine and also so that we do not confuse it, I mean after all we were doing xy coordinates and so on, so that we do not confuse it with the xy there.

So we are basically solving the system Ax = b, okay, and there were 2 schemes that we saw. One was the Jacobi iteration and the other was basically Gauss-Seidel, you may not have realized it, the other scheme that we basically used to solve that system of equations was Gauss-Seidel, okay, now what I want to do is both of these schemes are iterative schemes as opposed to direct schemes like Gaussian elimination or an equivalent which is LU decomposition.

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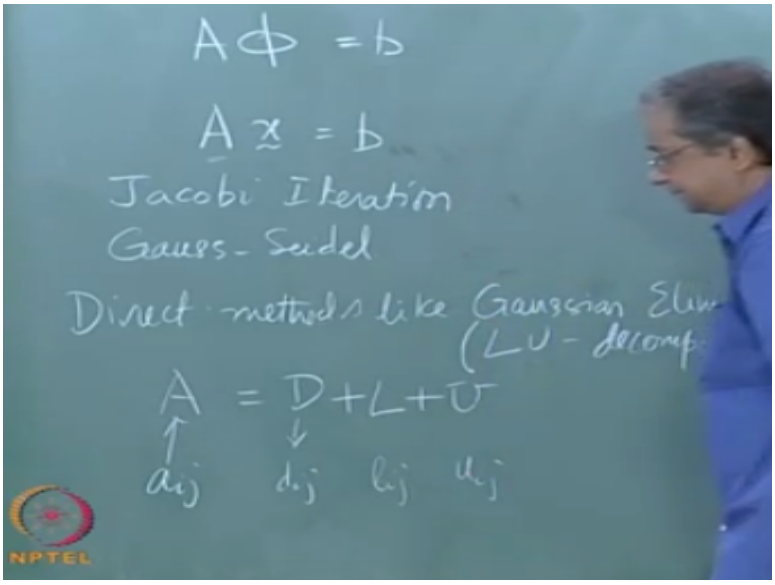

Okay, so we are not talking about using a direct method. Direct methods like we would not do this like either Gaussian elimination, we are not going to do this, or equivalently LU decomposition. You can actually show that these 2 are equivalent to each other, okay, the

elimination and the back substitution part turns out to be the same as doing the LU decomposition.

So we are going with iterative schemes, okay we are going with iterative schemes to solve this in the context of Laplace equation it is called a relaxation scheme, okay, now what we will do is, we will try to write these in the iterative form that we have actually written for phi, okay, in a recursive form that we have actually written for phi, in order to do that I need to partition A.

I want to partition A that means I want to write A as the sum of various parts as the sum of 3 parts D, L and U. This is basically where we ended in the last class, so if the entry is in A or aij, the entry is in D or dij, correspondingly you have lij and uij. I am defining my DS the diagonal terms D or my defining D: dij = aig, if i = j = 0 otherwise, so those of you have already seen upper triangular and lower triangular matrices this may just be a repeat, but just for completion let me just write this out.

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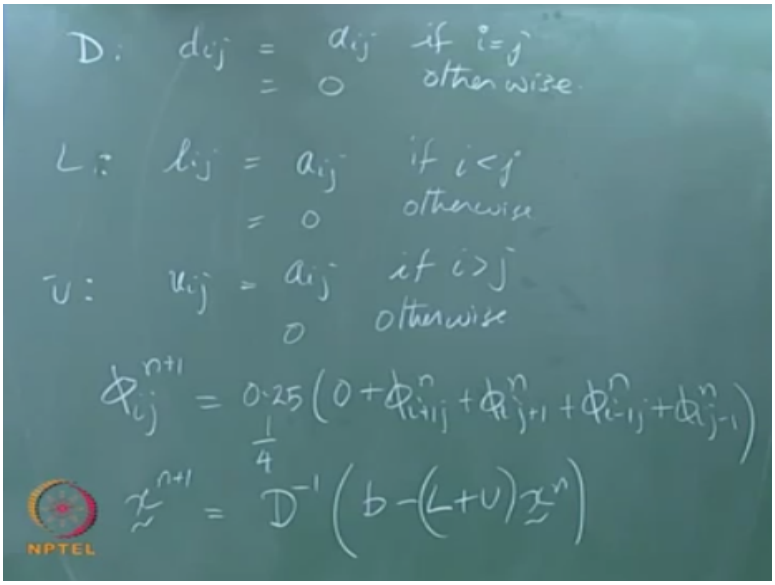

So for L: lij is aij, if i < j, okay, and = 0 otherwise, and U: which is uij = aij, if i > j and 0 otherwise. So basically this will give me the diagonal elements, this will give me the elements below the diagonal, this will give me the elements above the diagonal. Okay, so I have partitioned my matrix A in this fashion. Let us just take one equation that we did from Gauss Jacobi and see what we are talking about.

So the Laplace equation when we were doing the regular Jacobi iteration basically was phi ij at n+1 = 0.25, I will write a 0 here which was the righthand side of the equation lambda squared phi = 0 + phi i+1j n + phi i j+1 n + phi i–1j n + phi i j–1 n. This would basically be, the 0 here is basically the righthand side. The righthand side happen to be 0, okay, that is the entry in b. Okay, so if I look at this Jacobi iteration basically which is this 0.25 that is 1/4.

This is 1/4 which is essentially D inverse that corresponds to D inverse. So looking at this by inspection, I say that my x at n+1 is D inverse b – L+U times x at n, is that fine, everybody, so I can rearrange terms, I get xn+1 = sum P times xn + C, right. What is P, what is the matrix P? P = -D inverse L + U, okay, this is called iteration matrix. So P is the iteration matrix, we can write it in this form. So this is P that would be Jacobi, I put a J there P Jacobi.

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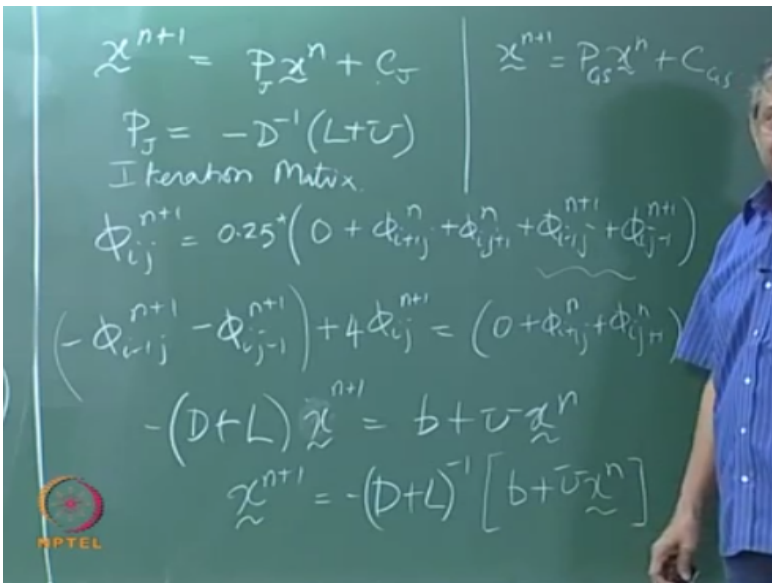

What about Gauss-Seidel, Gauss-Seidel is going to have, if we did Gauss-Seidel instead we had phi ij n+1 = 0.25 times my righthand side happens to be 0, I am just picking some arbitrary entry + phi i+1j n + phi i j+1 n + phi i-1j n+1 + phi i j-1 n+1 this is of course assuming that I am going from the lower lefthand corner right, left to right, bottom to top. There is implicit assumption on the sweep direction.

Okay, so these basically comes from the n+1 side, okay, in a sense that is like saying that we are solving, let me take these over to that side, okay, so that would be like saying, so I am undoing, whatever we did I am undoing it right to just get the matrix form, okay, so what do we have, so we have -phi i-1j n+1 or –phi ij–1 n+1 can either multiplied by 0.25 or we can get that 4 out, 4 times phi ij n+1 = 0 + phi i+1jn + phi i j+1 n.

What is this? this is a diagonal term, this is D, this is L, right so, this is L + D +L * x tilde at n+1 = b + U times xn is that fine. So Gauss-Seidel turns out to be xn+1 = D+L inverse b+u xn tilde. We have to be a bit careful with the signs, because really the way we had written 8 this would have been -4 and those would have been +1 okay you have to be a bit careful with the signs.

The diagonals in the off diagonal terms are of opposite sign in the original matrix, okay, so actually in the original matrix from the original matrix this is actually – of D+L, okay you have to be a bit careful, okay that is fine, right. Yet again we are able to write that matrix for Gauss-Seidel as xn+1 is P Gauss-Seidel xn + C, the C is of course a C Gauss-Seidel, C is also dependent on, is that fine, everybody, okay.

And the P is called the iteration matrix, we can actually figure out now that you can ask the question if you are going through we have the iteration matrix, if you are going through iterations. You are generating given the x0 you are generating the x1, given the x1 you are generating an x2 you are generating a sequence, now we have a form that relates one to the other in a simple equation.

Something at least that the chalk does looks as though it can be written simply, we can ask ourselves the question is there a way for proven analysis of this, okay, is there a way for us to figure out to analyze this problem, so does it converge, does the sequence that we get, does it converge. In the last class we saw that we could use the Cauchy's test to decide whether convergence occurs or not.

Here we will see what else we can do, so we start with this equation okay, so I will write this in this general form either whether it is Gauss-Seidel or Jacobi iteration, whatever iteration, we will see how to do the analysis for that so coming back here what we basically do is so the general form that we have is x at n+1 = P times x at n + C, this is the general form that we have is that okay.

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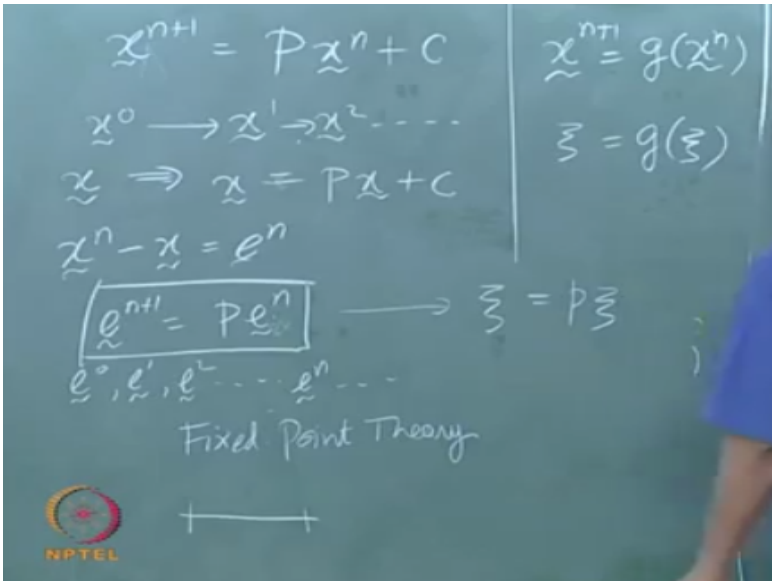

So you will guess an x not and get an x1 that is what you are going to do and then get an x2, from there you are going to get an x2 and so on. You are going to generate the sequence x. The actual solution to this equation we will represent by x so the solution that we want is x, is that fine, what do I mean by that, that means that if x is the solution this implies x = Px + C fine.

So from this equation if I subtract out the equation with the solutions that satisfies the solution, from this if I subtract out that I will get an error essentially what I am saying is if at any given time I have a candidate solution xn right, I know my actual solution x the difference between them I will define as the error en. This also happens to be a vector, is that okay, everyone, that is fine.

So if I subtract from this equation, I subtract this equation, both of them are linear fortunately I am using that fact then I get the iteration equation en+1 = P times en and by converting to en what I have achieved through what is the difference? There is a difference, the C is gone. And now you can ask the question do the sequence if en go to 0. I am generating a sequence.

So my iterations are basically generating e0, e1, e2, en and the question is does the sequence converge and you wanted to converge to 0 in this case we know earlier it had to converge to a solution and we do not know what the solution is okay in this case we know that this has to go to 0, fine. So in order to do that if this were scalar equation you do a ratio test okay, so let us look at how we go about analyzing this.

So please bear this in mind, we are going to come back to this equation now, I am not actually going to do fixed point theory, but I am going to talk about something called a fixed point theory, we are not actually going to do fixed point theory, right, but we are actually going to talk about fixed points, so in general an equation of this form, okay, so what is the fixed point because I suddenly introduced this idea of a fixed point.

So x is the fixed point or e here is a fixed point, if it turns out that we will actually get xi = P times xi, xi is a fixed point, so if you substitute xi back into the equation on the right hand side you get back xi, so when do we do this? In general, this equation in general, looks something of the form xn+1 = g of xn, g is some arbitrary function so most iterations will look like this.

We will have some function, you give me an xn, you give me a guess we perform some magic and you will get a new xn=1, so and this a particular case of that, so the question is if you have something of this nature when does it have a fixed point, that is when will have xi so that xi = g of xi, okay, I am not going to talk about it in the general context, I will still talk about it only in this context, but I want you to see that.

But I want you to see that, so that is the question that you can ask and what do g and P what do they do, given xi, what g and P do is if you take xi in the domain of definition of g you understand what I am saying, so those are the set of values for which g will give you an meaningful answer, it is defined, g is defined on those set of values, will return xi that you can plug back in that means g maps xi basically back into the domain.

The domain in the range of the same it is important, so this equation, the ways we have set it up for it to work g has to map any argument of it is own back into the domain, am I making sense? right. So g or P for instance if you say that I have some interval on which g is defined so what g has to basically do is, let us take x constraint, if we constraint x to 0 to pi/2 and I am looking at x = sin of x, xn +1 = sin of xn, am I making sense, okay.

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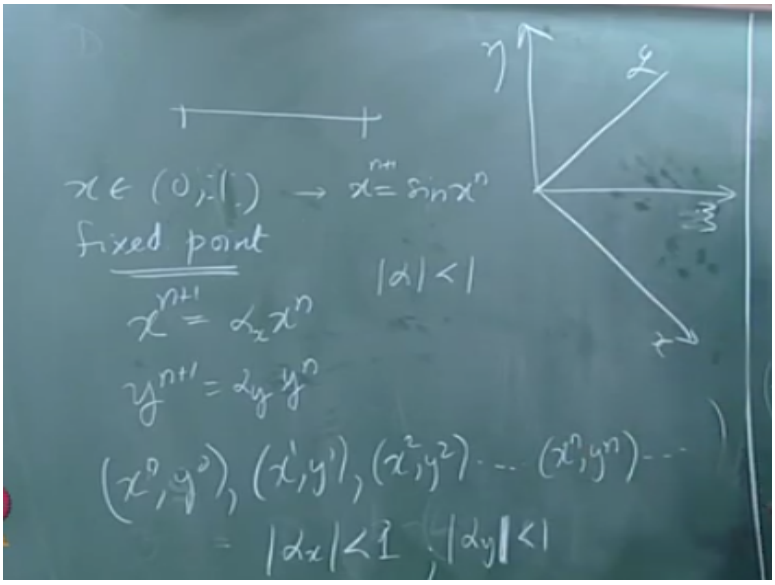

In fact, instead of pi/2 I could even taken 1, 0 to 1 then it turns out that right if I had even taken x is in 0 to 1 then this will constrain, okay, this will map back not constrain, sin x maps back into interval 01 right for this if I start off here and guaranteed that I am going to be in that interval, sin x is not going to take me out of that interval does that make sense, okay.

So first that it maps back into itself the second question that we have is, is that enough for us to be able to guarantee that there is xi which is g of xi and that we are going to get to it, okay, so the fixed point theory basically tells you when you have a fixed point as I said I am not going to really sit down and state and prove the theorem, but we will just look at the essentials of fixed point theory that we require because it is pretty intuitive.

Let us take an example similar to that but let us take a scaler so I will remove the tilde, I have an x no tilde underneath, this is a scalar, xn+1 = alpha xn these are just numbers, what will this generate for a given alpha? What is the kind of series this will generate? What is the sequence it generates? It will generate a geometric sequence, okay, so and when does that converge.

So you want mod alpha < 1, so mod alpha less than 1 will guarantee that this will converge to there is a fixed point, it will converge to 0, okay, so there is another situation where there are fixed points, if alpha = 1 then every point is a fixed point, right, if alpha = 1 every point is a fixed point, okay, so mod alpha < 1 and alpha = 1 right, both of them will work.

We will see where this takes us so it is possible that if mod alpha < 1 you will generate a sequence that converges and basically there in the sense you did a ratio test, actually doing a ratio test, this is not the Cauchy's test, okay, what if I had a second sequence, yn+1 now I will make this alpha x alpha yn, so you can ask me the question where are these sequences coming from, so these sequences corresponds.

I am not going to give you an actual problem but let us in our mind imagine that you are solving a problem and because of knowledge that I have, right, I pick a coordinate system and I pick a convenient coordinate system, I look at my problem and I pick a coordinate system, I have some knowledge right in fluid mechanics or whatever it is, I look at the problem and say this is a convenient coordinate system and in this coordinate system I get these 2 equations.

Okay, which if I iterate gives me some answer to a fluid mechanics problem it does not matter what is the problem okay, what is important is that I picked a convenient coordinate system and that I have deliberately set the convenient coordinate system at an angle here obviously, right, coordinate system is not convenient to the problem so I have picked the convenient coordinate system.

I have these 2 equations that come out of my problem just like we got a system of equation from Laplace equation, right, in a similar fashion I do some magic in my fluid mechanics or whatever and I end up with these 2 equations and if I solve these 2 equation I solved my problem that is all we need to know we do not need to know what the problem is, so I can now generate iterates, right.

So I will generate a sequence (x0, y0), (x1, y1), (x2, y2) … (xn, yn) … and when do these converge? You want mod alpha x <1 and you want mod alpha y < 1, is that fine, or in general, I can write it in a different way because I have (()) (25:48) you can say max of mod alpha x, mod alpha y <1 max form of those 2 is < 1 then that x0 y0 that sequence will converge, fine, any questions.

So if I have 3 equations instead of 2 equations then it will be max of alpha x, alpha y and alpha z and so on, okay, the largest one, the magnitude has been < 1 okay, but see in real life this does not always happen this way, normally what happens is we have a problem we may

not know of a convenient coordinate system to start with okay, so this is basic this coordinate system here I have clearly drawn it this way it is a setup, right.

In reality what you would have done and what you have seen we always do is I will draw a coordinate system this way. I will have an x prime I would not call it something else, xi eta coordinate system this way, in real life that is what would happen, you draw a coordinate system typically parallel to your page because you have no other reason to pick a coordinate system right.

If you had knowledge about the solution you would pick a convenient coordinate system, but sometimes it does not happen, sometimes we think deep, think of a coordinate system draw it and it turns out it is not really the convenient coordinate system right, we try our best, so let us say you had picked this coordinate system instead okay, but this is the underlying sequence that you are going to generate.

So how do I transform this into the new coordinate system, I just have to rotate the coordinates, in order to rotate the coordinates, I will first write that xn+1 I will write those in a matrix form because I am going to do some matrix manipulation here, right, is that okay.

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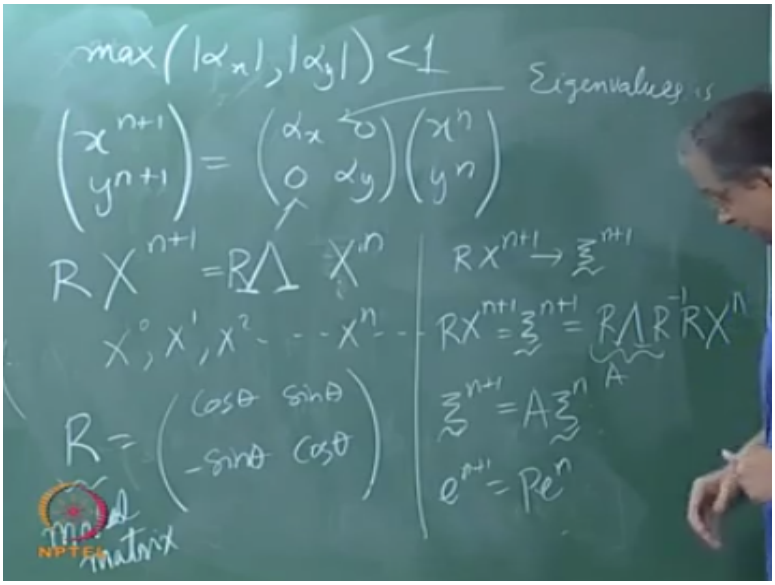

This is the same, same iteration matrix and these converge these, that is if I call this xn+1 let me make it capital lambda, this is this matrix, it contains alpha right, Xn, so the matrix form of this equation can be written in this fashion and it looks suspiciously like what we are

talking about earlier, this looks, the reason why I am doing this it looks very much like en+1 is P times en that is where we are going.

And this sequence of x is x0, x1, x2, xn converges if the maximum of alpha x and alpha y are <1 that is what we have. Now if we had picked a different coordinate system which was at an angle theta with respect to the convenient coordinate system xy if you had actually picked the different coordinate system then we can go from one coordinate system to another coordinate system by performing a rotation okay.

So the rotation matrix R basically is cos theta sin theta these are entries in the unit vector, right –sin theta, cos theta orthogonal matrix so if I take this equation and pre-multiplied by R, so I pre-multiplied by R, RXn+1 will give me xi, okay, the vector xi which consists of the entries so RXn+1, let me see, I will write back here, so this equation turns out to be RXn+1 we need to give it some nice symbol.

The capital xi does not look that good so I will use xi with tilde underneath okay just bear with the notation, right, so that if I take this equation RXn+1 = xi n+1 = R lambda so I stick an R inverse R there xn. So this tells me xi n+1 = if I call this matrix A, A times xi n, okay, these are the vectors that we get in the second set of coordinates, right, so depending on your coordinate system the equations that you get will change.

The iteration goes on, but if you could rotate from one coordinate system to another coordinate system you may actually find that there is a convenient coordinate system in which the equations decouple, right, so all of you are, you know where I am headed now, right, so this matrix, R matrix is often called the model matrix. We will come back to this later, we will need this okay, and the alpha x and alpha y what are these called?

Alpha x and alpha y are Eigen values and these of course will correspond to Eigen vectors, right, these are Eigen values so if you have matrix A it has Eigen values which are alpha x and alpha y, so if you have to do this iteration xi n+1 = A xi n, if you have to do this iteration, right or you have to do the iteration en+1 is Pen, what we are going to say is if this sequence of xi is that you get will converge if max of the Eigen value is less than 1.

The modules of the Eigen value is <1. The largest Eigen value is <1, is that fine everyone, okay, right, so en+1 = Pen this iteration matrix generates a sequence and that sequence converges if the modules of the largest Eigen value is <1 okay, so now we have reduced finding that fixed point to finding out what is the largest Eigen value and if you know that the largest Eigen value is <1 you are set.

In the sense that at least you know that you are going to get to the fixed point. So how does this works, this by the way, this mapping, I forgot to mention this earlier, this mapping or that xn+1 = g of xn so this maps into itself but this mapping when mod alpha x is < 1, right basically what is happening is there is a shrinking that is happening, it is called a contraction mapping, that is if you take values in a certain interval.

It will map into a smaller interval, it is called contraction mapping and we just write that here, contraction mapping, so the mapping is not only into itself it is a contraction mapping, the typical example that I would give for this would be you can imagine a steel rod but it is easier to think of a piece of sponge, right so you take a prismatic piece of foam or something of that sort that you can squeeze with your hand easily.

So you sit down and make markings on it at equivalent rules so if I take that foam and the contraction mapping that I do is that is easy, I squeeze, right so I hold the foam in my hand or I hold it in a wise something of that sort and I squeeze, so when I squeeze the right extreme moves in, the left extreme moves in all of the points move, there is one point that does not change, do you understand what, there is a fixed point.

Even in that example there is a fixed point, so if I go through a contraction mapping there is still a fixed point. There is one point that does not move, am I making sense, right, I mean I do not have the foam in my hand but I think you can imagine it, right, okay, there is one point that does not change. So if you have a contraction mapping right, well ya you could have a translation which is why I got it off the C, okay.

So you see that is the issue, so if I have to translate if I did not push them together in the proper fashion it is possible that I could have a translation now you will know why I move to the en+1 okay because I had got rid of the C. I want something that is linear, I want

something that is linear. Remember this is a standard confusion, this is outside the normal discussion y = mx + c is not linear, it is a straight line.

Unfortunately, linear also means line, right, so curvilinear does not mean then you will get confused if you assume linear means straight line, curvilinear does not make sense, curvilinear means a curved line, right okay, y = mx + c is not linear in the sense of function is linear okay, is that fine, remember that. So we got rid of that C by going switching to the error.

It is a little game that we play. Now so if you have, so the fixed point theorem basically says, right you have those map, I am not going to go through all the detail but you have a map back into the domain and it is the contraction mapping, right, then there is a fixed point and you can get to that fixed point in this case because we are generating a sequence that will help us converge, so they are automate on will actually converge because of the fact that we have a contraction mapping, is that fine, are there any questions? okay, now that is fine.

So now the thing is how do we deal with this, we have reduced ourselves going from en+1 = P times en how do we find out what are the largest Eigen values and Eigen vectors instead of dealing with this directly I will deal with Laplace equation itself directly so in Laplace equation if phi is the solution and phi tilde or capital phi is the candidate right and this of course will give us a corresponding e, it will give us an e, okay.

And we have talked about uniqueness, what are the boundary conditions that e satisfies, well let us call it en, what is the boundary condition that en satisfies? It will be 0 right; the boundary condition will be 0 because we have subtracted from the original which satisfies the boundary condition. See the candidate has to satisfy the boundary condition exactly, you cannot violate.

The boundary conditions are inviolate, you cannot violate the boundary condition, to do whatever you want on the interior, you cannot violate the boundary conditions, okay, so the candidate solution also satisfy the boundary condition therefore this is 0, this has homogenous boundary conditions, it is 0 on the boundaries, okay.

So how are we going to go about solving this, I cook up, I do use my Fourier series, I will write Fourier series in 2 dimensions if you have not seen this we will see how this works. So I look at this function e (x, y) = alm you know that these are Fourier coefficients are not the entries of the matrix aij, you understand what I am saying, do not confuse them with the entries with the matrix say alm exponent i pi lx/L.

L is the length of the domain in our case it is 1, but I will just leave it as L exponent i pi my/L right so instead of 1/1 unit square I have an L/L this is summed over m, summed over l. I know I am going to use a grid of a certain size, right if I have n intervals, then what is the highest frequency that I can represent, so here I do not have a 2 pi, it should have been 2 pi right, it is only pi.

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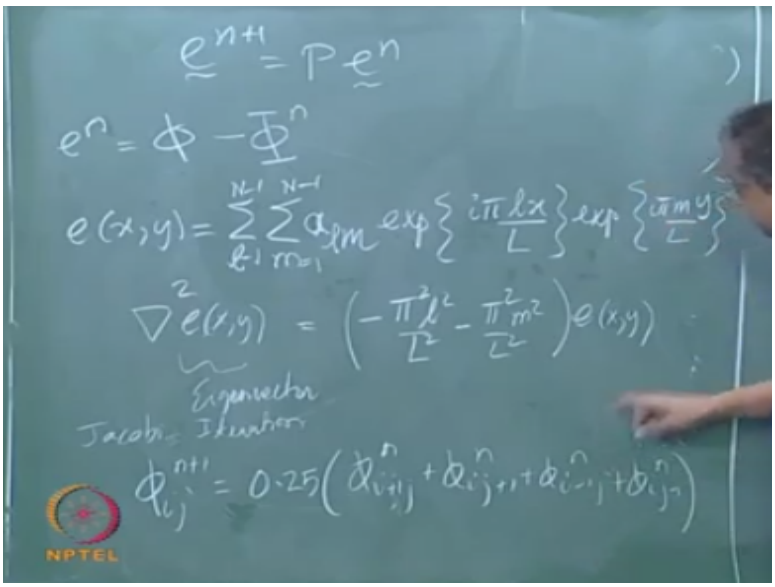

So this goes from m = 1 through n–1, l = 1 through n-1, I have dropped the m=0 and l=0 because my boundary conditions are 0 right my boundary conditions are 0 Laplace equation maximum minimum occurs in the boundary, okay fine, we know that it is going to go to 0 anyway so I dropped the dc component m = 0, so I have m = 1, so what is the relationship between exy and Laplace equation?

If I substitute this into Laplace equation what will happen, if I substitute this in the Laplace equation what will happen, lambda square. What is lambda squared of exy = -pi squared l squared/L squared – pi squared m squared/L squared, is that right, you can just check to make sure that I have not, so if I take any one of these for m = 1 or 2 or 3 or what if I take any one of these and this is what I am going to get, is that fine, just substitute it and try it.

So these are basically, now look at what we have just done these are basically Eigen function, these are Eigen values and these are Eigen vectors or Eigen functions, okay, fine so these are Eigen functions or Eigen vectors, why am I doing this, why is it important that they are Eigen vectors or Eigen functions because I know that I want to get my Eigen values, the largest Eigen value, am I making sense.

I know from my iteration matrix somewhere along the line I want to do the ratio test, let me not just say that I want to get the largest Eigen value, the largest Eigen value apparently comes from the ratio test so I do the ratio test in some fashion then ask the question for what value will the ratio test, right, give me the smallest change, the smallest ratio or I should say the least change, right okay.

So it turns out that anyone of these right, which we call elm or whatever or anyone of these can be now substituted into our differential equation, our finite difference scheme and we will see what it is that we get, okay, so what do we have, so we have let us pick Jacobi, we will do Jacobi iteration, so I will write this n+1, we have done this so many times okay, these i i+1 are spacial coordinates.

So we can actually substitute, we can sample the function exy right and substitute in here, am I making sense. In a similar fashion I guess I should not have written it this way because this will cause you to think that there is a V also maybe I should not have written it in terms of phi, I should have written it in terms of after all en also satisfies, e also satisfies del squared e, does it satisfies Laplace equation? is del squared e = 0? So we want this will be for any n okay.

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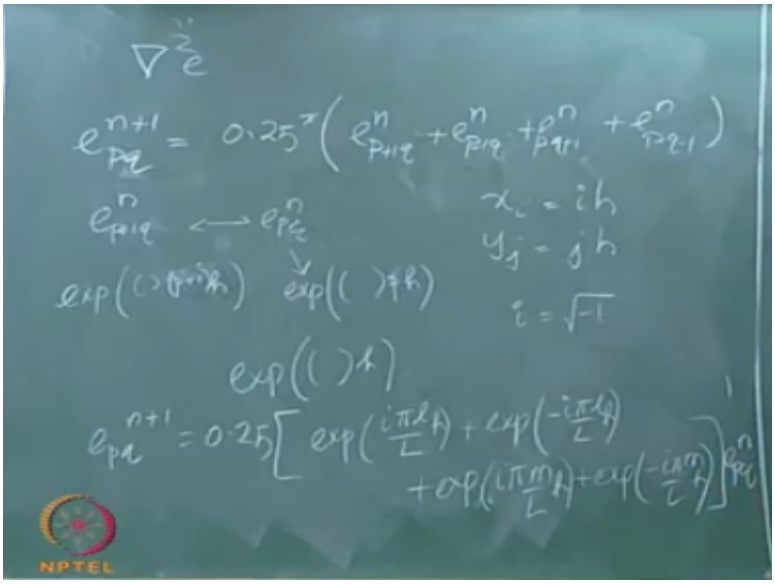

Del squared e = 0 what is this going to give us. Let me see if we go through with this en +1 = I am going to substitute now, the function phi I am going to represent in terms of, the function e I am going to represent in terms of these exponentials so what do I have en+1 at ij is 0.25 times en at i+1j + en at i-1j + en at ij+1 + en at ij-1 they are all equivalent rules so what does this going to give me i+1j i-1j.

So what is the relationship between ey+1j at n and eij is there a relationship between these 2? One will be exponent sum constant times i+1h, xi = i times h because we are taking equivalent rules, I am slowly getting ahead of myself here and yj = j times h we are taking equivalent rules, okay, so the relationship and eij would be of the form exponent everything else is the same ih okay.

So the relationship between ei+1j and eij would be just one exponent of all of this coefficient times h. Oh I have made a mistake here you guys have not corrected me, okay, what is the mistake that I have made? I am using i to be square root of -1, I am also using i as the subscript, I should not do that, you have to be very careful, so henceforth I will switch to the notation pq, p+1q, p-1q, pq+1, pq-1.

You have to be very careful right, p+1q, pq, p+1q, pq, is that fine okay, because right now all of the sudden when I introduced Fourier series I have decided that i = square root of -1 when I could use a different i but I am likely to forget so we will just switch the notation fine and when your electrical sciences you may have used j instead of an i but it does not matter, we will stick to i = square root of -1

So this equation using this information then becomes epq at n+1 = 0.25 times exponent of i pi l/L + exponent of –i pi l/L + exponent of i pi m/L + exponent of –i pi m/L the whole into epq at n, is that fine, did I make a mistake, i pi l * h, h is very important here i pi l * h, h is very critical for me. Okay, so what we will do is in the next class we will see where we can take this, right.

So in the next class what we are going to do is we will take the ratio of the 2 and find out what is the rate at which what is the growth, the geometric growth that we are getting over the decade, what does Gauss-Seidel, what does Gauss-Jordan do we will see if we can do something similar to similar by way of convergence to Gauss-Seidel and so on, is that fine, okay, thank you.