Gas Dynamics
Prof. T.M. Muruganandam
Department of Aerospace Engineering
Indian Institute of Science, Madras

Module - 3
Lecture - 9
Stagnation properties Differential form Equations for One-dimensional Flow, Isentropic Flow with area variation

Hello everyone, welcome back. We last time saw that when we accelerate a gas from near stagnation to very high Mach numbers actually just accelerate, we would not talk about mach number immediately; if we just accelerate the gas, then we found that when velocity increases temperature decreases and because of that Mach number increases much faster than the way velocity increases or the way temperature decreases.

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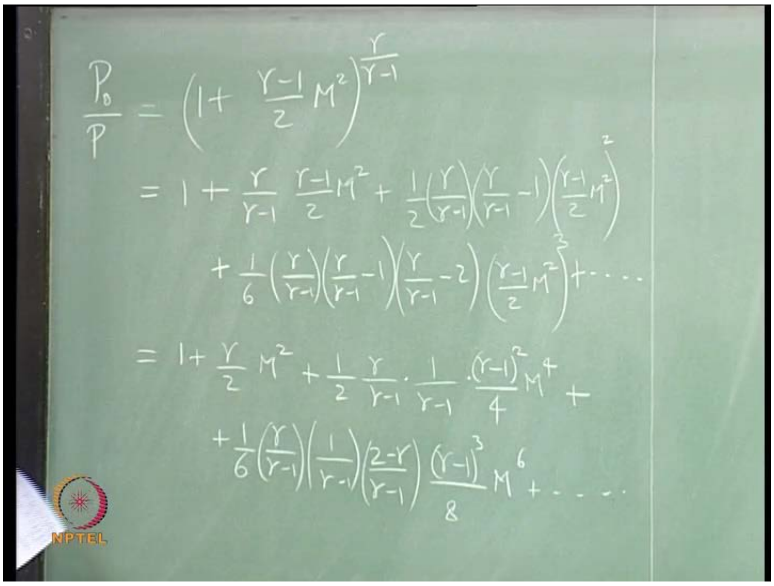

Now we will go and see the p 0 by p expression which we wrote last time. We have this expression. Now this expression can be expanded just by using binomial series, and if we say that Mach number is small enough, then we can remove higher-order terms of m square; let us say we do that. We would not very about, why; we will just do it right now. I am continuing the series and at last I am putting plus dot dot dot. This is just binomial series expansion of this whole term sitting here. It is a plus b to the power n which I am using binomial series to expand. Now we will simplify these four terms of the binomial series. The first one gamma minus 1 will get cancelled.

Here this can be simplified a little more. So, I will wait one more round or will this come out to be gamma minus r gamma minus gamma plus 1. So, that will just come out to be 1 by gamma minus 1 into plus I will fold and write the next one. This we already did there; that is 1 by gamma minus 1. The next term will be gamma minus 2 gamma. So, that will be a minus gamma plus 2. So, that will be 2 minus gamma into gamma minus 1 cube by 8 times m power 6 plus dot dot dot; this is what we have. I will go to the next board.

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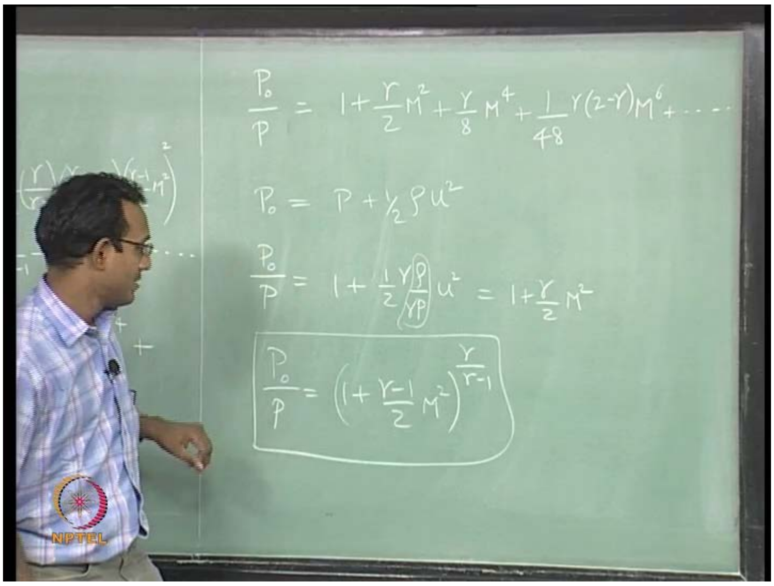

I will simplify this further. So, we will get; it will become this expression. If I go and look at our Bernoulli's equation, now I will rewrite this Bernoulli's equation in our form p naught by p; I got to this form. Now I want to make this gamma p by rho. So, I will put gamma here and gamma here. This gamma p by rho becomes a square. So, u square by a square that will happen to be m square. So, this comes out to be1 plus gamma by 2 m square. This is the first two terms of this full expansion we have. So, what we have done essentially is when we use Bernoulli's equation, we are using only the first two terms of our full compressible flow expression which we derived without using Bernoulli's equation anywhere in the middle.

So, this is the full expression; of course, when I say Mach number is less than very very small then I can neglect all these numbers. If Mach number is more than one, I cannot neglect m power 4, m power 6, m power 20, whatever I am going to get very high numbers; I cannot neglect any of them. So, when Mach number is very very small, then I can neglect m power 4, m power 6, etcetera and so I am okay with using Bernoulli's equation when it is very low Mach number cases only for linking the stagnation and static pressures; that is the way we have to look at Bernoulli's equation. Actually what we have to do is compressible flows fully.

So it so happens that when Mach number becomes higher, this is not very accurate. I have to have the next term gamma by 8 m power 4 in here immediately after that. So, instead of doing that when Mach number increases, instead of going and adding one more term each time, we will just use the most accurate thing which is our full expression. This is the most accurate one. If we want for simplicity sake, we can use this for very low Mach numbers; that is what we are seeing. This is just one way of saying that for actual cases we always have to use this expression.

If it is very low Mach number flows, then I am okay with using Bernoulli's equation which is what is done by most of the people who deal with water flowing over some bodies. Typically the velocity will be very low, then we will just start using this expression. It will not be very bad for us.

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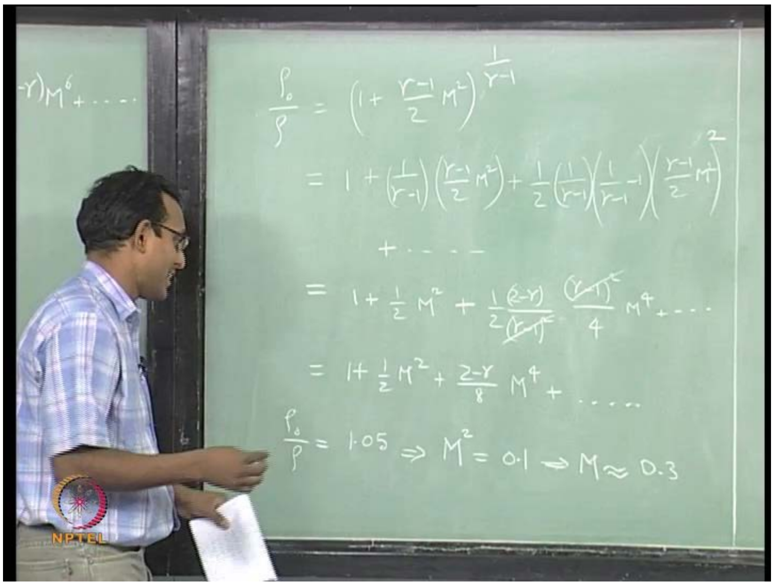

Now we will go look at the density variation. These expressions should just be staying in your mind forever; these are the expressions you need to remember. What is rho naught by rho? It is just t naught by t which is inside the bracket to the power 1 by gamma minus 1. It is just get used to that. What is 1 by gamma minus 1 really? C v by r that is what I want you to remember c v by r. In case if it is not air, then you have to go back and think about of course, yes, the value happens to be 2.5 which is what somebody had told. But we want to remember this as c v by r, because that is how we derived it. E d s equal to you remember that full expressions c v log p 2 by p 1 minus r log t 2 by t 1; from there only we are trying to get to this r log v 2 by v 1 by the way.

So, from there only we are trying to get to these formulae where we are saying t d s equal to zero; we are setting that equal to zero, from there we are getting to all this. So, remember that and that is how you got to this c v by r formula here. Now anyway we will try and expand this; again the same formula as before binomial series. Again I am folding over. Actually let us ignore; I do not want to write any more terms. We will just leave it like this; anyway we are going to cancel even the last term. Now I am going to cancel this gamma minus 1 and this one we already know as 1 by gamma minus 1; we did it just the previous step. Sorry, I made a mistake here; till here everything is right. I just made that mistake here.

Previously, it was gamma by gamma minus 1 minus 1, now it is not. So, it should just stay as it is 1 by gamma minus 1 minus, minus gamma plus 1 will be there. So, it will become 2 minus gamma divided by this. So, 2 minus gamma divided by gamma minus one whole squared. This is what it should become; now everything is in order again. So, now we will cancel these. So, finally I am getting an expression 1 plus half Mach numbers square plus 2 minus gamma by 8 Mach number power 4 plus we ignored the next term. So, it will just be this, something like this.

Now if I want to say I am allowing five percent variation in density to be neglected; if there is density variation between stagnation and static of the order of five percent change, which one will be higher? Stagnation will be higher, why? When I accelerate the flow, the flow expands; we saw that last time. So, it will be lesser density. So, this ratio is always more than 1. We already saw that from temperature point of view also. This term is always more than one. So, I am going to say this is more than one which means my density has to decrease when I accelerate the flow. And I am willing to neglect five percent change in density; that is the engineering approximation five percent, we always keep five percent as engineering.

If there is only five percent change or there is only fluctuation of five percent, we will say it is still constant; that is the engineer's way of putting things together. We do not want to worry about something that is varying less than five percent. So, let us assume my rho naught by rho is 1.05 roughly. If I have some such variation, what will my Mach number be 1.05. So, this will be 0.05 into 2.1 square root of 0.1. What is square root of 0.1? Roughly 0.32, something, 0.31; so, this will give you Mach number or I will say m square equal to 0.1 which will give you Mach number of the order of 0.3. Actually it should be 0.316 something like that; we will just say 0.3.

This is how remember the very first class somebody in here introduced saying like if my flow is incompressible, then we will take the Mach number to be less than this value. If my Mach number is less than this number, then what will happen? This whole expression will be smaller than 1.05. So, if it is less than five percent change, then I can ignore it. So, I can say there is not much of change in density; that is the way we are going to handle things. So, if my Mach number is less than 0.3, then the density change in the flow is very very small, and I can ignore it; this is what most of the books will say if my Mach number is less than 0.3 it is considered incompressible.

It is not really incompressible; it can be considered as incompressible. The wording in the book will be very clear; they will simply say can be considered incompressible. So, that is what we will be doing. We can say that if Mach number is less than 0.3, then of course, you can go back and use your Bernoulli's equation. It will most likely be correct; it will be close to the correct answer, but this is better. But what if engineers turn around and say five percent that is too stringent, let us make it ten percent; anything less than ten percent, I am not bothered. So, I will say roughly that will come down to 1 by 0.9. I will give you 1.1 is most likely 1 by 0.9.

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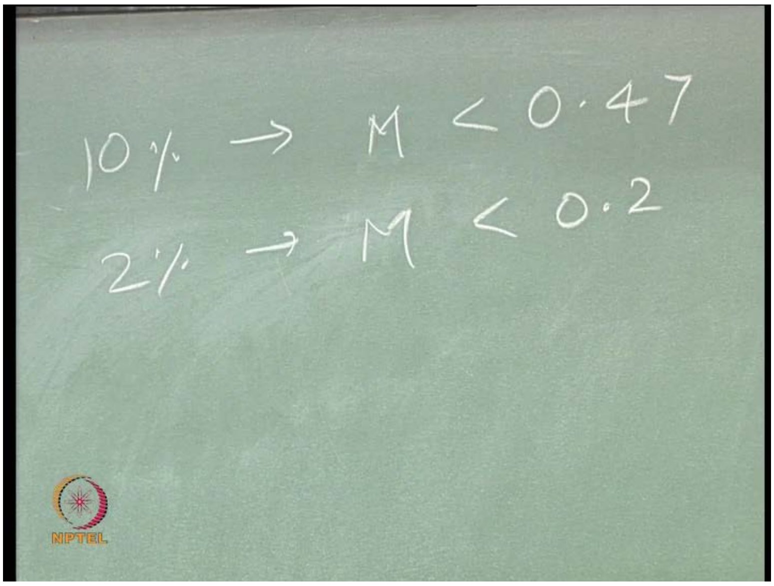

If I do that kind of calculation if I say I am okay with ten percent variation, my criterion is ten percent let us say. Density can vary ten percent in my fluid. If I say that then my Mach number anything less than 0.47 will be considered incompressible if I am okay with adjusting whatever variation is happening there; that is if I am okay with saying, oh, there is just ten percent; we will ignore that ten percent. If we say something like that, then I can go all the way up to Mach 0.47 and then say my flow is incompressible. If I am more strict and say, oh five percent variation is too high a change I have to take into account, then i will pick a two percent let us say.

I do the same set of calculations, and I will get my Mach number less than 0.2. Anything above 0.2, my density variation will be higher, and I have to consider that as compressible. Depending on what my setting is, how much is my tolerance, how much am I ready to tolerate. Based on that I will set different numbers; most commonly agreed numbers for engineering happens to be five percent, and because of that we are stuck with Mach 0.3 which was here. That was the connection which we wanted to make at the very first class which we did not make long back. Now we have linked all of them together, right.

Then previously we said there were two definitions for incompressible flow. One is based on whether my density is changing due to flow or not, and the second one was based on Mach number which is speed of sound and speed of the flow ratio of that. And we wanted to find how these two are connected; this is the connection. And we had to wait so many classes to find the connection actually; without these expressions you could not have gone any further. Now after all this we will go back to how to solve 1D flow. If there is any area change in my flow tube, am not talking specifically about any particular flow problem yet.

Once we have the ideas the tools developed we will just go and apply the tools in every possible formula, every problem possible in gas dynamics books. We will go and start solving one by one after some time; initially we are just developing the tools. So, we already have the momentum equation ready. We just have to say that it is steady 1D flow with area variation. We have developed our equations for area variation possible already. We had a d a possible d u everything.

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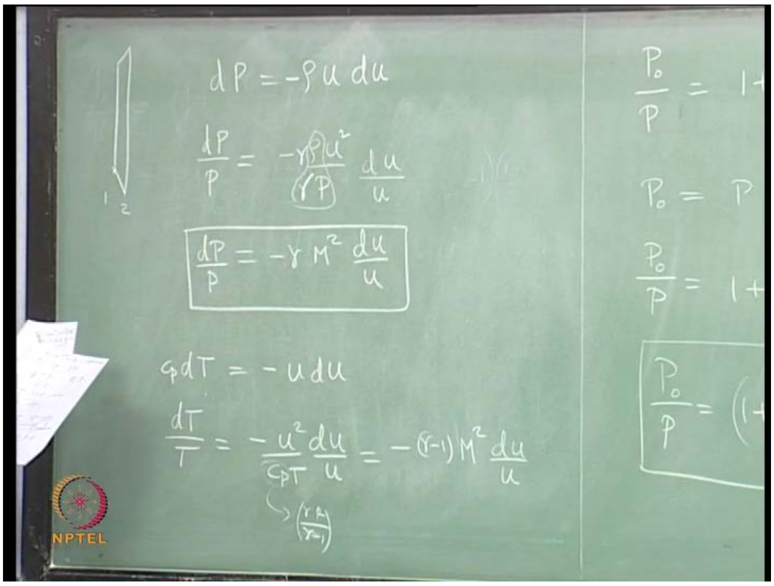

We started with control volumes that are looking like this, right, where area variation is possible; one to two there was an area variation. It is possible; we already had that taken into account. So, all I have to do is just go start using that. There is a minus sign. We had this as one of the equations. Now I am interested in d p by p. So, I will just divide this by p and I want to say d u by u. So, I will put a u square by u here which is equivalent to u. When I do something like this, I have already assumed that u is not zero. Then only I can multiply and divide by u, remember mathematically correct we want to be. I have already assumed pressure can never be zero which is correct pressure can never be zero.

When pressure is zero, there are no molecules, there is no gas, there is no flow. Here I am also saying the gas cannot be staying still; gas is moving in my case. Now I want to rearrange this again in the same way as what we did some time back, multiply and divide by gamma. And choose this as gamma p by rho reciprocal is 1 by a square. So, I will get this to be minus gamma m square d u by u. This is one of the important expressions, so I will write that separately. Next thing we want to do is go to energy equation and there we already had this kind of expression. So, I will just say energy expression was written as c p d t is equal to minus u d u.

We had this form before where I am assuming there is no other heat transfer. There is only this exchange of heat within the gas itself, nothing else outside. Now I want to find d t by t in terms of d u by u. I am going to do similar stuff. I will take the c p to that side. I will write d t by t u square d u by u, and I will put c p t here. Now I want to write c p in terms of a, speed of sound. So, I would write this in terms of gamma and r, because there is a t here. If I put gamma r t, it will become a square. So, I will put gamma r. So, what is this? Gamma r by gamma minus 1; c p is gamma r by gamma minus 1, not r minus 1, gamma minus 1; this is just the c p. So, I will take this gamma t together, make it a square; there will be a gamma minus 1 sitting there. So, I can write this equal to minus gamma minus 1 times m square d u by u. I have another expression here; I want to put that also in a work.

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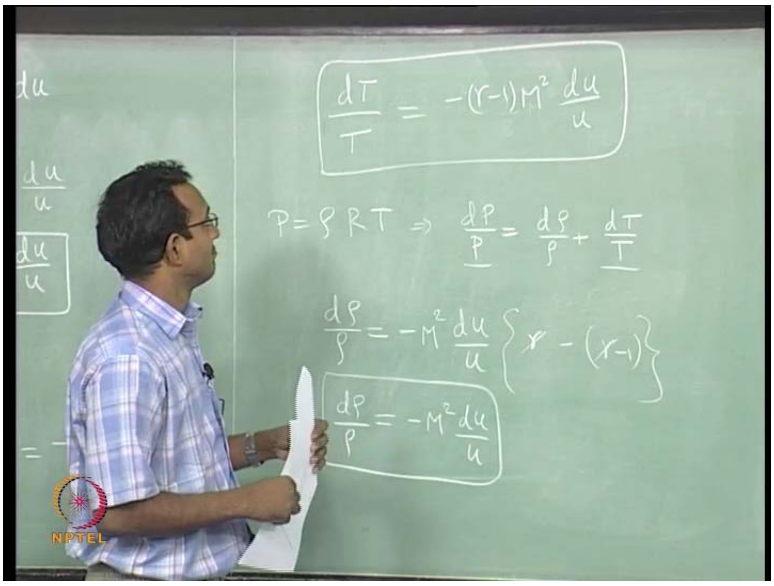

So, I will write it again. This is another expression we got. Now I want to find d density by density; of course, we will take the shortcut. I have p equal to rho r t; from here we already derived that d p by p is equal to d rho by rho plus d t by t. If you do not remember the derivation, you can derivative in any time; it is not very difficult to derive. So, now I just have to get we have these two expressions; this one and this one we already have, we want to find d rho by rho. So, d rho by dho rho will be d p by p minus d t by t. So, I have this box here this one, and I have this box here; everything has d u by u m square term. So, I will just put in fact it has both has minus sign also minus m square d u by u as a common factor.

Now d p by p has a gamma there, and this one has a of course, it is a minus d t by t. So, I have to put a minus sign of gamma minus 1; this is what I have, and this minus 1 will become plus 1 now. So, I am having minus m square d u by u. I got one more expression here. I started with saying area variation, but as of now I did not even introduce any area variation. D a I did not tell yet, but already we see a lot of insight into the flow problem already, how?

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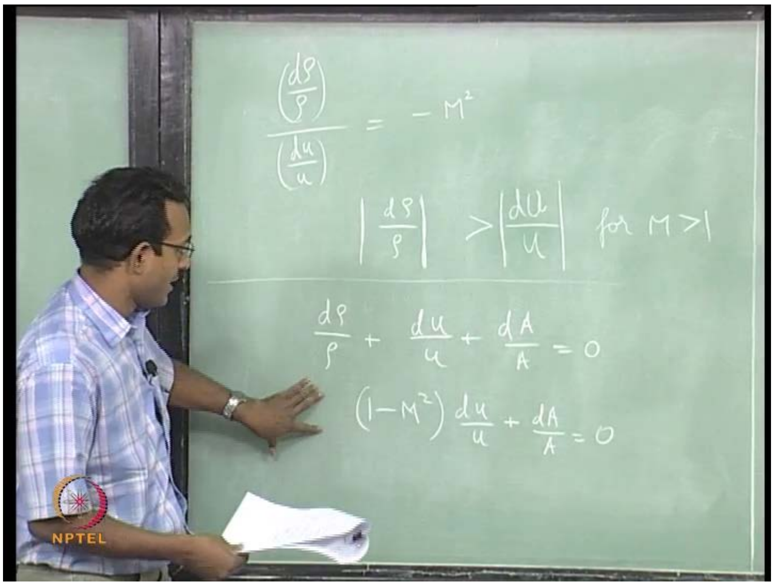

Let us go look at it in terms of the last expression we derived. I will write that last expression like this, d rho by rho by rho divided by d u by u is equal to minus m square. This is your percentage change or a fractional change in density per fractional change in velocity is given to be this, right. This is actually the change in density divided by current density, change in velocity divided by current velocity. So, it is actually the fractional change in velocity fractional change in density. So, what we are saying is the fractional change in density per fractional change in velocity is given to be minus m square. One thing it directly tells you is it is negative which is if my fractional change in velocity is positive; I am going to get fractional change in density as negative.

If I accelerate my density drops which we already saw in some other form last class. It is giving you the same answer, but the next thing is more important which is I am going to say if my Mach number is less than 0.3, if it is 0.3 what will be this value? 0.3 square is 0.09; if it is 0.316, it will come out to be 0.1. Then the change in density will be ten percent of change in velocity which we are ready to neglect. If we are ready to neglect, then we will call this flow incompressible. I am again giving you connection with incompressibility always; you should know which is compressible which is incompressible. And we will say that d rho is approximately equal to zero for small m's, but when I go my Mach number increasing.

As Mach number increases and gets closer to one, this number becomes very significant. If it is more than one, it is very very dramatically high. Mach number is two, it is four times. So, if there is a ten percent change, there is a forty percent change in density if Mach number is two. There is more serious change. If there is my Mach number three, then it is becoming minus nine. There is ten percent change; there is ninety percent change in density, very very significant change. So, because of that in fact, this is going to tell you the physics of why nozzle flows go one particular direction. We will go and look at nozzles later, but remember that this is very very important.

I will write it in a different form. I am going to say modulus of d rho by rho, modulus as in the magnitude is greater than d u by u modulus the magnitude or I will say for m greater than one. In supersonic flows the change in density due to flow is higher than the actual flow velocity change itself. That is the important thing which we are going to look at. So, if I change my density if I increase velocity by twice density drops by more than half. This is the reason why you have to expand your supersonic flow if you want to accelerate. You will see that again when we go to next mass equation which is the next thing immediately.

Now we are going to write the mass equation; we had all these equation already derived. So, I just have to go write one by one. This is one form of mass equation we derived. Now we want to write everything in terms of Mach number which we have been deriving today. It will come out to be this form. Now there is one special thing that is happening here. If I put m equal to one, what will happen? If I put m equal to one this whole term goes away; suddenly it is telling d a equals to zero which means area is not changing. That is the first thing I wanted to say, and somebody here said area can be minimum. I do not want to accept that area can be minimum; all you know is area is an extremum. It could be maximum or minimum.

We will go deal with maximum or minimum later some other time. Yes, there is a possibility of thinking about it that way. I want to think about it in a new dimension. I want to say that the stream tube does not want to change its area if my Mach number is one. This is a very special way of looking at it. My teacher professor Shanta Kumar told me this. If you are in transonic flow then stream tubes are like metal sheets. They do not want to move just because you put an airfoil in there. They just want to go straight; it is like that. So, they do not want to change; they do not want to response to any changes you want to make in the flow if my flow is close to Mach number one. That is what makes transonic flow very very difficult to handle.

We will go deal with that probably sometime very late in the course. We would not do much of transonic flow and simple gas dynamics, but I will just tell you that there is a lot of problem when m equal to one.

Student: In previous expression d rho by rho, if my m is more than four, three means it is 90 percent, for four means?

Okay. So, the question is if my Mach number becomes very very high, what happens? We cannot think about it as my change is going to be ten percent; I just gave as an example. Typical changes the flow is going to respond to will be like 0.01 percent itself. You are consulting a case where Mach number if it is ten, then when this is going to be minus hundred, then if there is ten percent change here, there is thousand percent change that is not possible yes that is right. But actually it is not percentage; I am not thinking about percent really. Thousand percent is ten times, yes; it is going to be such a big change in reality but nobody can ever change the flow that very high-level immediately.

Even a small change will immediately be having an effect. It would have changed by then; that is the supersonic flow. That is just any flow in fact; this is just coming from flow equations. I just used mass momentum energy equation till now, nothing else; mass momentum energy and p equal to rho r t. And the flow equations are telling that this is what will happen. According to flow conditions, there will be some change in flow condition; automatically that will just take care of itself. So, we are here.

So, now we are going to make one extra assumption. I am going to say yes, when m equal to one, things are going to go different. Things are going to be very difficult to solve, but let us say we will never consider a case m equal to one for now, very important assumption; remember this sometime later it will come back and hurt you if you forget this. I am going to say m is not equal to one which means I can now divide this equation by 1 minus m square; otherwise I cannot.

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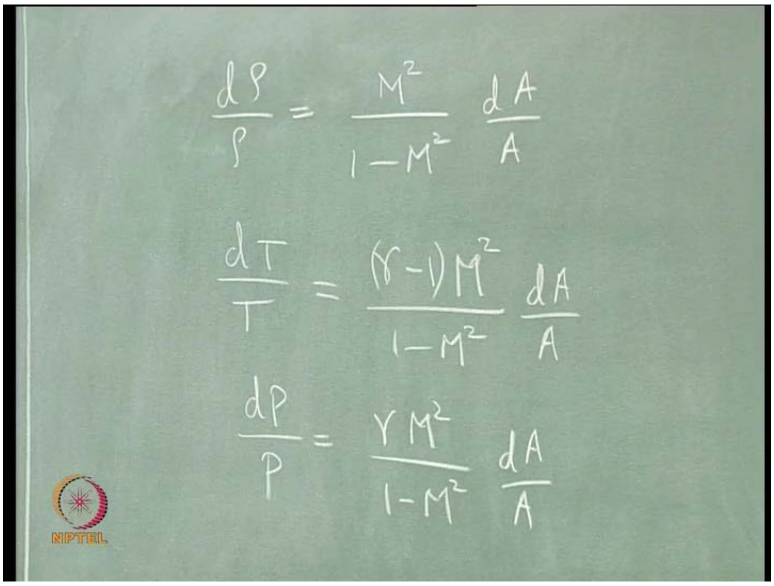

So, now, I will go write some set of expressions coming because of that. This is the expression I just directly get dividing the previous expression by 1 minus m square. There is a minus sign there also. This is one expression I get. Now whatever I derived today I can write everything else in terms of d u by u; substitute this in place of d u by u in everything I did today. So, if I do that I will get a whole bunch of expressions; I will write one by one. So, we have written all these expressions; all this has been derived only today. Now for any given change in my stream tube cross-section area because we had 1D flow; stream tube cross-section area is our a.

I pick any stream tube and the cross-section area is a and if there is a small change in that, we are seeing what will happen; that is the idea here. Now we are seeing that this whole set of expression on the left the d p, d t, d rho, d u, everything will change sign across m equal to one. If m is less than one, this term is positive; this term is positive; this term is positive; this term is negative, for a given d a by a let us say it is positive. If I go and change m greater than one this one minus m square changes sign. So, this tells me that the flow does not behave the same way for the same change in area for subsonic and supersonic.

They are very different and these expressions cannot tell me what the flow will do for m equal to one, why? I assumed m is not equal to one already and then divided by one minus m square to get to this expression; okay I assumed it, I forced that m cannot be equal to one. So, it cannot be solving m equal to one cases remember that, common mistakes; I am just telling you that also. Now some more extra details that you can see from here for a given change in area; it looks like the least change happens for velocity magnitude wise, least change happens to be for velocity. The next least change I do not know whether it is this or this, I think it is temperature gamma minus 1 will be less than 1.

So, it is going to be this the next change, and then the next one is this, and then the next one is this. Temperature changes the least, next higher is density, and overall much higher is your pressure. This is the way it is going to go. So, the least is this, not velocity by the way. This is having magnitude of the order of one; this is having less than that gamma minus 1 for gamma equal to 1.4 will be 0.4; that is less than oh, but there is m square. So, I cannot really tell that sure, okay. So, this might be the least for small Mach numbers.

So, depending on Mach number it can change, but the effect is this. Temperature change is the least, next will be density, next highest change will be pressure; this is what we are seeing from here. This will help you later. It is giving you a lot more other physical feel; I will stay next to these expressions. Now we want to see the effect of area change for different cases.

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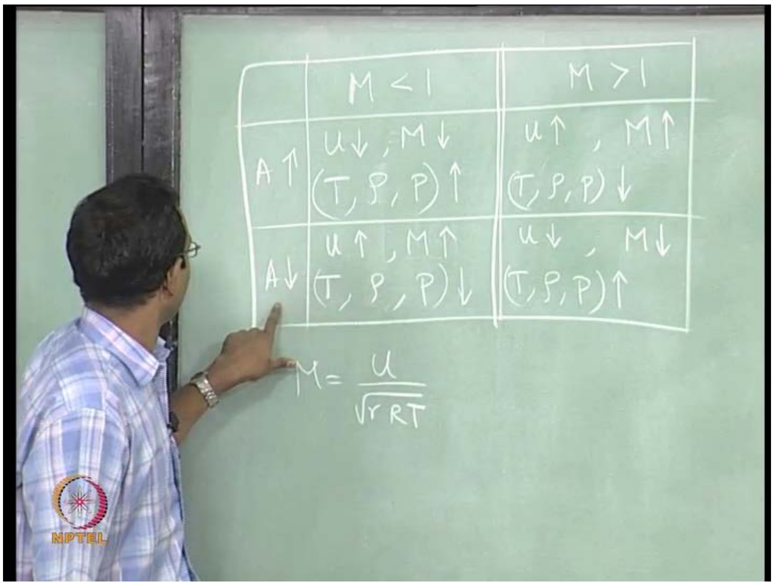

So, I am going to form this table; it is standard available in most of the books. I want to consider four cases, area increasing or decreasing, Mach number in less than one or more than one; we said these are the different effects. So, what we want to see is let us look at the subsonic case, simple case; area is increasing. Actually from garden hose point of view area decreasing will increase velocity. So, we will start there; that is easy to understand, area decreasing case is easy to understand. We know that in garden hose, we now it is a subsonic flow there, and I am pressing and velocity increases. So, I know velocity increases there.

Now I will go back to these expressions again; is that what it is telling? I am pressing means I decrease area; d a is negative. That negative will cancel that; we said subsonic, this is positive. So, overall d u is positive which is what we are getting here. D u is positive means velocity increases. I put that or I will go back to that expression again for other cases. Let us say I picked this temperature. What is going to happen? I decreased area, I decreased area subsonic. This is positive; m square is always positive; gamma minus one is positive; whole term is negative, because d a is negative; same thing here. This is all positive terms; d a is negative. So, density is dropping, temperature is dropping, density is dropping, pressure is also dropping.

So, I am going to write this. I will write in that particular order; temperature, density, pressure, all of them dropping in the increasing order of magnitude of dropping. This will drop the maximum, temperature will drop the least, but it will definitely drop. Now I go to the one picture above still Mach number less than one area increases, exact opposite of this, why? All the expressions had d a by a. I just have to change sign of d a by a; every expression will change sign. So easy to think about t rho and p. Subsonic flow if I increase area velocity decreases.

What happens to Mach number? We did not put that in this whole list. Let us pick this case. What happens to Mach number if I increase area? I increase area, velocity decreases, temperature increases. I have formula for Mach number, velocity decreases, temperature increases, denominator increases, numerator decreases, whole thing decreases faster. So, m decreases and in this case of course, m increases. Now we will go to the other case supersonic case. We do not have a m equal to 1 line here. We already assumed that m is not equal to 1 for my gas dynamics currently. So, I go back here. If it is m supersonic, all the denominators are going to be negative; all the denominators are going to be negative.

So, this negative sign will be cancelled by that; so, d a changes directly proportional to u d u. So, if d a is positive, d u is positive, and I will go back here. I am going to say area increases, d a is positive. So, velocity will increase; that is the main thing. If area decreases, velocity will decrease, easy to write; that is the easiest thing. Next we will look at the other terms here. So, when I look here now I have denominator negative we said supersonic flow; everything else is positive. Let us say d a is positive, then d p will be negative because this is negative. Here this is a positive quantity; this is positive; this denominator is negative. If d a is positive then this is negative, same thing here.

This is negative, numerator is positive; if d a is positive d rho is negative. D rho has change is negative which means it is decreasing; change is negative. D a is positive, change is positive means it is increasing. So, if my area is increasing t rho and p are decreasing. And if it is the opposite then this will also be opposite. Now we will go back and look at Mach number again. I am having supersonic flow and I am increasing area. I am having velocity increasing and temperature decreasing, velocity increasing, temperature decreasing, Mach number will increase; here Mach number will decrease.

So, what we are seeing is overall the flow behaves very differently for subsonic; I will draw one more line here across this border subsonic. It is behaving in one particular way, and it is behaving straight opposite in supersonic, very important thing to remember. If we go and look at the expressions for various cases as and if I plot all these functions, if I take this functions and plot them for a given change d a. Then you will find that the functions will switch sign, and it will all go to infinity when m equal to one which is obvious right; 1 by m square is here, when m equal to one this whole function goes to infinity.

There is too much change that is needed for flows very close to m equal to one. And of course, our theory is not valid at m equal to one remember that; these expressions are all invalid at m equal to one. That is also there one more thing we need to think about. Now there is only one part left which is I want to link what should I do to make my flow go from subsonic to supersonic; that is the only part left. Let us see what can I do. I am starting with subsonic we said, and I want to go end up in supersonic. So, I am here m less than one. I start with very low velocity; I want to increase velocity or increase Mach number. So, I have to go this path; area should decrease for me to increase Mach number. Till where can I go maximum, only up to m equal to one because beyond that this is not true.

So, I will keep on decreasing area till m equal to one. After that I still want to increase Mach number. What should I do? I go to this column, and suddenly I see only if m is greater than one, this will work. Is my m greater than one yet? No; it just became slightly less than one and I do not know what happens at m equals to one. There is a small gap, but let us say we ignore it; somehow it got pushed to the other side. So, I am sitting here slightly more than one 1, 0, 1 or something. And now for my Mach number to increase this will decrease now. This will be the one increasing. If it increases only if only if a is greater than one. So, now, all this time I was decreasing my area.

Now I have to increase my area for it to go accelerate further. Why is this so; why is it so? The answer is already in the expressions I give you today and one point I told you notice that this is happening. Those are the points where there is answer; go and look at it. It looks like there is not enough time for me to complete the next thing which I started. So, I will do that next class. You guys have any other questions? Just go think about that particular part why is it so. Why should I expand past m equal to 1 for me to go supersonic; answer is already there on the board in fact. See you guy’s next class.