Flight Dynamics II (Stability)

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Module No. # 09

Perturbed (linear) Aircraft Model

Lecture No. # 30

Linear Model and Aircraft Dynamics Modes

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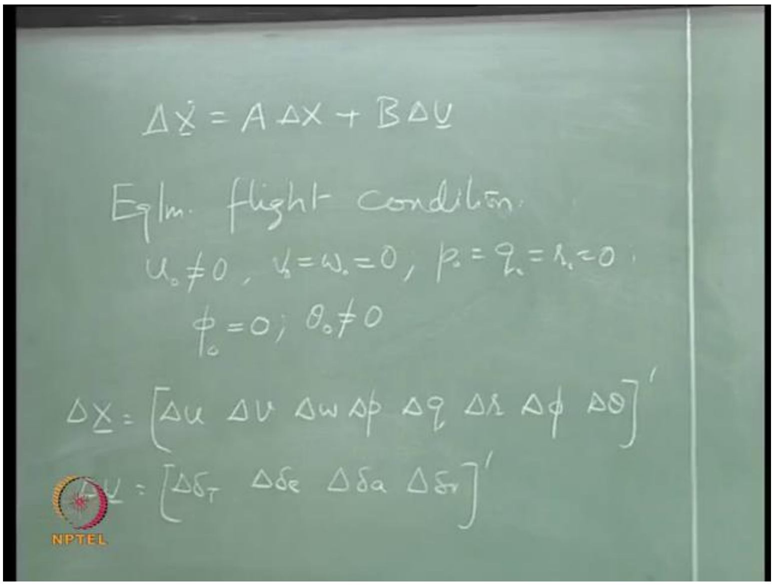

$$\Delta \dot{\underline{x}} = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_0 \Delta \underline{x} + \left.\frac{\partial \underline{f}}{\partial \underline{U}}\right|_0 \Delta \underline{U} = A\Delta \underline{x} + B\Delta \underline{U};$$

$$'0': u_0 \neq 0, v_0 = w_0 = 0, p_0 = q_0 = r_0 = 0, \phi_0 = 0, \theta_0 \neq 0 \qquad (1)$$

$$\Delta \underline{x} = [\Delta u\ \Delta v\ \Delta w\ \Delta p\ \Delta q\ \Delta r\ \Delta\phi\ \Delta\theta]'; \Delta \underline{U} = [\Delta\delta_T\ \Delta\delta e\ \Delta\delta a\ \Delta\delta r]'$$

So, we have our linear model now,....(Refer Eq(1)) which we have obtained by linearising our aircraft equations of motion, around the equilibrium flying condition, ... equilibrium flight condition which is ..... (Refer Eq(1))

So, aircraft is having a forward nonzero speed and other two speeds are 0, and the rates are 0, bank angle is 0, and pitch angle is nonzero and the variables are these .... delta represents the perturbation from the actual (trim values of) aircraft variables.

And you will observe, that because of this flight condition, where all non-longitudinal variables are 0, what we arrive at .....

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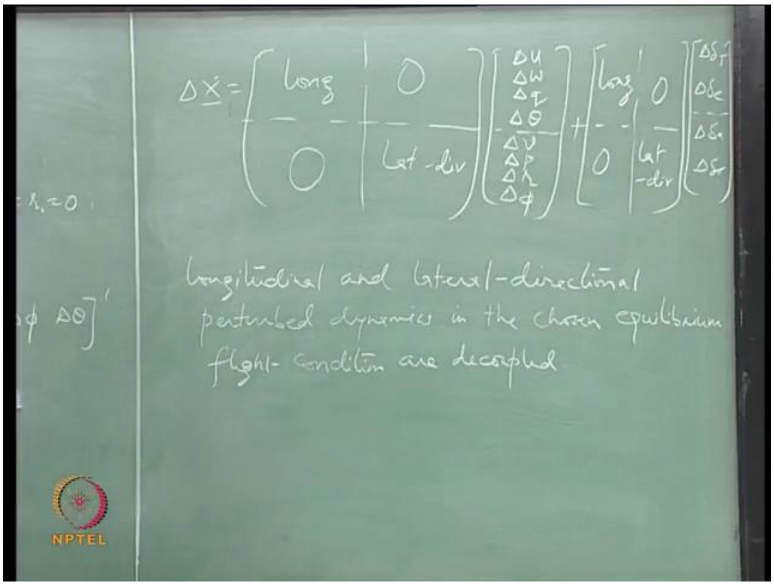

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$$\Delta \underline{\dot{x}} = \begin{bmatrix} A_{Long} & 0 \\ 0 & A_{Lat-Dir} \end{bmatrix} \begin{bmatrix} \Delta \underline{x}_{Long} \\ \Delta \underline{x}_{Lat-Dir} \end{bmatrix} + \begin{bmatrix} B_{Long} & 0 \\ 0 & B_{Lat-Dir} \end{bmatrix} \begin{bmatrix} \Delta \underline{U}_{Long} \\ \Delta \underline{U}_{Lat-Dir} \end{bmatrix} \tag{2}$$

$$\Delta \underline{x}_{Long} = [\Delta u \; \Delta w \; \Delta q \; \Delta \theta]'; \; \Delta \underline{U}_{Long} = [\Delta \delta_T \; \Delta \delta e]'$$
$$\Delta \underline{x}_{Lat-Dir} = [\Delta v \; \Delta p \; \Delta r \; \Delta \phi]'; \; \Delta \underline{U}_{Lat-Dir} = [\Delta \delta a \; \Delta \delta r]'$$

What we arrive at is, this (Refer Eq(2)) two decoupled set of equations. So, if I now want to look at the motion of the perturbed variables from the equilibrium flying condition, then actually I can look at the two dynamics separately, because the equations are decoupled. ..... (Refer Eq(2)).

... So, longitudinal (Long in Eq(2)) and lateral directional (Lat-Dir in Eq(2)) perturbed dynamics in the chosen equilibrium flight condition are decoupled. So, we can study them separately. So, what I mean to say is that these variables (Longitudinal) are not going to be affected by this (Lateral-Directional) set of variables, and this set of variables are not going to be affected by this set of variables. So, we can look at the two dynamics, two perturbed dynamics separately.

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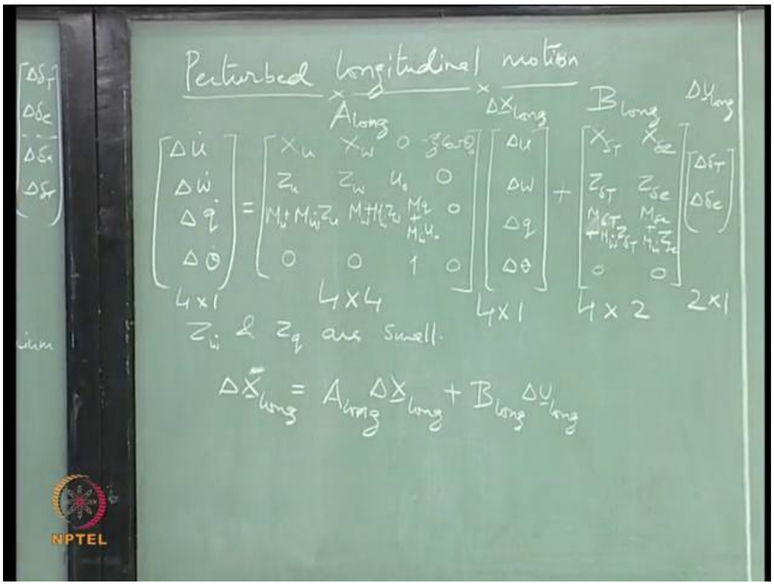

So, in the first part we will look at the perturbed .... longitudinal …

Longitudinal motion and the equations .... are these.

$$\underbrace{\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix}}_{\Delta\dot{\underline{x}}_{Long}}=\underbrace{\begin{bmatrix}X_u & X_w & 0 & -g\cos\theta_0\\ Z_u & Z_w & u_0 & 0\\ M_u+M_{\dot{w}}Z_u & M_w+M_{\dot{w}}Z_w & M_q+M_{\dot{w}}u_0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}}_{A_{Long}}\underbrace{\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix}}_{\Delta\underline{x}_{Long}}+\underbrace{\begin{bmatrix}X_{\delta_T} & X_{\delta e}\\ Z_{\delta_T} & Z_{\delta e}\\ M_{\delta_T}+M_{\dot{w}}Z_{\delta_T} & M_{\delta e}+M_{\dot{w}}Z_{\delta e}\\ 0 & 0\end{bmatrix}}_{B_{Long}}\underbrace{\begin{bmatrix}\Delta\delta_T\\ \Delta\delta e\end{bmatrix}}_{\Delta\underline{U}_{Long}} \quad (3)$$

X delta T, X delta e, Z delta T, Z delta e, M delta e, delta T plus M w dot Z delta T, M delta e plus M w dot plus Z delta e and 0 0...

9:02

X u, X w, 0 minus g cos of theta naught, Z u, Z w, u naught, 0, M u plus M w dot Z u, M w plus M w dot Z w plus Z w and the third term in this row is M q plus M w dot u naught 0, 0, 0, 1, 0. Here we have assumed that Z w dot and Zq are small,... this neglected them from this set of equations (Refer Eq(3)). If they are significant you have to keep them. So, I am only trying to look at a form, where I can do some analysis, using you know, we can do some analysis analytically. ... I do not have to depend upon computer for doing an analysis in this case. So, this (Refer Eq(3)) is my A matrix for the longitudinal perturbed dynamics, and this (Refer Eq(3)) is my B matrix, and size is - size of A is 4 by 4 ....,... and my interest is to find out how these variables - perturbed variables are going to change in time ....-staring from an initial condition....

So, initial conditions could be you know including the .... effect coming from the gust or or the perturbation ..., and that perturbation be could be in ... one of these variables or in all the variables, is it not? So, I have this …(Refer Eq(3)).

12:33

So, I want to look at the stability first. To study stability - dynamic stability, you assume that this delta U is fixed, is 0 ...., I am trimmed at some flying condition and I do not allow this delta U to change, then all I will get by solving this part ($\Delta \underline{\dot{x}}_{Long} = A_{Long} \Delta \underline{x}_{Long}$) is the free response of aircraft to perturbations.

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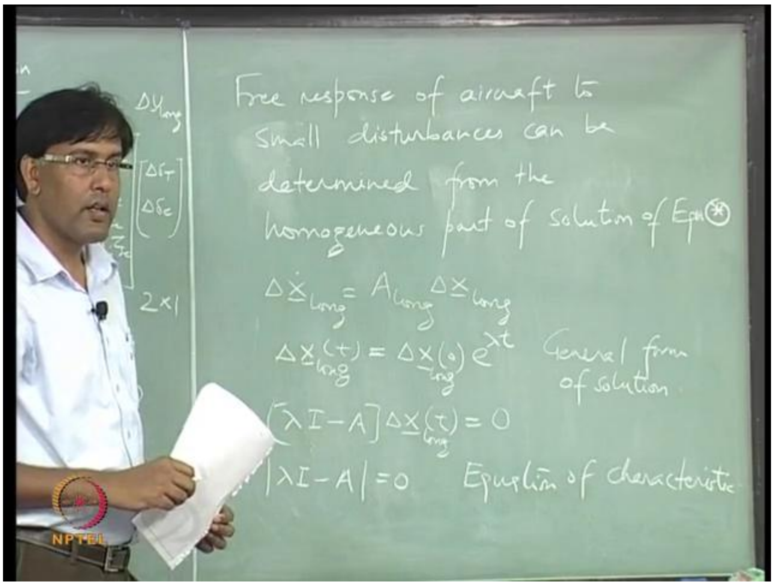

So, let us look at that.

Right, controls are fixed.

Student: (( )).

Yeah, you cannot change the control you are keeping it fixed, and then there is some disturbance which is going to come and hit the aircraft. So, there is ... change in variables.

Control surfaces all (( )).

Yeah, they are fixed. .... Control surfaces are all trimmed at the equilibrium flying condition. ... I am not looking at the response of aircraft to control inputs here, here we just want to look at the free response and that is what defines the stability. Now, this is a linear set of equations. So, you can assume ....(Refer Eq(4)) the solution in this form, you know general form of solution. And plug this solution in this equation. ..... (Refer Eq(4))

$$\Delta\dot{\underline{x}}_{Long} = A_{Long}\Delta\underline{x}_{Long} \Rightarrow \Delta\underline{x}_{Long}(t) = e^{\lambda t}\Delta\underline{x}_{Long}(t=0)$$

$$\Rightarrow \frac{d(e^{\lambda t}\Delta\underline{x}_{Long}(t=0))}{dt} = A_{Long}\Delta\underline{x}_{Long} \Rightarrow \lambda e^{\lambda t}\Delta\underline{x}_{Long}(t=0) = A_{Long}\Delta\underline{x}_{Long} \qquad (4)$$
$$\Rightarrow \lambda\Delta\underline{x}_{Long} - A_{Long}\Delta\underline{x}_{Long} = 0$$
$$\Rightarrow (\lambda I - A_{Long})\Delta\underline{x}_{Long} = 0 \Rightarrow \det(\lambda I - A_{Long}) = 0 \quad for \;\; \Delta\underline{x}_{Long} \neq 0$$

What you get is this. .. (Refer Eq(4)). So, what we are looking at is solution of this equation (Refer Eq(4)) starting from an initial condition which is not 0. ... If it is, the initial condition is 0, what it means is, there are no perturbations and it is a trivial solution. If there are perturbations and the initial conditions are not all 0, then this vector is nonzero, and then this equation (Refer Eq(4)) is satisfied only when this part is 0 .... So, that is where we start looking at solution of an equation, which is called equation of characteristic. .... $\det(\lambda I - A_{Long}) = 0$

Right, determinant of this matrix has to be 0, you know.

Student: (( ))

I will answer your question, afterwards we can discuss. So, this is the general form of solution for this equation ... , and we are plugging this solution - general solution in this equation and this is what we get. ... If this $(\Delta \underline{x}_{Long})$.. vector is 0 ... then this is identically equal to 0 (Refer Eq(4)). But we are looking at, and that $(\Delta \underline{x}_{Long} = 0)$ is the trivial .... solution. If we are looking at solution to this equation when delta $\underline{x}$ vector is nonzero, ... that is when you are introducing the perturbation. If the perturbations are all zero then I am at the equilibrium flying condition.

Student: (( ))

Let us, now deviate from here, I can discuss this with you afterwards.

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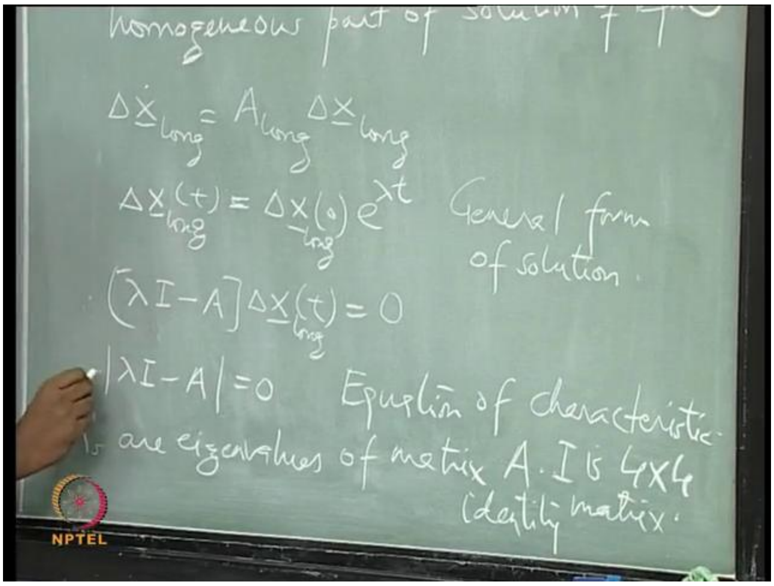

So, finally, you arrive at this equation which is the equation of characteristic... $\det(\lambda I - A_{Long}) = 0$, and lambdas (solution of equation of characteristics are called eigenvalues).

So, this is nothing, but the determinant of ..... of lambda I minus A equal to 0.

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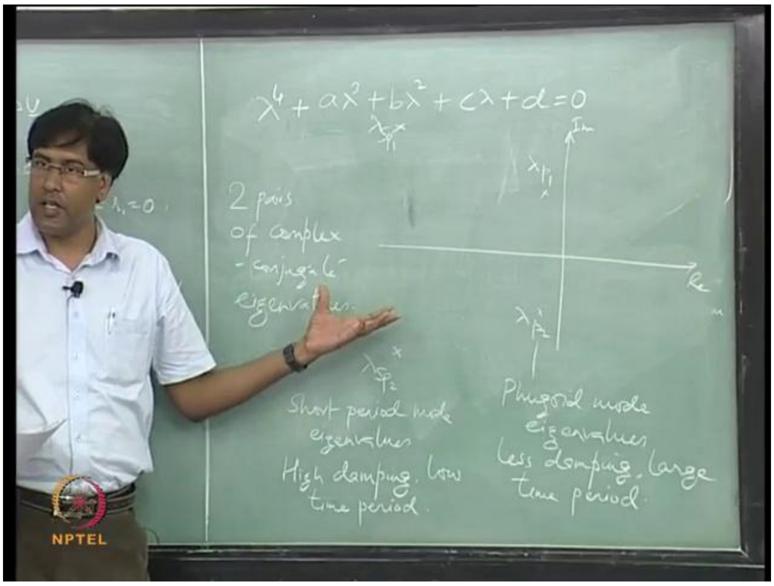

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \qquad (5)$$

Now, if you expand this equation, you will .....get this... (Refer Eq(5)), And in general, you will have, you know, how many values of lambda? How many lambdas you will get? 4 right.

Four Eigenvalues ..., and these Eigenvalues will depend upon these coefficients (inside matrix A_{Long} Eq(3)) and these coefficients are functions of aircraft parameters .... So, lambdas are going to define the perturbed dynamics, and they are functions of aircraft parameters. If I can arrive at simple functions where I can see which parameter of aircraft is affecting which Eigenvalue, then I will understand, you know which parameter is going to play a role in which dynamics.

So, you have to solve this. ... You think you can solve this (Refer Eq(5)) by hand? It is not really possible, it is not possible to solve it by hand .... So, earlier when people did not have computers how they would have done this, they would have simplified it further ..., assuming something else .... So, let us look at another simplification. In the longitudinal dynamics, the Eigenvalues, for the angle of attack that we are talking about they are well separated. So, if you look at the, the 2 pairs of Eigenvalues, we get 2 pairs of Eigenvalues when we solve this equation for longitudinal perturbed dynamics. ......

One pair is located close to the imaginary axis, and the other pair is located very far from the imaginary axis. So, other pair is having lot of damping, ... that is what it means, is it not? And the frequency is also high ..., this pair is called phugoid, because …. It is associated with a longer time period, it is closer to the imaginary axis, less damping ... and the frequency is low. So, time period is high.

The other pair of Eigenvalues belongs to what is called short period mode of aircraft dynamics.

27:17

So, if you look at these two dynamic modes of aircraft, they are well separated in time. The order of magnitude, if you look at the time period - time period will have, of this mode will have values, which is of the order of 3 to 4 seconds, and if you look at the phugoid mode, the time period is of the order of 25 to 35 seconds depending upon the angle of attack.... I am only talking about one particular flying condition, and this whole exercise you have to actually repeat, you know for different flying conditions.

So, ... there are people who actually observed the perturbed motion of aircraft, you know, in .... early 20th century. So, 1910, you know, ... some time ... frame like that. And they found that this short period motion actually, so, if you look at the equilibrium flying condition, aircraft is going with constant velocity ... and there is a perturbation and the short period motion is a pitching motion, you know, very fast pitching motion and highly damped, so, it will damp very fast. When the aircraft is moving, a disturbance hits the aircraft, it starts doing this motion which damps out fast and that is what is the short period motion.

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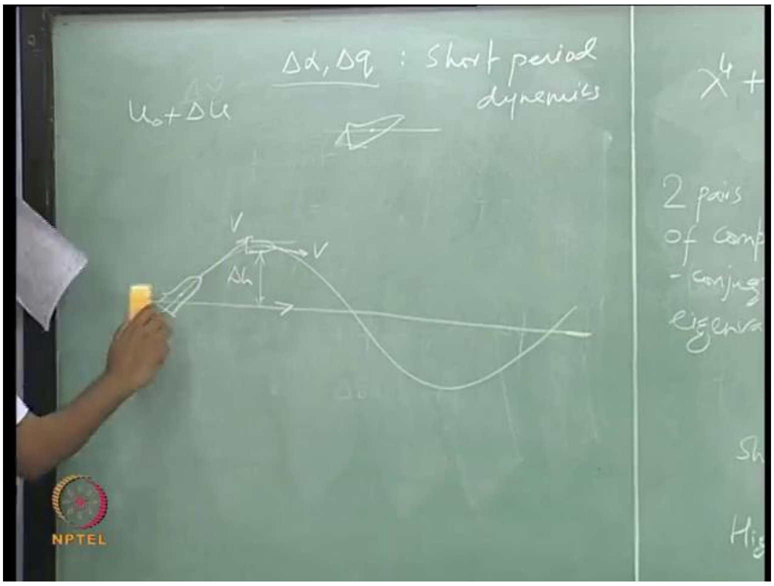

So, actual variables which play major role in this mode are delta alpha and delta q, there is hardly any change observed in this particular mode in the perturbation in forward speed. So, these two variables are associated with short period dynamics. The long period dynamics, or the phugoid is actually the motion where the aircraft seems to be following this sinusoidal curve .... So, what is happening is .... so, this is the velocity vector at any point.

So, what aircraft is doing is actually following this velocity vector, so, that means, the change in angle of attack is actually 0, you know, whether it will be 0 or not in actual situation, because here we are making some approximations, delta alpha may not be, may be very small and not 0, but it looks like that in this particular mode, the aircraft is, when it is perturbed from .... this flying condition, you know, you are going straight and then you are hit by some disturbance, the aircraft starts going up, because of added kinetic energy ... because of perturbation there is a increase in speed of the aircraft. So, u naught is now increased by delta u ... So, you have more kinetic energy. So, what aircraft starts doing is, it starts moving up and attains a height corresponding to that energy - extra energy .... And once it has exhausted that extra energy, because of this delta u, its starts going down and that is how it follows this sinusoidal path in ..... phugoid. So, what is happening is something like this .... going, ... angle of attack is not changed. So, aircraft will start following this velocity vector, and then going in this sinusoidal path.

There is an exchange of energy - potential energy and kinetic energy right. See, because of this delta u, you have more kinetic energy now. So, it starts going up, exhausts that energy, attains some height and afterwards it will start going down.

Student: (( ))

Why will it not, because this delta u is not going to be there all the time, if that ... delta u is going to be there all the time, then it is like an input to the aircraft, this is like a disturbance.

Student: (( )).

Yeah

Student: (( ))

And yeah and then, now ... it has attained this delta h.

Student: (( ))

Yeah so, but this perturbation is going to go down slowly,... this sinusoidal curve is changing in amplitude, because there is a damping associated with this motion. So, whatever perturbation is there that is going to change, you know, decay or grow only slowly.

Now, based on these assumptions, it was further possible to break this into 2 second order equations .... So, what I am going to do is, we are going to look at 2 different types of motion - one is this (phugoid mode), and the other one is this where aircraft is only, you know, pitching up and down (short period mode) around an (equilibrium) point. So, you are in flight and you start doing this (fast motion in pitch), and that motion is so fast that it will, it actually damps out in very less time that is what is short period dynamics.

Student: (( ))

It is always true actually in this particular case, and we are talking about low alpha flights. ... In some cases it has been observed that they will come closer and merge with each other and then give rise to another type of dynamics. There are all kind of things possible when you change aircraft parameters, because these eigenvalues are depending upon parameters. When you change the parameters, these eigenvalues are going to move in the complex plane. But at low alpha flying conditions, where we can relate our aircraft dynamics with some of the design parameters, that is what is our interest actually right now .... So, we are flying a condition which is alpha equal to 0 right now. We only have forward speed and theta naught which is not equal to 0.

So, what we will do is, we will now assume, make assumption that the short period dynamics can be represented by a pure pitching motion, and the phugoid motion can be represented by change in flight path angle, and the forward speed.... We can make these assumptions and ... assuming that these two dynamics are well separated.

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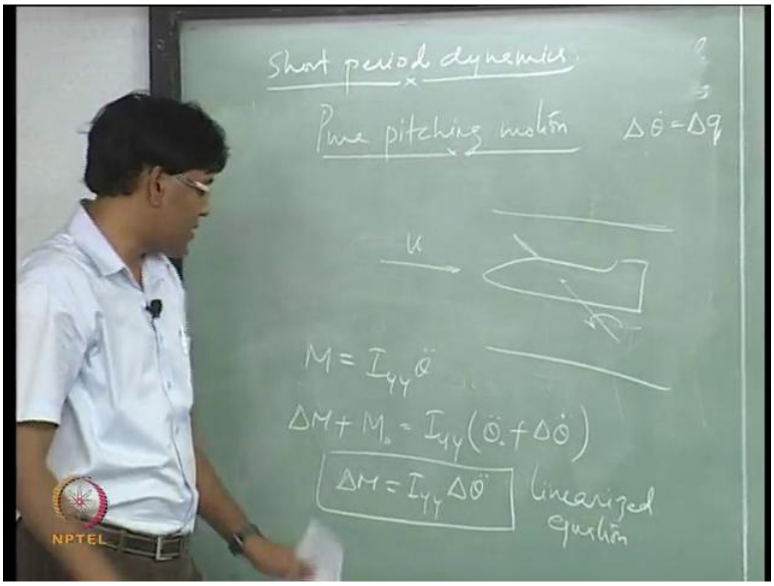

... Let us look at the dynamics of each one of them separately. Short period ..., so, I am assuming there that it is a pure pitching motion.

$$M = I_{yy}\ddot{\theta}; \quad \Delta M + \underbrace{M_0}_{0} = I_{yy}(\underbrace{\ddot{\theta}_0}_{0} + \Delta\ddot{\theta}) \Rightarrow \Delta M = I_{yy}\Delta\ddot{\theta} \Rightarrow \Delta\ddot{\theta} = \frac{\Delta M}{I_{yy}} \qquad (6)$$

.... This is a very interesting thing attached to this, you know, why I am doing this study separately is, we can also observe this dynamics or you can simulate this dynamics in wind tunnel. If you assume that it is going to be a pure pitching motion ..... So, you have a wind tunnel and this is your model which is aligned with this flow. So, you are flying cruise, alpha is 0, and then you give a perturbation in pitch. So, you have motion around this point, you know, through which we have passed the rod in the wind tunnel. So, aircraft will start pitching about this, ... it is actually constrained to only have pitching motion and nothing else. So, you cannot have a forward speed or any other motion. So, only pure pitching motion is possible in the wind tunnel. If you pass a rod through this point.

... So, let us look at this particular motion - pure pitching motion. The equations representing this dynamics is given by this...Refer Eq(6).

So, ... this is actual equation of motion, I am only interested in looking at the perturbed dynamics. So, what I do is … I linearise the equations.

We have already shown that from a level flying condition, you know, when we linearise the equation of motion what we get is .... this. This we have shown earlier.

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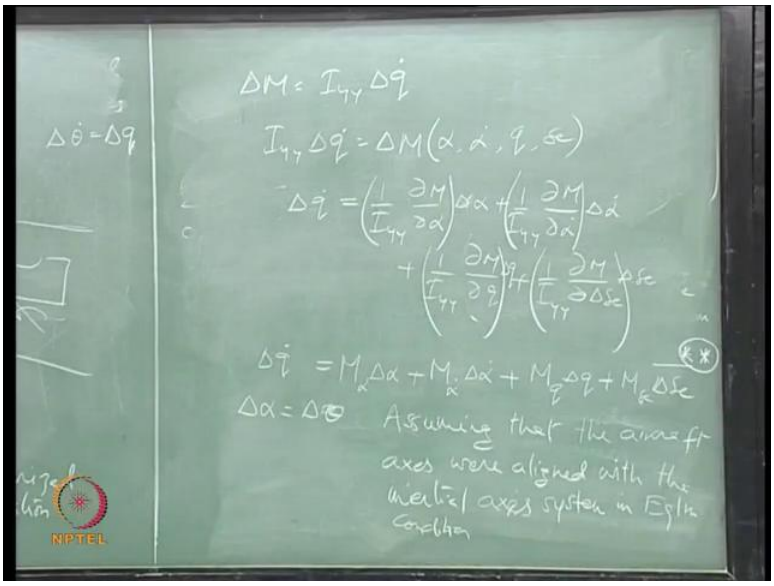

$$\Delta\dot{\theta} = \Delta\dot{\alpha} = \Delta q$$

$$\Delta\dot{q} = \Delta\ddot{\alpha} = \Delta\ddot{\theta} = \frac{\Delta M}{I_{yy}} = \underbrace{\left(\frac{1}{I_{yy}}.\frac{\partial M}{\partial \alpha}\right)}_{M_\alpha}\Delta\alpha + \underbrace{\left(\frac{1}{I_{yy}}.\frac{\partial M}{\partial \dot{\alpha}}\right)}_{M_{\dot{\alpha}}}\Delta\dot{\alpha} + \underbrace{\left(\frac{1}{I_{yy}}.\frac{\partial M}{\partial q}\right)}_{M_q}\Delta q + \underbrace{\left(\frac{1}{I_{yy}}.\frac{\partial M}{\partial \delta e}\right)}_{M_{\delta e}}\Delta\delta e$$

$$\Delta\ddot{\alpha} = M_\alpha\Delta\alpha + M_{\dot{\alpha}}\Delta\dot{\alpha} + M_q\Delta q + M_{\delta e}\Delta\delta e \qquad (7)$$

And this delta M is going to be a function of alpha, alpha dot .... anything else? ....Refer Eq(7).

Our q dot equation was …. We have already linearised the equations of .... and also write this as ..... Refer Eq(7).

43:08

Let us make another assumption, and that is, let us say to start with, this model was aligned with the Earth fixed axis system ..., which is also valid....If I put this horizontally ..., to start with, which is aligned with the you know, this air speed ... then I can also say that delta alpha is nothing but delta theta Refer Eq(7). So, to start with ... the body axis is aligned with the Earth fixed inertial axis system, and then,.. there is a pitch from this equilibrium condition. Then delta theta is also going to be equal to delta alpha. So, look ... at this, if it takes, if there is a pitch introduced, then the X axis of the aircraft will start making an angle with this velocity vector which will give the delta alpha, and because initially it was aligned with the X axis of the Earth fixed frame, this is also going to be delta theta. ......

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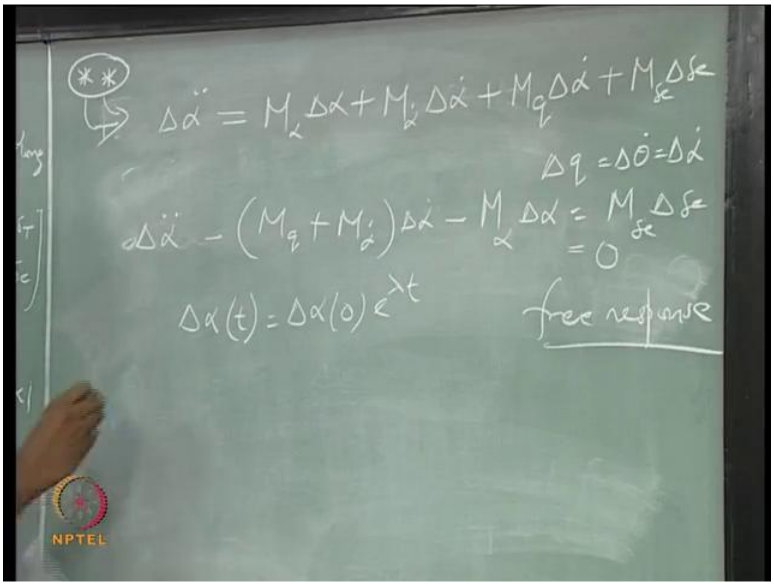

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So, I can write this linearised equation ….. in terms of delta alpha ..... Refer Eq(7).

$$\Delta\ddot{\alpha} - M_{\alpha}\Delta\alpha - M_{\dot{\alpha}}\Delta\dot{\alpha} - M_{q}\Delta\dot{\alpha} = M_{\delta e}\Delta\delta e$$
$$\Rightarrow \Delta\ddot{\alpha} - (M_{\dot{\alpha}} + M_{q})\Delta\dot{\alpha} - M_{\alpha}\Delta\alpha = M_{\delta e}\Delta\delta e \quad (8)$$

What is the solution of this? So, you can assume a solution which is for the homogenous or the free response. So, this is a general form of solution for the free response. Now, you substitute this, in this part equal to 0.

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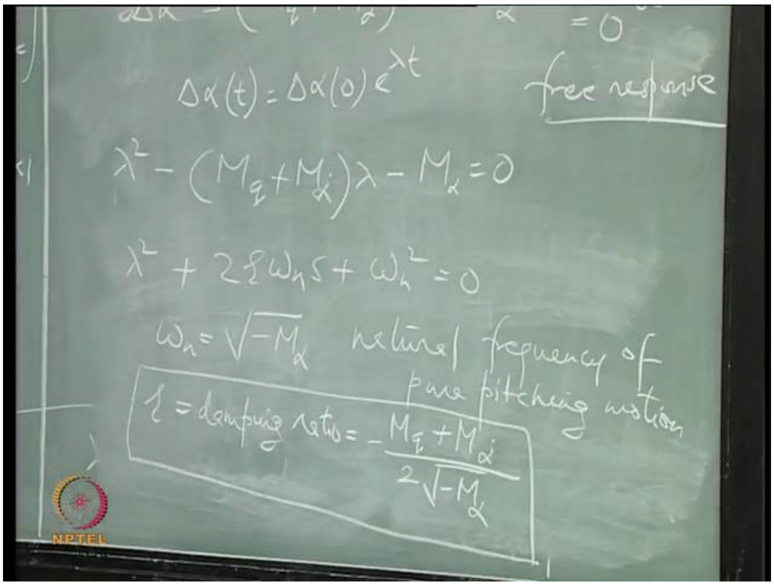

$$\Delta\ddot{\alpha} - (M_{\dot{\alpha}} + M_{q})\Delta\dot{\alpha} - M_{\alpha}\Delta\alpha = M_{\delta e}\Delta\delta e = 0; \Delta\alpha(t) = e^{\lambda t}\Delta\alpha(0)$$
$$\Rightarrow \lambda^2 - (M_{\dot{\alpha}} + M_{q})\lambda - M_{\alpha} = 0; \Delta\alpha(t) \neq 0 \quad (9)$$

What will arrive at is an eigenvalue problem ..., an equation which is second order equation in lambda .....Refer Eq(9).

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \quad (10)$$

So, the frequency of motion is given by M alpha term. So, you compare with, compare this with the general form of characteristic equation for a second order system .....Refer Eq(10). And then you can find out the frequency, and the damping parameter of, you know, corresponding to the pitching motion .....Refer Eq(11).

$$\omega_n = \sqrt{-M_\alpha}; \quad \zeta = -\frac{M_q + M_{\dot{\alpha}}}{2\sqrt{-M_\alpha}} \tag{11}$$

See, you see that the frequency of the motion only depends upon the stiffness in pitch, and this damping ratio depends upon these three parameters.

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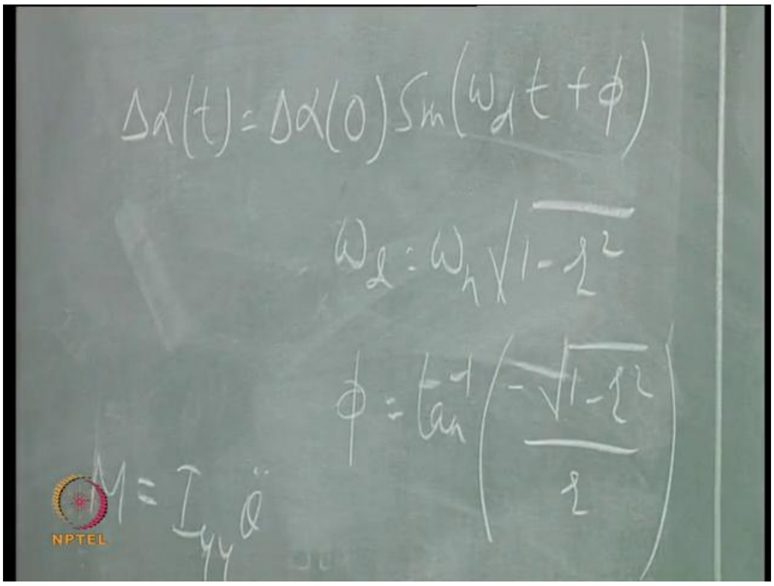

$$\Delta\alpha(t) = \Delta\alpha(0)\sin(\omega_d t + \varphi); \quad \omega_d = \omega_n\sqrt{1-\zeta^2}, \varphi = \tan^{-1}\left(\frac{-\sqrt{1-\zeta^2}}{\zeta}\right) \tag{12}$$

And the final solution for the free pitching motion can be written as .....Refer Eq(12), delta alpha in t delta alpha at 0 into sin of omega d t plus phi. omega d is ..... Refer Eq(12).

So, we will stop at this point and continue in the next class.