Flight Dynamic II (Stability)
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Module No. # 02

Equilibrium and Stability

Lecture No. # 03

Equilibrium States, Static Stability

(No Audio from 00:11 to 00:30) One of the things that an aircraft designer always have to keep in mind is about, how an aircraft is going to behave in motion with respect to external disturbances, and the disturbances can be small or large. When we talk about disturbances and the motion, the aircraft response with respect to those disturbances, we are talking about aircraft stability.

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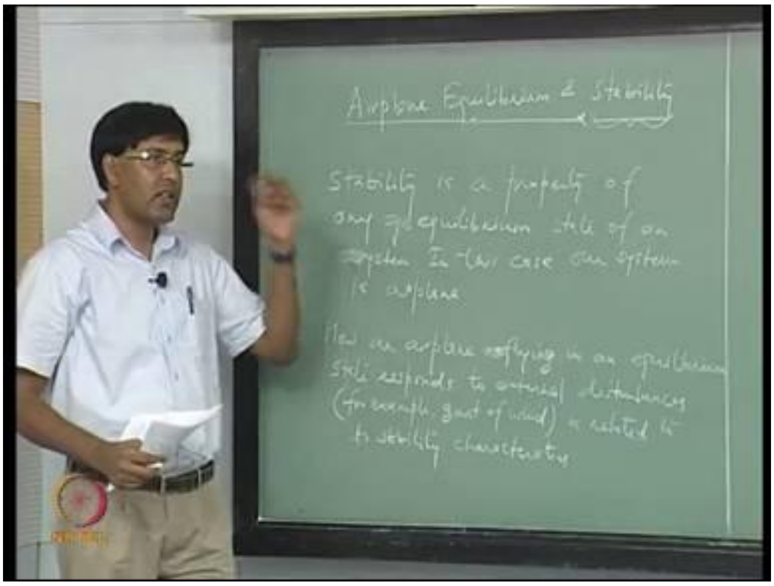

Stability is a property of an equilibrium state of a system (No audio from 01:07 to 01:17). In this case, our system is airplane. Let me repeat, what we are interested in looking at? We are interested in looking at how an aircraft which is flying in equilibrium responds to any external disturbance which can be coming from wind, for example. How an airplane flying in an equilibrium state responds to external disturbances, for example gust of wind, is related to its stability characteristics; and that stability characteristic is a property of that particular equilibrium condition that the aircraft is flying in. So let us try to define what (an) equilibrium state is (No audio from 03:48 to 03:58).

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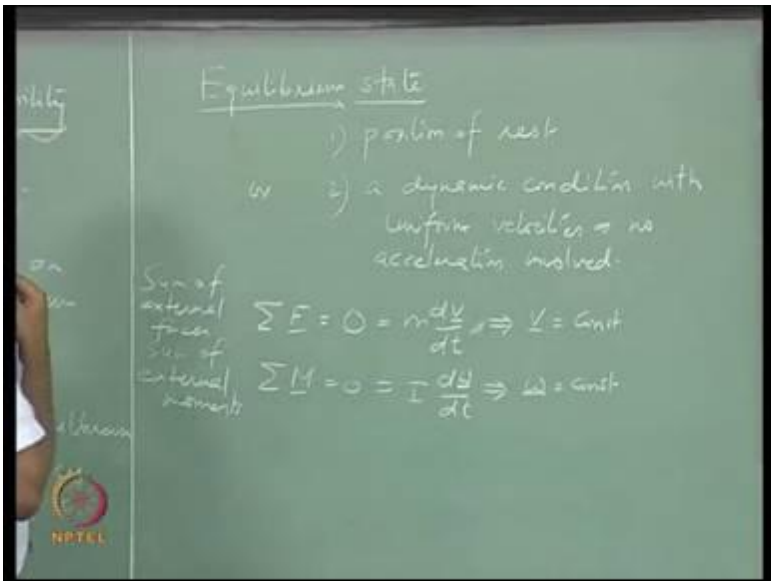

As we know, equilibrium state can be a position of rest or a dynamic condition with uniform velocities, that also means that no acceleration is involved. Mathematically this translates into these two equations ($\sum \underline{F} = 0; \sum \underline{M} = 0$). Sum of all external forces acting on the body are balanced ($\sum \underline{F} = 0$), so that net acceleration is 0 and, that gives, that results in V equal to constant.

So, if an aircraft is flying at constant velocity, uniform velocity, then we say that the aircraft is in equilibrium. Added to that, we should also have this condition ($\sum \underline{M} = 0$) satisfied which is added to that condition ($\sum \underline{F} = 0$). We should also have this ($\sum \underline{M} = 0$) satisfied because the aircraft is (having) 6 degree of freedom, it is having 6 degree of freedom motion, 3 in translation described by these 3 force equations. And this $\underline{F}$ is a vector consisting of the motion along the 3 axes of the aircraft and the moments which are resulting in 3 angular motions of aircraft (Refer Slide Time: 06:00). This also means, if omega is the angular velocity vector, multiply that by this inertia, that equal to 0, this ratio ($d\underline{\omega}/dt$) is giving the angular acceleration. So, if the sum of external moments is 0, then what we arrive at is this ($\underline{\omega} = const.$) (Refer Slide Time: 07:07). A dynamic equilibrium condition will be represented by a uniform velocity that can be translational or angular.

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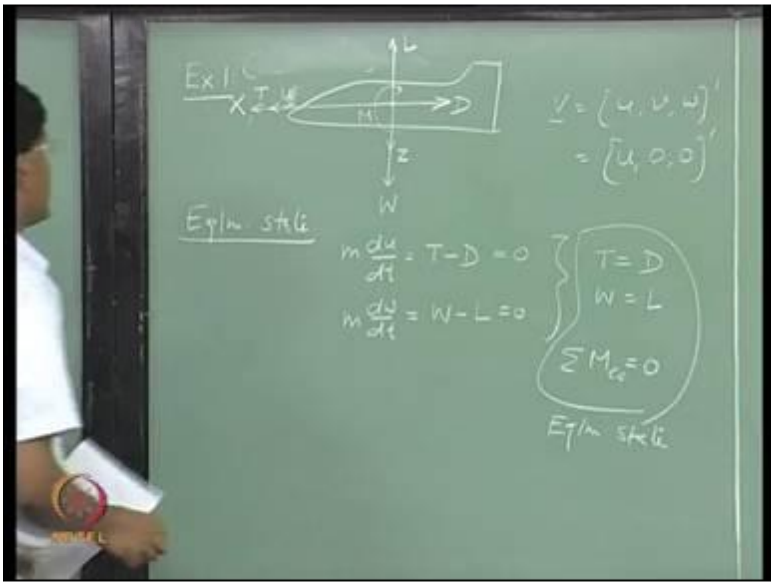

Let us look at some of the flying equilibrium states of aircraft. Example of such equilibrium states could be, for example, this cruise level flight. Aircraft is moving with velocity V along its X axis and thrust is also acting along the X axis of the aircraft. The equilibrium state and the corresponding equations are, just balance these forces in the X and Z directions and what we get is. So if I say u is the forward velocity, so this velocity vector is ($V = [u\ 0\ 0]'$), or we can just write this as u.

So, if you write down the balance of forces in the X and Z directions, what you get is ($m\dfrac{du}{dt} = T - D$), and this change in u with respect to time, we have to set to 0 to find the equilibrium condition. In the Z direction similarly you have ($m\dfrac{dw}{dt} = W - L$), w is the velocity of the aircraft in the Z direction, and that gives you, and this also has to be set to 0 to get the equilibrium condition. So, equilibrium conditions with the degrees of freedom both in translation and rotation, for this case, so here we are talking about longitudinal flying condition which is cruise level flying condition.

The equilibrium state can be defined by these two balance of forces ($T = D$; $L = W$); and the moments about the CG also should be 0. So, these three equations, this M here is the pitching moment, moment about the C G in this longitudinal plane of aircraft is the pitching moment and, these are the forces acting on the aircraft. So, these three equations define the equilibrium state (Refer Slide Time: 11:10). Let us look at another example.

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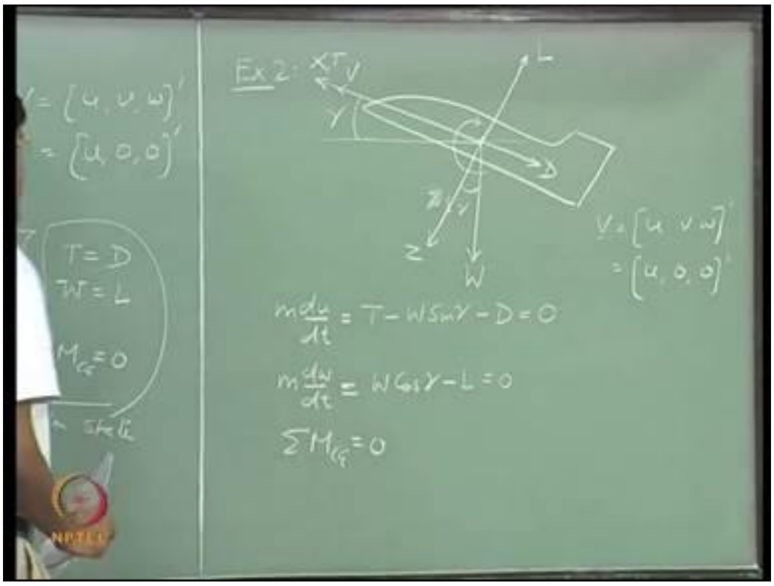

This example is also from longitudinal flying condition and here the aircraft is climbing, gamma is the flight path angle, this *W* is the weight of the aircraft, *L* is the lift, drag is acting against the forward motion, this is the Z axis. Let us try to write down the components of forces acting along X and Z directions.

We will again assume that thrust is acting along the X axis of the aircraft and *V* is velocity vector, which is again having only the velocity component in the X direction. So, you can write the equation of motion along the X axis. Thrust which is taking it in the forward direction and there are forces opposing this motion and there is component of weight along drag which is opposite to this X axis.

This ($m\dfrac{du}{dt} = T - W\sin\gamma - D$) has to be set to 0 to find the equilibrium state and the motion in the Z direction is given by ($m\dfrac{dw}{dt} = W\cos\gamma - L$), this has to be 0 at equilibrium state and the moments also have to be balanced, so this condition ($\sum M_{CG} = 0$) is still active (Refer Slide Time: 14:20). So, these three equations will define the equilibrium state of aircraft in steady climb. These three equations ($T = D$; $L = W$, $\sum M_{CG} = 0$) defined the equilibrium state in the level cruise condition and if you want to look at a non-longitudinal equilibrium state, it can be, for example, a steady level turn state.

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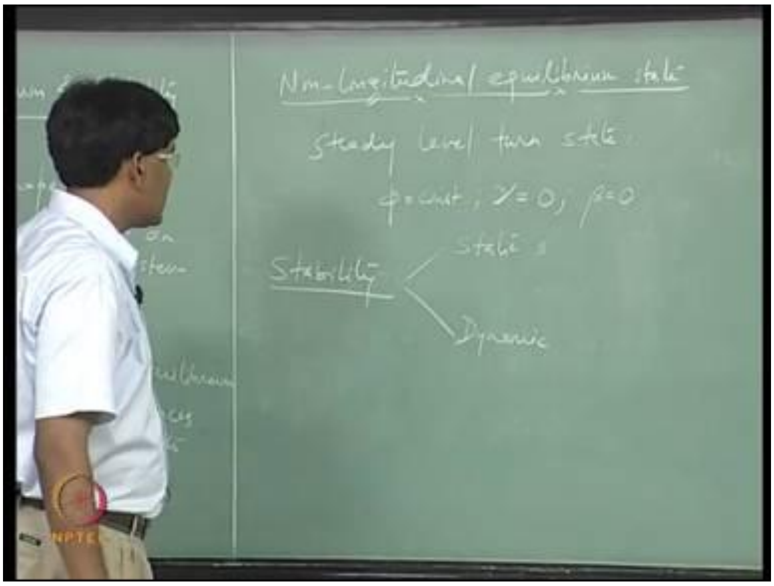

What is happening in this case is, bank angle is a constant, level condition: flight path angle has to be 0. On top of that if you want to say it's a coordinated level turn state, then we can also add this condition which is side slip angle equal to 0. This will also constitute one equilibrium state, which is a non longitudinal equilibrium state. What (do) we want to understand now through this is, we want to look at when the aircraft is flying in one of these equilibrium states and a small disturbance hits the aircraft, how it is going to respond to that disturbance and that is the subject of stability.

Stability has two notions, one is static and another is dynamic. So, we will first look at what static stability is, static stability of an equilibrium point. And what is dynamic stability? We will try to understand the difference between the two and see where they are useful in flight dynamics.

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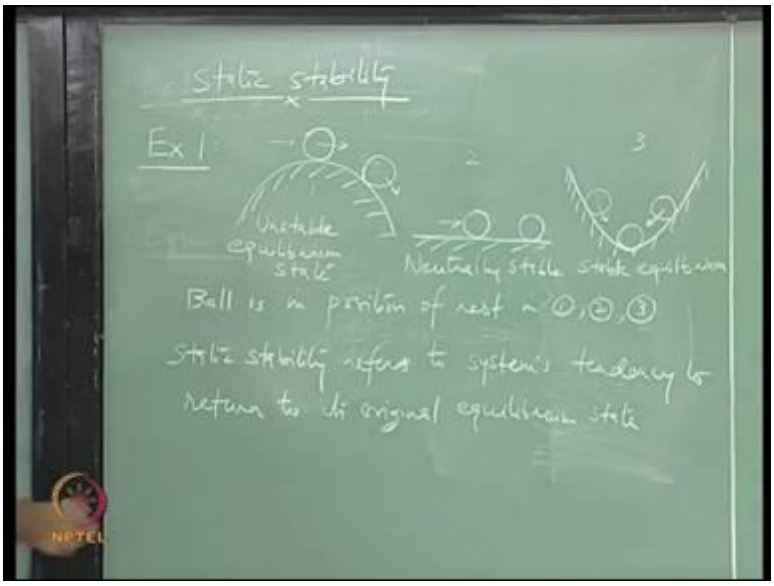

These concepts are applicable to any system, any physical system, not just to the airplane. Let us look at, what I said in the beginning was that stability is a property of an equilibrium state. So, we have to look at first an equilibrium state and see how a system when disturbed from equilibrium state responds to the disturbance. The stability refers to the system's property to come back to its original equilibrium state, when it is disturbed from there.

Let us look at simple examples, let us look at example 1. We have let us say, in this case, a ball sitting on top of this hill. There is another situation when this surface is now flat and the ball is sitting at this position. This position is a static position of rest or you know equilibrium position of rest. We can also have this kind of curved surface which is like a well and the ball is sitting at the bottom of this. Now let us look at what happens, these 3 positions are equilibrium positions, so ball is in position of rest in (positions) 1, 2 and 3.

So, this is 1, 2 and 3. We will also assume that, this surface is smooth and it does not offer any resistance to the motion of the ball. This is a very, not realistic assumption, but this simplifies lot of things. So, we want to look at how this ball is going to behave, when the ball is hit from or disturbed from this equilibrium position. So the ball is hit.

We know that when the ball is hit from this side, ball is going to have a motion which is going to make it fall. What is happening in this case is, when we disturb this ball from its equilibrium position of rest, it starts going away from this position; when this happens we say that, this equilibrium position of this ball is unstable.

When you hit it from the other side, it is going to fall to the other side. So, this position of equilibrium is unstable for this ball. Now let us look at what is happening here? Here when you hit the ball, it is going to go away from this equilibrium position and going to acquire another position. So, in such a case, we say that the ball is neutrally stable, or the equilibrium state of this ball is neutrally stable. In this case when you hit the ball from this position, it uses that energy of the disturbance to rise along this wall of the well and goes up to some point and, once it has lost that energy, it starts coming back, similar thing will happen on the other side.

What is happening in this case is, the ball is trying to come back to this equilibrium position. So, this equilibrium position is a stable equilibrium (No audio from 23:45 to23:57). Right now what we are saying is, we want to look at how the system is responding to a disturbance. We are not saying whether it has really achieved its original state of equilibrium or not (No audio from 24:15 to 24:25)

Static stability refers to system's, ball or the aircraft, tendency to return to its original (equilibrium) state, static stability refers to system's tendency to return to its original equilibrium state. Clearly in this case, it does not have any tendency to come back to this equilibrium state (position 1), so we say that (position 1) is unstable. In this case (2; on the flat surface) also its going to go acquire another equilibrium state, so that is why it is neutrally stable. Here (position 3), it has a tendency to come back to its original equilibrium state. So, we say that this (position 3) is a stable equilibrium state for this ball (Refer Slide Time: 25:10).

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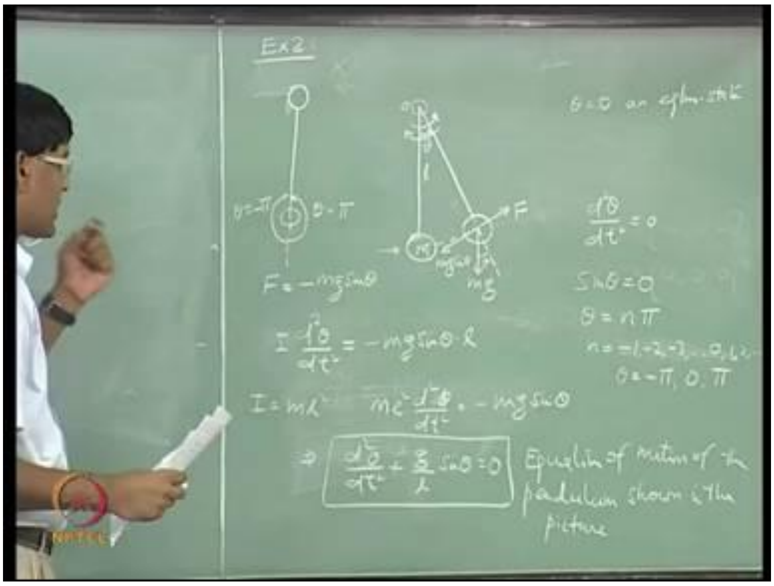

Let us look at another example. So here we have a pendulum. Bob of mass m hanging at one end of this wire of length l and this O is the suspension point (Refer Slide Time: 25:55). This clearly is an equilibrium state which is also the position of rest for this pendulum. Now, let us see what happens, when we hit this bob from this equilibrium state. It is going to go away from this equilibrium state and acquire, depending on the energy of the hit, it is going to take a new position. See what is happening at this position to the forces? We will also associate one variable θ with this displacement from this equilibrium state which is $(\theta = 0)$. We will see how we get this from the equation (No audio from 27:24 to 27:45).

Normal to this line is the path of this m representing its motion, and we want to write down the equation of motion for this motion of mass m. F is here $-mg\sin\theta$. Let us try to write down the moment equation for this case. We are trying to write down the equation of moment around the suspension point. This pendulum is having angular motion about the suspension point, and I is the inertia of mass m, rotational inertia of mass m, the equation of motion is $I\frac{d^2\theta}{dt^2} = -mg\sin\theta . l$. (No audio from 29:20 to 29:33). So positive θ is this (anticlockwise angular displacement) and positive moment is this (anticlockwise) and negative moment is this (clockwise) (Refer Slide Time: 29:35).

Here this ($\frac{d^2\theta}{dt^2}$) is the acceleration, positive acceleration and angular acceleration, (angular acceleration) into I is giving the moment in this positive direction, which is balanced by this force $mg\sin\theta$ into l in the negative direction, that is why this negative sign. I is ml^2. (No audio from 30:27 to 31:03). From this equation ($I\frac{d^2\theta}{dt^2} = -mg\sin\theta.l$) what we get is: ($\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$). So, this is the equation of motion of pendulum shown in the picture (Refer Slide Time: 30:50).

To define the equilibrium state for this pendulum, what we have to do is, we have to either set the sigma M to 0 or we can also set all the variables, derivative of variables with respect to time to 0. So, in this case, this rate of change of any variable with respect to time is 0, that is another definition of an equilibrium state. In this case what we get as the equilibrium state is the result of this equation ($\sin\theta = 0$).

sin theta equal to 0 has solutions, which is *n* into *pi* and *n* can be, *n* = 0, 1, 2, 3, ..., and so on. *n* is an integer. Physically looks like that there are two equilibrium solutions possible and they are. Actually *n* can be -1, -2, -3…0, 1, 2. The pendulum can go in the other direction also (No audio from 33:07 to 33:22).

You can have solutions for this case as: ($\theta = -\pi, 0, \pi$). These are physical solutions and this ($\theta = n\pi$) is the mathematical solution. So, here we have three equilibrium states, two of them are actually same. One equilibrium state is theta equal to 0 and the other equilibrium state is vertically up position of this mass m (No audio from 33:58 to 34:35).

Now, let us look at what is happening to this force when theta is changing. What we want to know is, how this force is changing with respect to the angular displacement of the pendulum.

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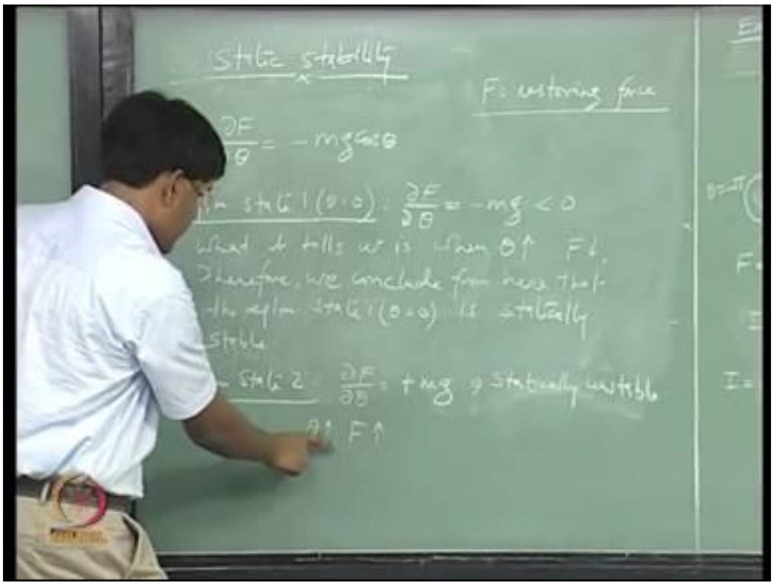

From this equation what we get is ($\frac{\partial F}{\partial \theta} = -mg\cos\theta$). As I said, the equilibrium, stability is a property of the equilibrium, so we have to look at how this force is changing around the equilibrium states.

Remember, this force is developing only because of the motion of this mass *m*. If this θ is 0 there is no force, this force is developing because of the displacement of the mass *m* and it is called restoring force. If you look at the derivative of this force, which is being developed because of the motion of the mass *m*, derivative of this force with respect to the displacement variable is -*mg* (at equilibrium position $\theta = 0$), which is less than 0. So, what it tells us is, when θ is going up or increasing, this *F* is going down, that is what this derivative is telling.

This force is actually acting in the direction to decrease the angle θ, and that is when we say that, this equilibrium state around which I have done this is a stable equilibrium state (No audio from 37:21 to 38:06). Now, let us look at what happens. This ($\theta = 0$ equilibrium state) is one and these ($\theta = -\pi, \pi$ equilibrium states) are two. Now let us look at what is happening at equilibrium state that is the vertically upward position ($\theta = -\pi, \pi$) of mass *m*. $\frac{\partial F}{\partial \theta}$ is +*mg* in this case.

Now, here what is happening is, when the ball is disturbed from this ($\theta = -\pi, \pi$) equilibrium condition, this force along the motion of the ball, in the increasing direction of θ, it is going to grow. So, it is actually taking the ball away from the equilibrium condition and that is when we say that the equilibrium state 2 is statically unstable. (No audio from 39:32 to 39:51).

What is happening in this case is, with θ, this force, in the direction of increasing θ is growing, so it is going to further increase θ, and θ is the disturbance it is a change from this equilibrium state ($\theta = -\pi, \pi$). What is happening in this case is, theta is growing, this force is also going up and helping theta to grow further. And that is when we say that (the ball) is going to go away from the equilibrium state 2, which is theta equal to *pi* or -*pi* and, therefore, the equilibrium state 2 is statically unstable.

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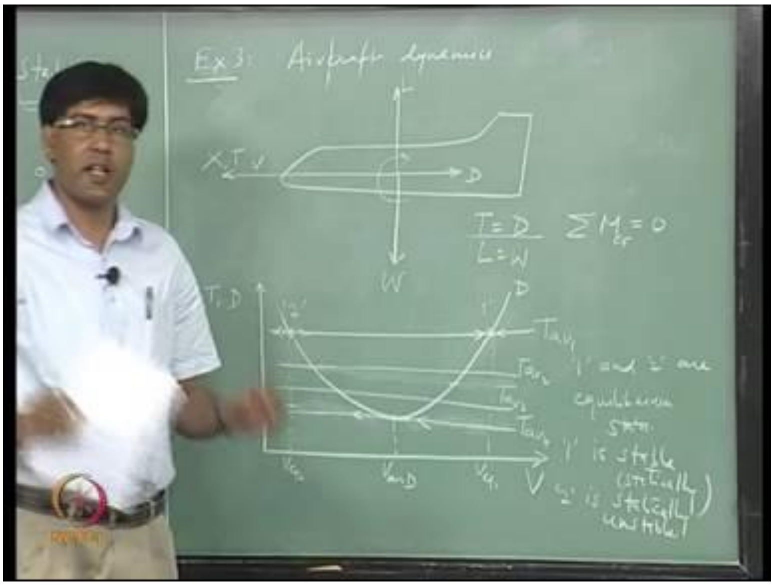

Now, this example 3 is from aircraft dynamics. Here we are looking at this aircraft which is flying level having only velocity component in this X direction. In equilibrium, thrust is equal to drag and lift is equal to the weight of the aircraft, and let us say sum of all external pitching moments is also equal to 0. In this situation, what we want to study is, to look at if the aircraft is statically stable with respect to disturbances or not. And this we are only doing graphically. So, here we plot thrust and drag versus speed of aircraft. Drag profile is parabolic and this drag has to be overcome according to this equation (*T* = *D*; to maintain equilibrium).

So, thrust available should be able to overcome the drag. Let us say, we draw this (thrust available) for a jet engine (on top of the drag curve). Thrust is almost constant with respect to the velocity (for jet engines), so this is my thrust available and this is drag (Refer Slide Time:

43:00). Where these two curves meet, we are satisfying thrust equal to drag condition and that intersection point is our equilibrium state. This (right intersection) is equilibrium state 1 and the other (left intersection) equilibrium state is here, which is marked as 2.

Now, let us try to look at the stability properties of these 2 equilibrium states. So, 1 and 2 are equilibrium states. We can look at stability with respect to all variables involved in the longitudinal plane. But here we restrict ourselves (only) to change in speed in the forward direction. Let us look at what is happening to this equilibrium state 1. So, let us say there is a change in speed, there is a disturbance which is causing a change in speed from this equilibrium state so that now aircraft speed has increased to this point from this equilibrium state.

Now, when this happens, what is happening here is, drag is going to go up, it is going to change because of this increase in speed, it is going to go up according to this curve, and it becomes more than the thrust which is available. So, automatically there is a deceleration of aircraft which is trying to bring it back (towards the equilibrium state from where it was disturbed in speed). For the similar reasons, when the perturbation results in change in velocity in the negative direction, velocity is reducing, speed of aircraft in the forward direction is reducing and the drag goes down, and the aircraft will accelerate because thrust available is more than the drag now. Let us put an arrow like this (to indicate direction of motion from perturbed condition).

So, in both situations we see that, the tendency of the aircraft is to come back to this equilibrium state 1. Change in speed is resulting in aircraft trying to reduce that change in speed, and then we say that this equilibrium condition 1, with respect to the change in speed, is stable. That is statically stable. We can give the same arguments on this side also for equilibrium state 2. This is your V equilibrium 2, V equilibrium 1. If there is a decrease in velocity in this case, now drag is again, it is more than the thrust available, so aircraft will decelerate.

It is going to take it away from this equilibrium condition 2, and the other side, giving the similar argument, the aircraft is going to go away from this equilibrium condition. So, this equilibrium condition 2 is statically unstable with respect to this change in speed due to the disturbance (No audio from 48:04 to 48:24). Now, let us look at what happens when you change the thrust available. Thrust available will also change with altitude and when you go to higher altitudes, thrust available is going to go down and there will be one, this line is going to be almost parallel to the first one (No audio from 48:58 to 49:10).

So, let us try to change the thrust available. You see that there is one thrust condition, which is resulting in a single equilibrium state, and that is associated with the velocity which is resulting in the minimum drag. Now, this case is of a neutrally stable equilibrium state, because here, what is happening is, when the speed is increasing on this (right) side because of the perturbation, and drag is more than the thrust available and this is what is happening (aircraft is decelerating).

On the other side also, it is the same thing (aircraft is decelerating). So this is neutrally stable equilibrium state, and any equilibrium state for the thrust available higher than the minimum drag, any equilibrium state on the backside of this curve is unstable, and (on) the other (right) side it is stable. But in reality, what it tells us is that, whole of this part of this curve (left side of the minimum drag point), where we can find several trim points (equilibrium sates) by changing the trust available, are not available for flying, they are all unstable, and on this (right) side is all (trims or equilibrium sates) stable. In reality, this is not what happens and we will talk about it when we come to dynamic stability.