Flight Dynamics II (Stability)
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Module No. # 08
Equations of Rigid Aircraft Six-Degree-of-Freedom Motion
Lecture No. # 26
Nonlinearities and Associated Aircraft Behavior

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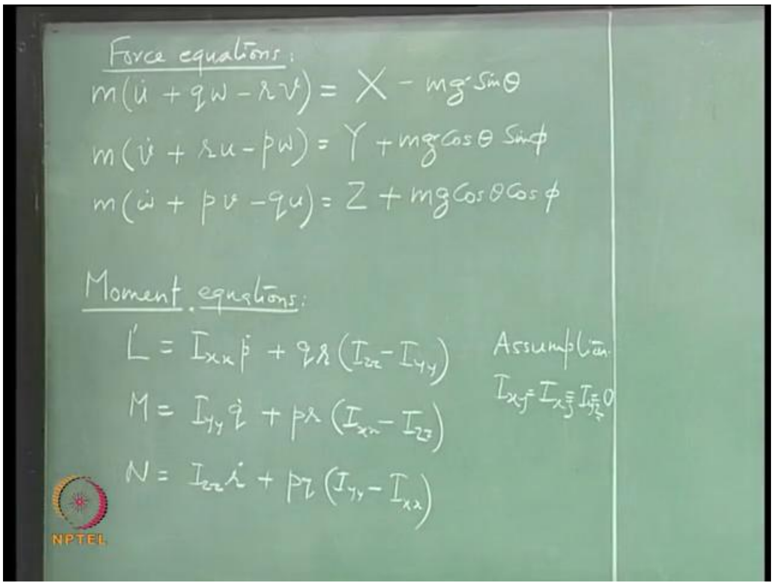

So, let me summarize the equations of motion. (No audio from 00:17 to 00:40) So, I am keeping here all the forces together. External forces acting along X axis of the aircraft and this is the gravitational force which I have kept separately.

$$
\begin{aligned}
m(\dot{u} + qw - rv) &= X - mg\sin\theta \\
m(\dot{v} + ru - pw) &= Y + mg\cos\theta\sin\phi \\
m(\dot{w} + pv - qu) &= Y + mg\cos\theta\sin\phi
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
L' &= I_{xx}\dot{p} + qr(I_{zz} - I_{yy}) \\
M &= I_{yy}\dot{q} + pr(I_{xx} - I_{zz}) \\
N &= I_{zz}\dot{r} + pq(I_{yy} - I_{xx})
\end{aligned}
\tag{2}
$$

This dot over a variable will indicate time rate of that derivative. .... (No Audio from 01:16 to 02:08) So, these three force equations. ...(Refer Eq(1)) this L prime is for rolling moment. All of you, and I thought I will just put a prime here for, you know, differentiating it from the lift. .... (Refer Eq(2)) This M is for pitching moment. .... (No audio from 03:25 to 04:29) This is assuming that the body fixed axis system is coinciding with the principle axis system of the aircraft, which is not a bad assumption because, you know, it depends where you define your origin and origin of your body fixed axis system and then, the origin will be placed at the center of mass. So, we can, safely take this assumption.

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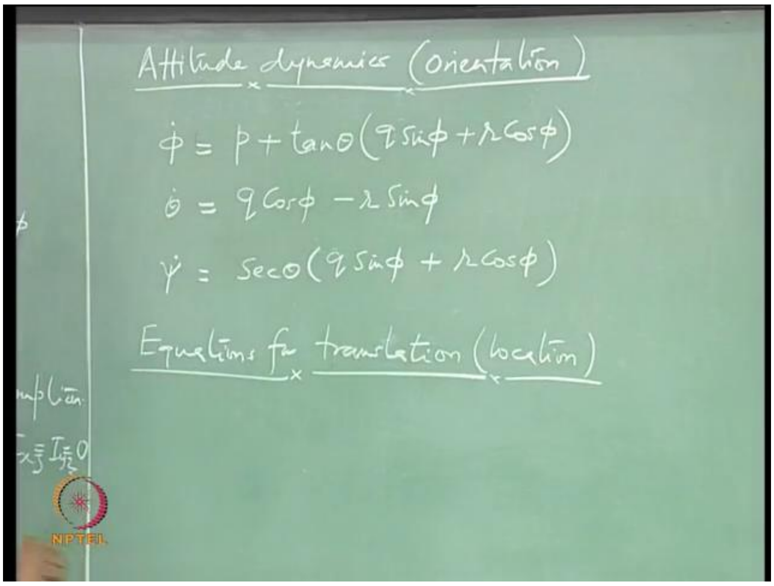

$$\dot{\phi} = p + \tan\theta(q\sin\phi + r\cos\phi)$$
$$\dot{\theta} = q\cos\phi - r\sin\phi \qquad (3)$$
$$\dot{\psi} = \sec\theta(q\sin\phi + r\cos\phi)$$

So, assumption here is that (Refer Slide Time: 05:13) $I_{xy} = I_{xz} = I_{yz} = 0$. ..... and we have equations for the attitude (kinematics) ....(Refer Eq(3)) or the orientation.

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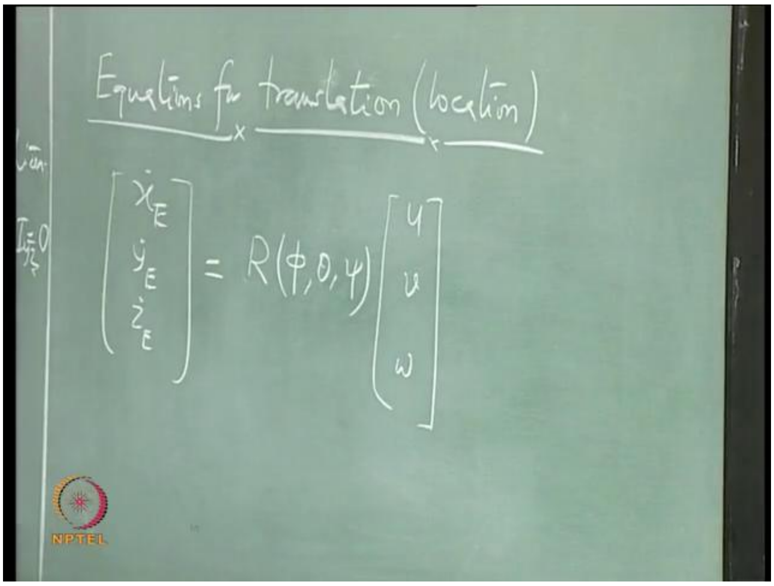

$$
\begin{bmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{bmatrix} = R(\phi,\theta,\psi) \begin{bmatrix} u \\ v \\ w \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad (4)
$$

So, equation for the evolution of attitude (angles) in time, ...(Refer Eq(3)) (Refer Slide time: 06:12) Equations for finding out the location of the aircraft with respect to the initial frame of reference (Refer Slide Time: 08:11) .....(Refer Eq(4)) are these. So, now, we have twelve equations, and what we want to do is, we want to look at the dynamic stability of aircraft using these equations. Before I start doing that, let me write these force equations again in a convenient axis system.

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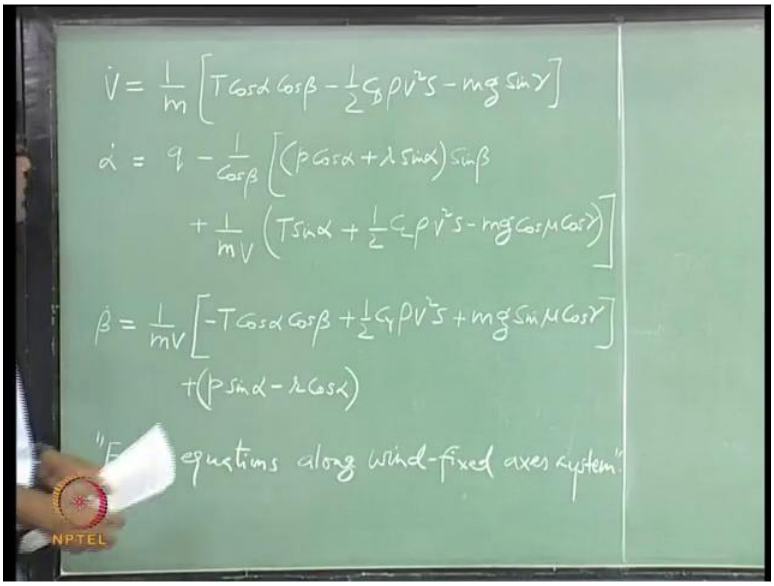

$$
\begin{aligned}
\dot{V} &= \frac{1}{m}\left[T\cos\alpha\cos\beta - \tfrac{1}{2}C_D\rho V^2 S - mg\sin\gamma\right] \\
\dot{\alpha} &= q - \frac{1}{\cos\beta}\left[(p\cos\alpha + r\sin\alpha)\sin\beta + \frac{1}{mV}\left(T\sin\alpha + \tfrac{1}{2}C_L\rho V^2 S - mg\cos\mu\cos\gamma\right)\right] \qquad (5) \\
\dot{\beta} &= \frac{1}{mV}\left[-T\cos\alpha\sin\beta + \tfrac{1}{2}C_Y\rho V^2 S + mg\sin\mu\cos\gamma\right] + (p\sin\alpha - r\cos\alpha)
\end{aligned}
$$

This is also perfectly all right. Now, we want to write these equations in a different axis system. So, I will try to write these equations in the wind fixed axis system because there we recognize the aerodynamic forces. We know that it is drag, lift and the side force. So, I am directly writing down the equations and I am assuming here that the thrust is acting along the X axis of the aircraft.

(No audio from 10:06 to 12:18)

.....(Refer Eq(5)) So, these are the force equations along body fixed, wind-fixed axis system. .... Of course, we recognize these angles. Gamma γ is what? What is gamma? Gamma is the flight path angle ..... and the way we define this angle is ..... (Refer Slide Time: 13:36)

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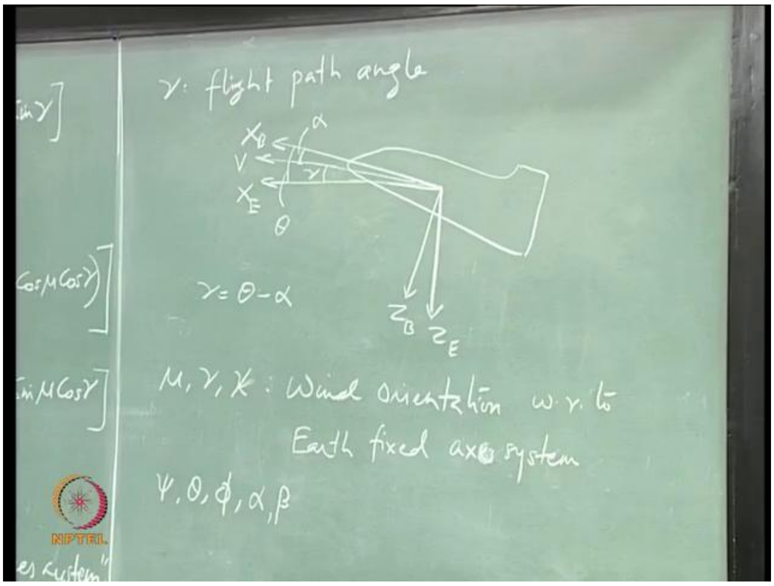

So, if this is the direction of your velocity vector or aircraft is flying along this vector and what is gamma γ? So, I have drawn here a local horizon. This is the angle which is called flight path angle in this 2 d plane (angle between velocity vector and local horizon X_E), longitudinal plane. What is this angle? What is this angle? Alpha α (angle between relative velocity vector and X_B axis) , and this angle? What is this angle? I am talking about the longitudinal plane. So, this angle is theta θ (angle between X_B axis and local horizon X_E axis). So, in this 2 d plane, you can see that gamma is nothing but theta minus alpha, $\gamma = \theta - \alpha$. So, in this 2 d plane it is actually the angle between the velocity vector which is wind fixed and the Earth fixed X axis.

Now, let us say my aircraft is having a velocity vector which is including angle of attack and side-slip. It is not in 2 d plane. Then you have to talk about three angles which will relate components of vectors in Earth fixed axis system and the wind fixed axis system.

We always have to talk about these three angles. In the case when we are talking about components of a vector in the body fixed axis system and the Earth fixed axis system, we have three angles and those angles are Euler angles, phi, theta and psi ($\phi\ \theta\ \psi$).

So, here, there are three angles – mu, gamma and chi ($\mu\ \gamma\ \chi$), which is giving you the wind, orientation of the wind fixed axis system with respect to the Earth fixed, you know, initial frame of reference. .... How do you find the relation? Now, between the vectors, components of vector in Earth fixed axis system and components of the same vector in the wind fixed axis system, how do you find the relation? You already know the relation between the components of vector in the body fixed axis system and the Earth fixed axis system. We have to include two more rotations which will include the angles alpha and beta.

So, you will, in all have rotations, psi, theta, phi and alpha, beta, five rotations and then you can relate, or you can find components of any vector, you know, vector which is which we know in the Earth fixed axis system. I can find the component of that vector in the wind fixed axis system using five rotation matrices. So, it is not the objective of this course to go into details of this, but I believe you know this part, you understand this part now.

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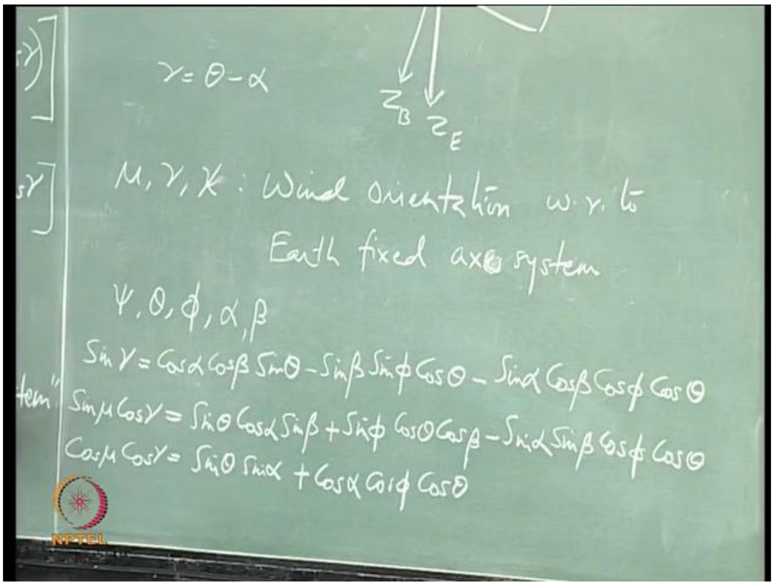

$$
\begin{aligned}
\sin\gamma &= \cos\alpha\cos\beta\sin\theta - \sin\beta\sin\phi\cos\theta - \sin\alpha\cos\beta\cos\phi\cos\theta \\
\sin\mu\cos\gamma &= \sin\theta\cos\alpha\sin\beta + \sin\phi\cos\theta\cos\beta - \sin\alpha\sin\beta\cos\phi\cos\theta \\
\cos\mu\cos\gamma &= \sin\theta\sin\alpha + \cos\alpha\cos\phi\cos\theta
\end{aligned}
\tag{6}
$$

Now, I want to write down the expressions for these in terms of the angles that we know. ......(Refer Eq(6))

(No audio from 19:40 to 21:42)

And when you substitute, let us say beta equal to 0, you will see that you are getting this expression $\gamma = \theta - \alpha$. Beta equal to 0 means you are talking about the longitudinal plane now. This mu angle is similar to the bank angle, you know, but this is about the velocity vector which is the X axis of the wind fixed axis system. So, now let us try to examine these Equations.

So, what kind of equations (Eqs 1-5) they are? They are differential equations, ordinary differential equations, is it not? Linear or non-linear? Non-linear, because you have product of two variables present in these equations. So, these equations actually are non-linear for many reasons not just this. That is something that you can see straight forward. What else you can think of? So, there are many couplings and many nonlinearities because of the coupling and also direct nonlinearities available in these equations.

Ofcourse we know that these trigonometric terms are also non-linear , $\sin\alpha$ $\cos\phi$ all of them are non-linear. Anything else you notice in these equations? Anything else you notice in these equations? See, we are trying to look at stability, dynamic stability of the aircraft through these equations, is it not? That is our objective. So, is there anything that you notice? We want to have simpler models, not really all twelve equations because doing any analysis using these twelve equations, will become, it will be more difficult.

Let us look at these equations carefully and tell me can you simplify these set of equations. Let me help you. Look at these three equations (Force Eqs 1 or 5). Do you see anywhere this angles psi ψ appearing in these three equations? You see it anywhere else? Even here (Eq 6) we do not see this angles psi ψ. What about these three (Moment Eq 1)? We do not see this psi appearing here also (Eq 3). Here, you see psi appearing in these two equations (Eq 4)? No. So, can we neglect this equation? This equation for psi dot is depending upon other variables, but other Equations are not depending upon this angle. So, this equation is actually decoupled from the rest of the equations. So, we can for now get rid of this equation.

X_E, Y_E, Z_E, do you see them appearing anywhere? These are only telling you the location of the aircraft. You do not see them appearing anywhere. They are not influencing the dynamics of aircraft. In a more complicated model, we have to actually worry about this z, because with z or the altitude, density is going to change. But for our simple model, we will not take density as a variable. We will only look at the sea level condition.

We will assume that rho is constant. So, we can neglect these three equations also. But in general, if you want to know the location of your aircraft at any point in time, then you have to have these three equations (Eq 4), but it does not affect the stability of the aircraft. So, we will, so, we can remove these three equations assuming that these first eight equations (Force Eqs 1 or 5, Moment Eqs 1 and Eq 3 minus psi dot Eq) are not depending upon the location of the aircraft.

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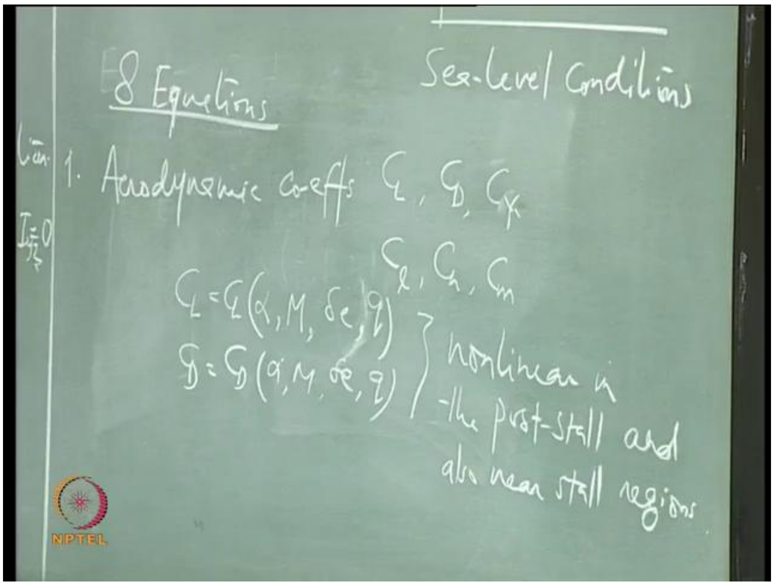

So, we are assuming that the density is constant. We are going to look at the dynamics only at sea-level conditions. So, now, 2 ($\dot{\phi}, \dot{\theta}$) plus 3 (Moment Eqs. 1) and those three force equations (in Eq. 1 or 5). So, we have now eight equations. We already looked at the direct nonlinearities because of these trigonometric terms and also because of the product of variables. Any other nonlinearities you see in these equations? What about these coefficients C_D, C_L, C_m? Are they linear? .... Are these linear? What these coefficients are going to depend upon? C_L is a function of alpha and Reynolds number which is related to Mach number, (()) no we have fixed the camber, ... and elevator.

So, C_D is also going to be a function of these three variables, and anything else? So, we know that C_L is linearly varying with respect to alpha in the pre-stall region, but beyond that, it is not going to be a linear function of alpha. So, it is becoming non-linear.

Similarly, C_L is also going to vary with this Mach number, and it is going to be more; it is going to be non-linear, around what region? When you are approaching the transonic speeds or the supersonic speeds, this C_L is also going to be a non-linear function of the Mach number. Anything else? C_L is a function of elevator also, control input, and that can also become non-linear at higher angles of attack in the post-stall region, anything else? C_L may also depend upon the rates, angular rates of aircraft. So, CL will also be a function of q, which is the pitch rate, and this can also become non-linear in the post-stall regime of flight.

Similarly, we have all these aerodynamic coefficients depending upon all the variables particularly alpha, beta and the rates, because rates are going to affect the flow field around aircraft and that in turn is going to affect the forces and moments, is it not? Post-stall and also near stall. So, in fact, all of them, all these aerodynamics coefficients are going to be non-linear in the post-stall regions. This is one source of nonlinearity.

So, if you want to fly your aircraft in the post-stall regime, or alpha higher than the alpha stalls, then we are going to see non-linear behaviors of aircraft. So, that is one thing. Even at low alpha in the pre-stall regime, you have nonlinearities and that will be due to, you see any other nonlinearity here, any other source of nonlinearity at low alpha?

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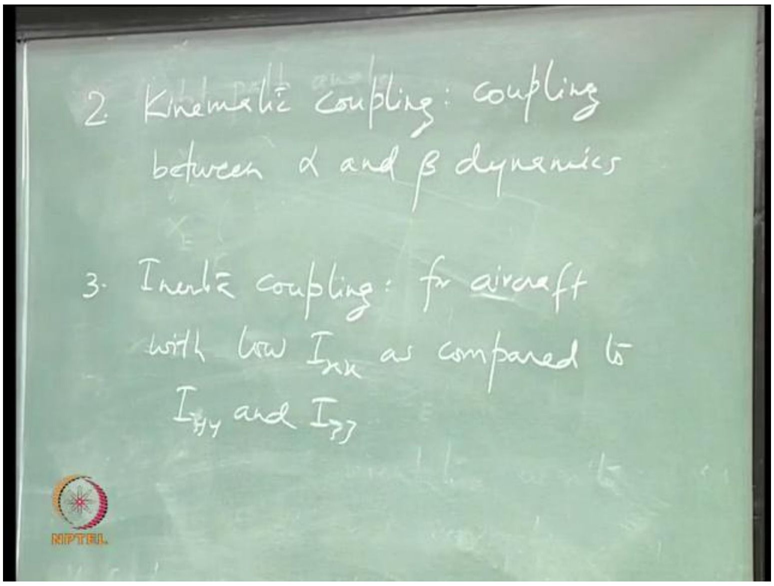

Kinematic coupling - this is the coupling of alpha and beta dynamics. So, at high alpha, you will have, because of alpha, you have beta generated and beta is going to affect alpha in turn. So, these two equations become strongly coupled at high alpha. Third one is related to low alpha flights and this is because of the coupling due to inertia terms. So, we have aircraft with different inertia values. So, for example, if you look at, look at a combat aircraft, there this I_{xx} term is going to be low because all the mass is concentrated towards the centerline (belly of aircraft). .....

(No Audio from 36:00 to 36:41)

What happens here is, because of this low I_{xx} for significantly higher value of the roll rate, p, q and r (dynamics) get strongly coupled. So, you have the roll rate which is going to affect the pitch rate and yaw rate and they are affecting each other. I will tell you what are the non-linear behaviors of aircraft that you are going to see because of nonlinearities in the aircraft equations of motion.

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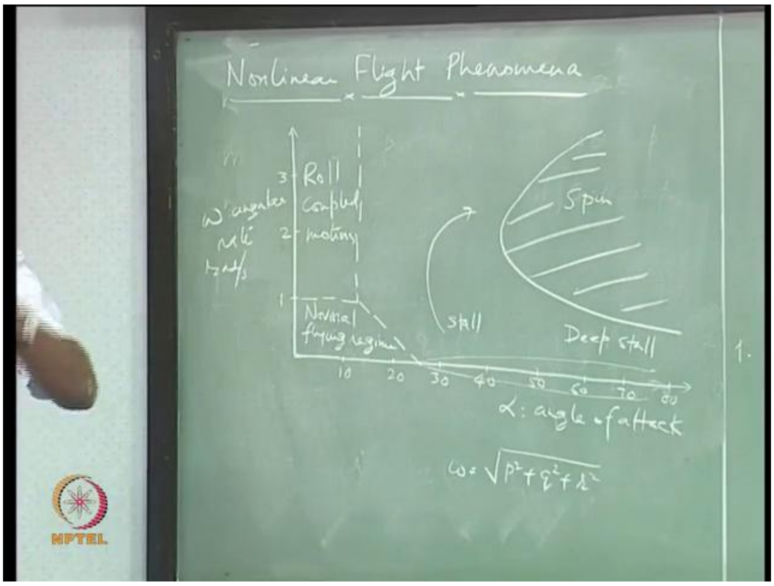

As I said, dynamics is going to depend upon the angle of attack and the angular rotation of the aircraft. So, I am trying to list down all non-linear flight phenomena and, also the normal flight phenomena on this plot, which is a plot, on this omega and alpha axes; this omega is absolute omega. So, this is this is equal to $\omega = \sqrt{p^2 + q^2 + r^2}$ … (Refer Slide Time: 39:25)

(No Audio from 39:32 to 40:12)

So, this is your normal flying regime, this is your normal flying regime (Refer the slide above) when you are flying below this alpha and below the, this value of the absolute angular rates. .... And at low alpha, further because of that inertia coupling term, you see what is known as roll coupled motions at low alpha because of high angular rates and this is mainly because of this I_{xx} term becoming very small.

Then you have stall beyond this point, you have stall. At higher alpha, really high alpha going beyond, you know, around 60 or 70 degrees, ... you go into what is known as deep stall regime, and both these, when your aircraft is in these conditions, it can further lead you into what is called spin. Spin is a, is an aerodynamic phenomenon. So, it happens because you have taken your aircraft or you have increased your aircraft angle of attack, either you do it yourself or it may so happen that you have lot of, you know, wind which causes sharp increase in angle of attack. It is like a thunder bolt or microburst condition where the vertical column of wind is coming down and spreading. So, in such a situation, you can have a condition which is called spin, and spin is basically a high rotation in yaw of the aircraft and this is purely because of aerodynamic reasons.

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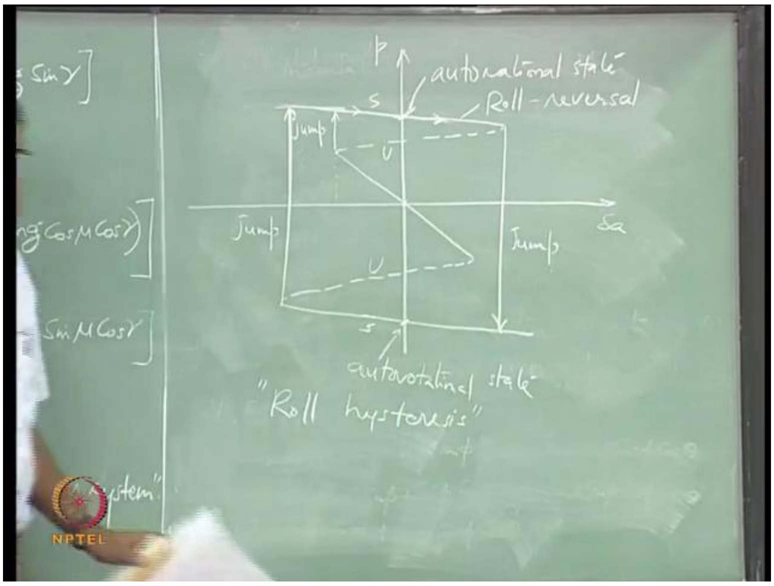

Let us also quickly talk about the roll coupled problem, and this is because of the inertia-coupling term. So, we are deflecting aileron and we are expecting some roll rate and that is going to be linear up to some limiting value of aileron deflection, not that you will keep getting this linear increase in roll rate (Refer slide above) when you are applying aileron for all values of aileron.

So, this is a positive roll, rolling to the right, when you are deflecting your aileron in the negative direction, and up to this point, your roll rate is increasing linearly with delta a which is aileron deflection (Refer slide above). Suddenly you see that your aircraft has jumped into a situation which is giving you higher roll rate than expected. You would expect this linear growth of roll rate when you are deflecting your aileron but that does not happen, and what you see is a jump in roll rate.... And this (Post-jump the roll rate solution branch, solid lines) is almost flat, it is not going down but it is almost flat.... And similar thing can happen also on the negative side. So, this is the complete picture.

Now, so, I am deflecting my aileron in the negative direction, I am getting a positive roll rate and you see that at one point, aircraft is having higher roll rate and, there is sudden jump in the roll rate. So, what happens now? When you start decreasing the aileron thinking that you will be able to reduce the roll rate, when you, when you get into this, you will think that you can now bring back the roll rate to 0 by changing the aileron and see what happens.

When you are doing that, you are decreasing aileron, you are actually moving onto this solution branch and there is situation when you have brought back your aileron to neutral level but you still see a positive roll rate, and this is called auto- rotational state (non zero roll rate for zero aileron). I have brought my aileron back to the neutral level, but still I am having a positive roll rate and that is like auto-rotation because that is unintended. So, automatically aircraft is rotating even without applying any control which can give you rotation.

Further, when you further decrease aileron or start changing the aileron in the opposite direction, and you see still positive roll rate. This positive roll rate is for positive aileron deflection. So, there is a condition, which is called roll-reversal, which is taking place. So, I am expecting negative aileron to give me positive roll rate and positive aileron to give me negative roll rate (in a normal scenario), but here what is happening? Because of this non-linear condition, I am seeing that a positive aileron is giving me positive roll rate and that is called roll-reversal.

So, you decrease your aileron further in the opposite direction and you see that you have a larger jump in roll rate, and now, you are landing yourself into this solution branch (lower solution branch below delta a axis), because this is what is available (stable equilibrium solutions) to you. All the solutions are here where you can go by applying aileron. .... So, you have jumped here. Now what do you want to do? What do you have? You have a negative roll rate. Now we want to bring back aileron and see what is happening. So, you start bringing back your aileron, and at this neutral condition when the aileron deflection is 0, you see a negative roll rate.

This is also, ... and so if you further change your aileron, you see that you are going on to the upper branch of solution. So, what is happening here? There is a hysteresis. So, you are never going to be able to come back to 0 roll rate unless you do something else.

So, with only aileron it is not possible to bring back your aircraft to this starting condition.

(( ))

This is because of the inertia coupling.

Sir, but that is because it is non-linear (( ))

No, it is going to give rise to side-slip when this happens. So, beyond this point, you have a large increase in side-slip, and slide-slip can also contribute to the roll, look at the vertical tail.

That, that, point (( )).

Solutions become unstable. So, we have these stable solution branches (marked by solid lines), and here, the solutions are unstable (marked by dashed lines). These are possible motions where aircraft can get into as a result of loss of stability. So, you see a hysteresis here. Now, unless you do something else, you know, to kill that side-slip and not really killing the roll rate. So, you have to kill the side-slip using rudder to bring your aircraft back to this 0 roll rate condition. So, using aileron and rudder both together, it is possible to bring your aircraft back to 0 roll rate, but not using aileron alone. (No Audio from 51:34 to 51:45) So, we will stop.