Flight Dynamics II (Stability)

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Module No.# 08

Equations of Rigid Aircraft Six-Degree-of-Freedom Motion

Lecture No. # 24

Derivation of Angular Motion Equations

We are writing rigid body aircraft equations of motion. We already wrote the force equations yesterday and we are half way through writing the moment equations, equation for the rotational motions of the aircraft.

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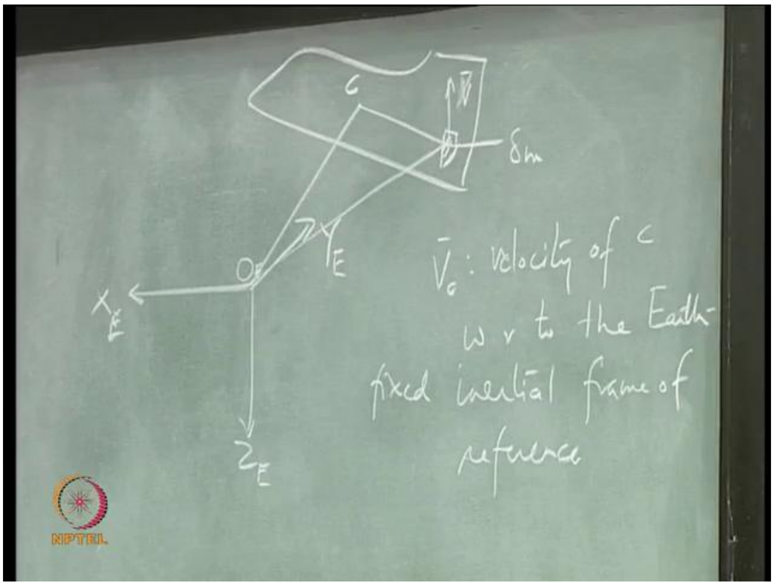

So, we are first writing the equation of motion for this elemental mass and then integrating it for the whole body. Earth is our inertial frame of reference. This E subscript is for Earth. This velocity of the center of mass is V c.

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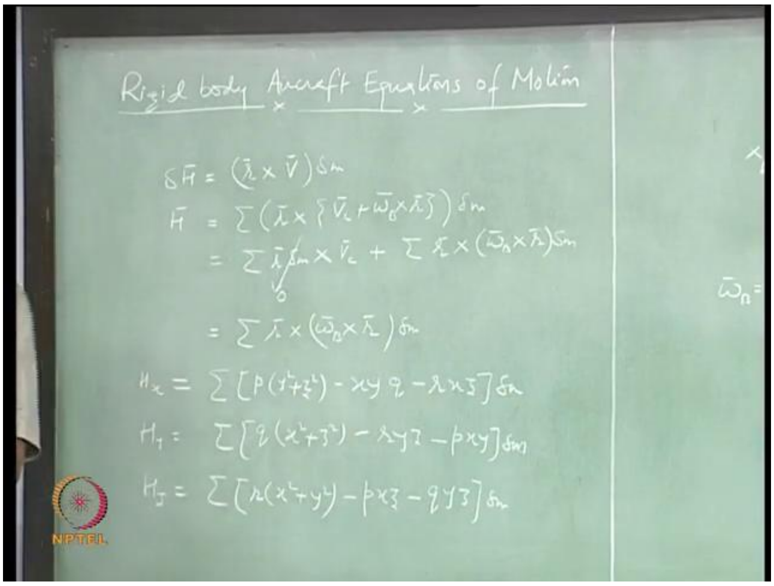

And we arrived at an expression for the angular momentum for this elemental mass delta m.

$$\bar{H} = \sum \bar{r} \times (\bar{\omega}_B \times \bar{r})\delta m$$

$$H_x = \sum \left[p(y^2 + z^2) - qxy - rzx \right] \delta m \qquad (1)$$
$$H_y = \sum \left[-pyx + q(x^2 + z^2) - ryz \right] \delta m$$
$$H_y = \sum \left[-pxz - qzy + r(x^2 + y^2) \right] \delta m$$

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We dropped this term saying that all the distances are being measured with respect to the center of mass and then we arrived at this expression. This omega B is the angular velocity of the aircraft, right, with respect to the body fixed coordinate system. p is the roll rate; q is pitch rate; r is yaw rate, and later on, we expanded this. So, we found the components of this angular momentum vector about aircraft body fixed X, Y and Z axes.

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In the event when we assume that this delta m is very small, this summation will become an integral and then you can write these components as …(Refer Eq(2)) (Refer Slide Time: 06:40)

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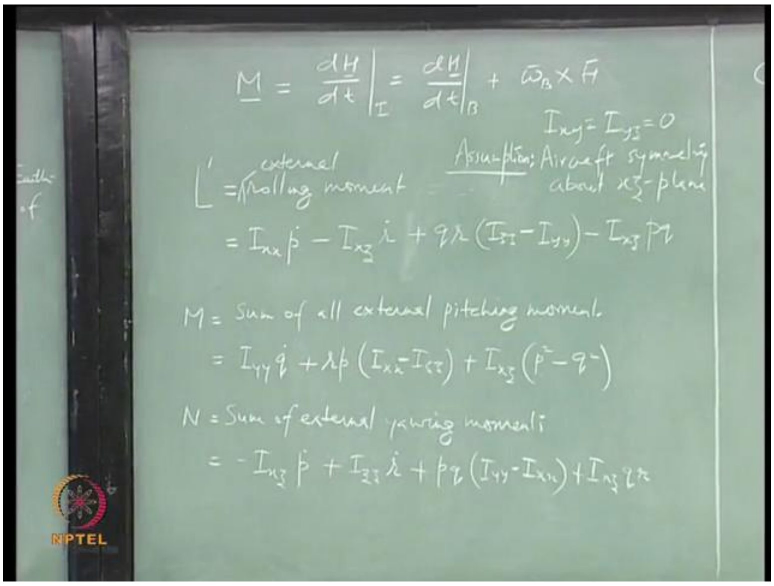

$$\bar{M} = \left.\frac{d\bar{H}}{dt}\right|_I = \left.\frac{d\bar{H}}{dt}\right|_B + \bar{\omega}_B \times \bar{H}; I_{xy} = I_{yz} = 0(\text{aircraft symmetric about XZ plane})$$

$$L' = I_{xx}\dot{p} - I_{xz}\dot{r} + qr(I_{zz} - I_{yy}) - I_{xz}pq \qquad (2)$$

$$M = I_{yy}\dot{q} + rp(I_{xx} - I_{zz}) + I_{xz}(p^2 - q^2)$$

$$N = -I_{xz}\dot{p} + I_{zz}\dot{r} + pq(I_{yy} - I_{xx}) + I_{xz}qr$$

Moment is nothing but the rate of change of angular momentum, and using that, now we are ready to write the aircraft equations of rotational motion. Now, because the body is rotating, all the vectors as seen from the inertial frame will have, rate of change of vector as seen from the inertial frame will have two components - one will involve the angular velocity of the body.

So, now, with this information available, I can directly write down now the equations for the moment about three axes of the aircraft. Let us assume that, aircraft has a symmetry about XZ plane. So, now I am writing the equations about the three axes of the aircraft.

L, is the rolling moment; we have been using L prime. So, I will put a prime there. So, some of all external rolling moments, right, equal to… (Refer Eq(2))

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Now, I also said that I am going to assume the symmetry about the XZ plane and that will give you this .... (Refer Eq(2)). So, if you find any mistakes with the equation, just stick to this.You have to use this, and this, to arrive at the equations (Refer Eq(2)). So, if there are … some minor mistakes with this subscript you can figure that out.

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An assumption which is a valid assumption. So, but there could be minor manufacturing asymmetry. So, if you want to take all that also into account, then you have to, you know, write the expanded version of the equation. We can also assume that the principle axis system is coinciding with the body fixed axis system of the aircraft (that means $I_{xz} = I_{zx} = 0$).

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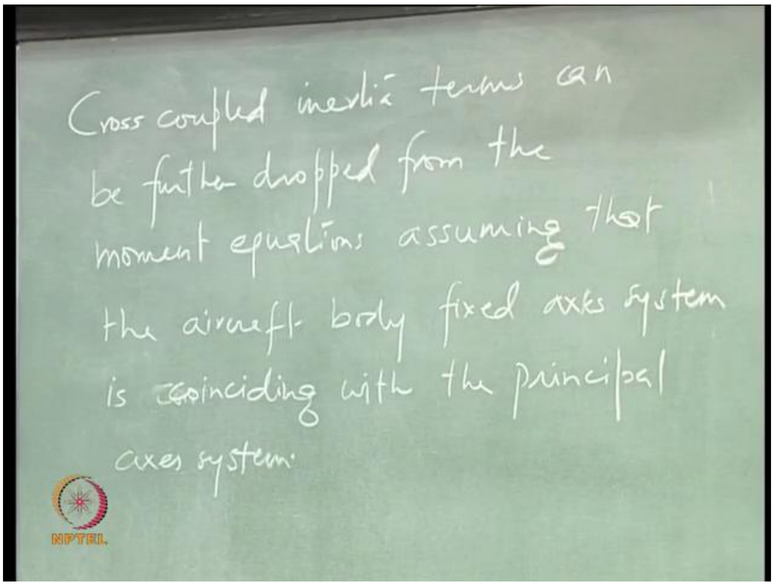

So, body fixed, or aircraft body fixed axis system, is also the principle axis system. In that case, we can also get rid of these coupled inertia terms. So, you can drop out this, cross coupled inertia terms can be further dropped from the moment equations (Refer Eq(2)), assuming that the aircraft body fixed axis system is coinciding with the principal axis system, and finally, I am going to retain the equation after getting rid of this coupled inertia terms to .... (()) there is another axis system about which this cross coupled terms will disappear.

You, how do you neglect this part? We assume that, you know, y is same on, you know, in the positive side and the negative side. (()) That is by symmetry. So, you can assume another symmetry. And which is kind of valid assumption and it simplifies things to a great deal. You know we are going to look at simpler models. So, you have the equations for the, the force equations and the moment equations for the translational motion and the rotational motion, anything else?

You also want to find out the location of the aircraft and the orientation at any point in time, with respect to the inertial frame. So, you want to say where your aircraft is located. So, you also need to write what is known as equations of kinematics.

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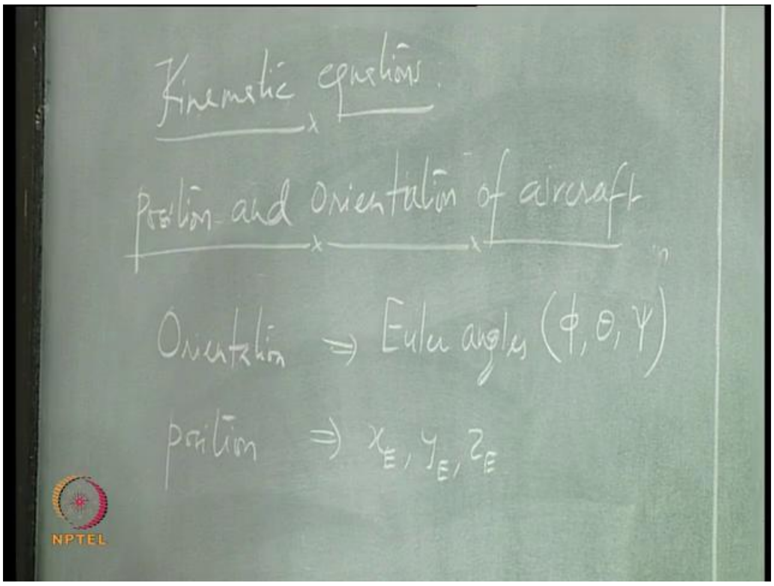

So ....(x_E, y_E, z_E) and that is also changing in time. So, we should get the differential equations. Orientation, or the attitude can be found by using the angles which are called Euler angles (ϕ, θ, ψ). ...... So, I am measuring the distances from the Earth fixed coordinate system. So, coordinate of airplane at any point in time. So, let us assume that the aircraft is, aircraft fixed, body...

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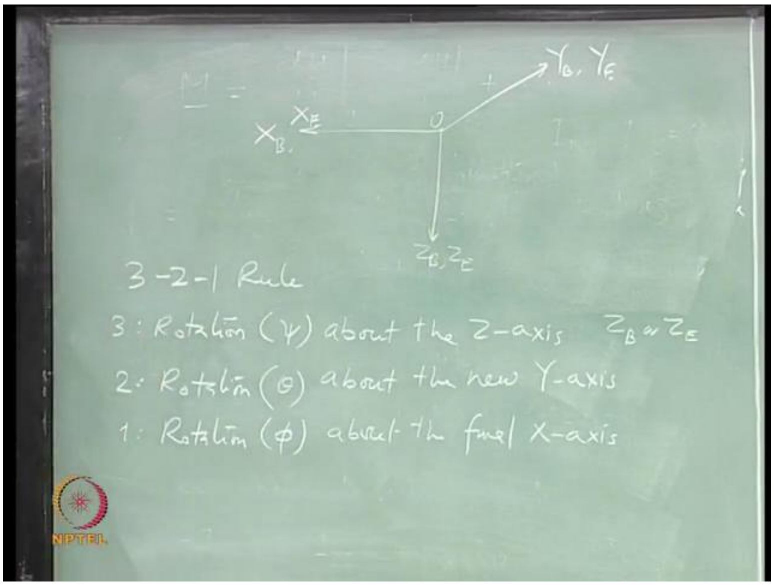

Aircraft body fixed axis system is coinciding with this Earth fixed axis system to start with. We can assume that so we have .... (Refer Slide Time: 21:25) and ..... So, it is coinciding with the Earth fixed axis system to start with. Now, I want to know the attitude of the aircraft with respect to this system. So, there is a rule, which says that, in order to find the orientation, what you should do is, you give a first rotation to the body fixed axis system about this Z axis and that is yaw. So, there is a rule.

So, 3 is the first rotation about the Z axis, and Z axis is at this time coinciding with the body fixed Z, Earth fixed Z axis. So, 3 is the rotation about the Z axis and this Z axis is Z axis of the body or the Earth, because to start with its coinciding, both of them are coinciding with each other.

So, this Z is Z_E. So, the second, now you go to a different axis system. Once you have rotated your airplane, yaw ψ about the Z axis, you go, you will go to a different coordinate system, because X and Y are going to be different from what you see here, is it not? Give the second rotation, theta θ, about the Y axis and that Y axis is indeed the Y axis of the aircraft because I am rotating the aircraft. So, rotation theta about the new Y axis, and the final rotation is rotation phi ϕ about the second new X axis. So, remember this rule, what we are essentially doing is, we are rotating the aircraft with respect to this inertial frame of reference.

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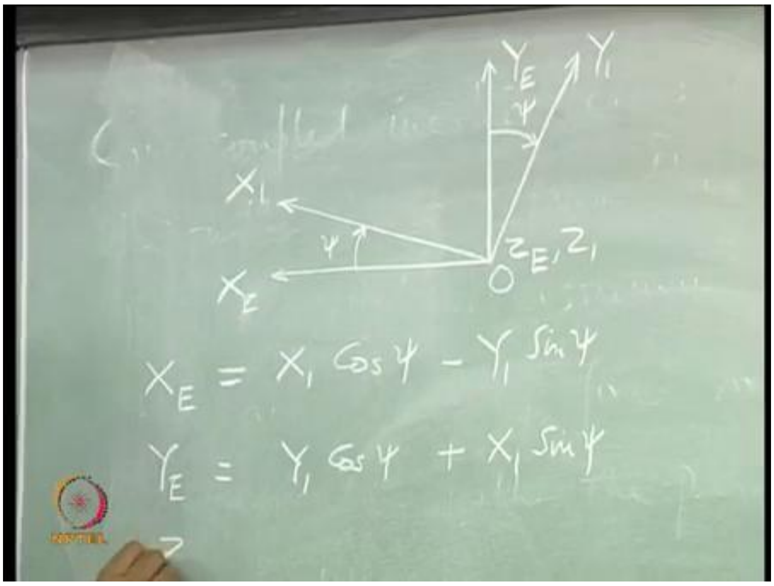

$$\left.\begin{aligned} X_E &= X_1\cos\psi - Y_1\sin\psi \\ Y_E &= Y_1\cos\psi + X_1\sin\psi \\ Z_E &= Z \end{aligned}\right\rangle \begin{bmatrix} X_E \\ Y_E \\ Z_E \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_3(\psi)} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} \qquad (3)$$

You will see what we get from doing this. So, first rotation is this yaw. So, so, the rule is that you are trying to actually rotate the Earth fixed system so that, you know, it coincides with the aircraft body fixed axes. So, finally, the final axis system that I get by doing this should be the aircraft body fixed axis system. I will call this Y_1 and X_1. And this rotation is about the Z axis. So, Z axis is into the board and coinciding, Z_E and Z_1, both are same.

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$$\left.\begin{aligned} X_E &= X_1\cos\psi - Y_1\sin\psi \\ Y_E &= Y_1\cos\psi + X_1\sin\psi \\ Z_E &= Z \end{aligned}\right\rangle \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_3^T} \begin{bmatrix} X_E \\ Y_E \\ Z_E \end{bmatrix}; R_3^T R_3 = R_3 R_3^T = I \qquad (4)$$

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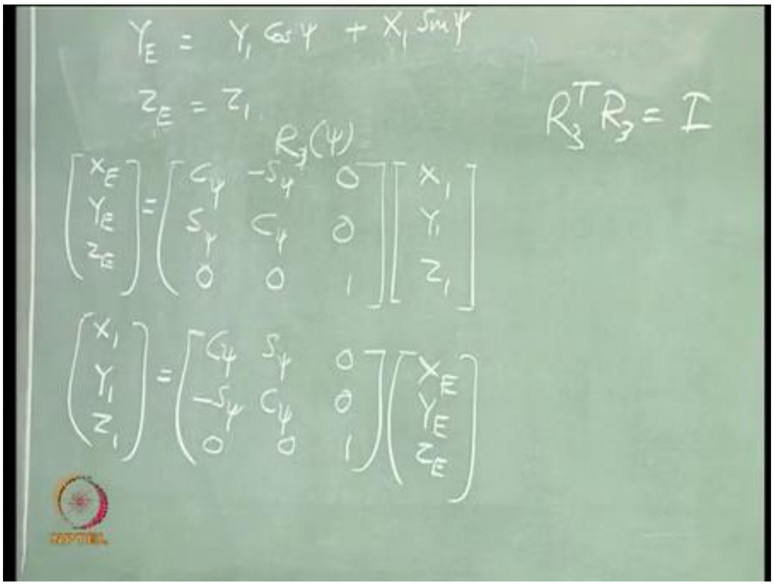

So, I am writing cos psi as this C of psi, ... and if you want to write it other way round, then you just have to invert this matrix, and this matrix has a special property; it is a orthogonal matrix.

So, this matrix, I can call this as, you can also call (Refer Eqs3,4) this as R_3. So, rotation about the third, I will call this as R_3 lets say. So, this matrix has a very interesting property, and that property is this, transpose of this matrix multiplied by this matrix is the identity matrix. And that is the orthogonality property of this matrix. So, I can directly write this matrix now, it is not difficult to write this matrix once you know this property. Now this next rotation, this theta is about the new Y axis which is Y_1.

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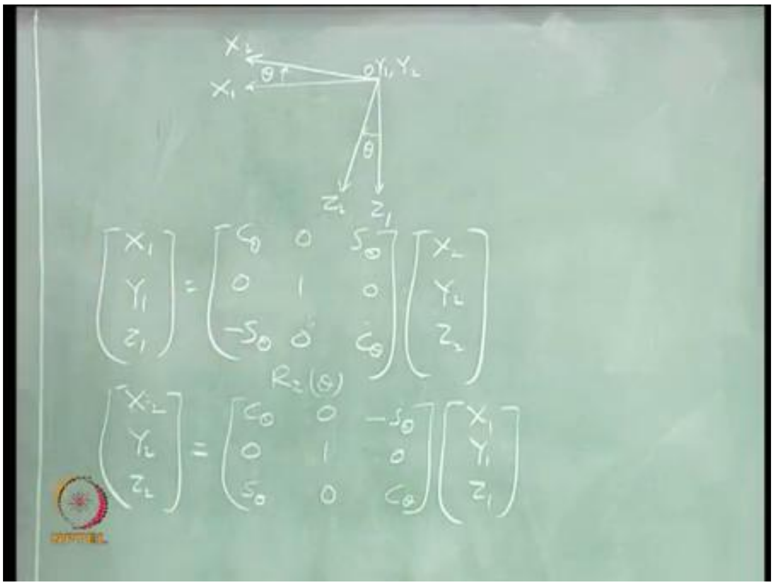

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}}_{R_2(\theta)} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}; \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}_{R_2^T(\theta)} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}; R_2^T R_2 = R_2 R_2^T = I \quad (5)$$

So, the rotation is about Y_1. So, Y_2 is going to be Y_1. So, repeat that exercise and you can find out. So, quickly let us just. So, this is R_2 and the final rotation is about this X_2 axis. So, X_2 is the body fixed X axis.

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$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}}_{R_1(\phi)} \begin{bmatrix} X_B \\ Y_B \\ Z_B \end{bmatrix}; \begin{bmatrix} X_B \\ Y_B \\ Z_B \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}}_{R_1^T(\phi)} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}; R_1^T R_1 = R_1 R_1^T = I \quad (6)$$

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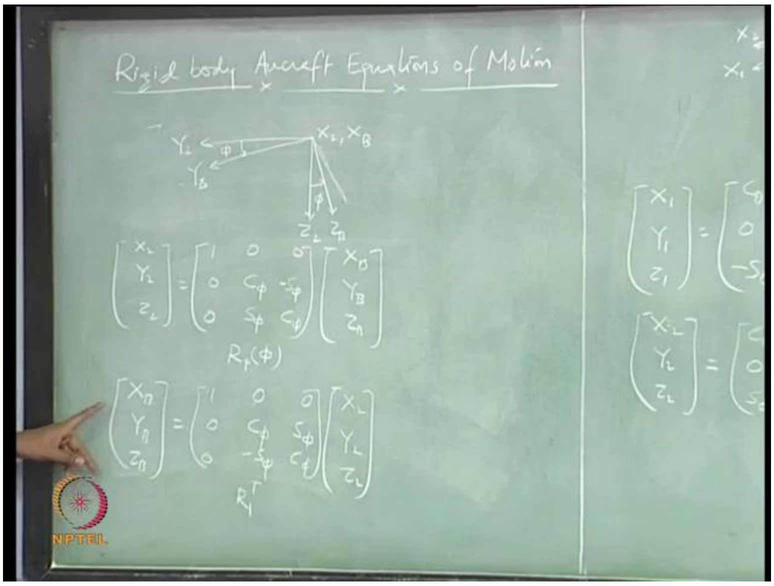

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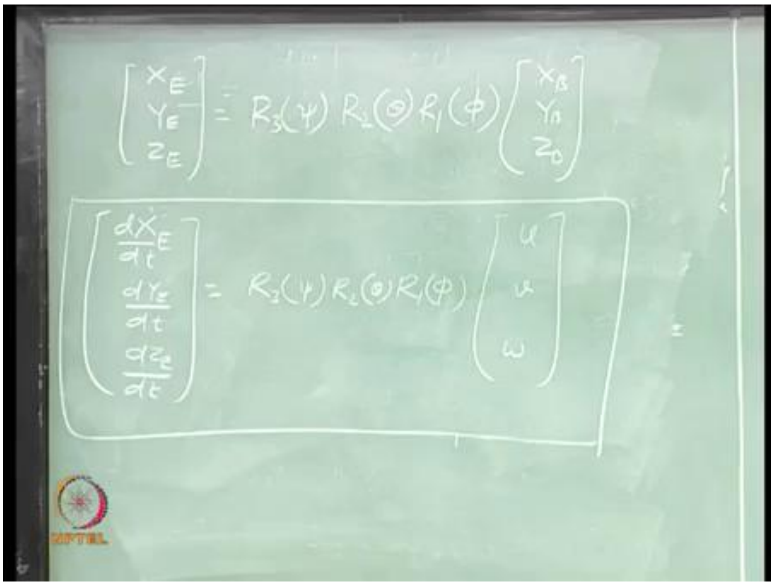

$$\begin{bmatrix} X_E \\ Y_E \\ Z_E \end{bmatrix} = R_3(\psi)R_2(\theta)R_1(\phi)\begin{bmatrix} X_B \\ Y_B \\ Z_B \end{bmatrix}; \begin{bmatrix} \frac{dX_E}{dt} \\ \frac{dY_E}{dt} \\ \frac{dZ_E}{dt} \end{bmatrix} = R_3(\psi)R_2(\theta)R_1(\phi)\begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{7}$$

Now, what I want to find out is the relation between this axis system and the Earth fixed axis system. That is what is going to give me the relative orientation of the aircraft with respect to the inertial frame of reference, ... and this into X_1, Y_1, Z_1, ... that is, R_2 into X_2 Y_2 Z_2(Refer Eq(5)). So, you have to take the derivative of this, you know, to find out the location of your aircraft center of mass. You want to find out the location of your center of mass of the aircraft at any time t. ..... u v w are velocity of the aircraft along X_B Y_B and Z_B axes.

So, another three equations, we already wrote six equations earlier plus three, now we have nine equations. So, so far we have written the equations only for knowing the location. Now, I want to know the orientation, and orientation is also going to evolve in time, (()) ... you should try this out. You will get this. So, work that out. So, little more complicated that, but you will get this finally. So, what he is asking is, why I have not taken the derivative of these angles with respect to time while taking the derivative of this. That is what he is asking. You should work this out and you will find that you arrive at this answer.

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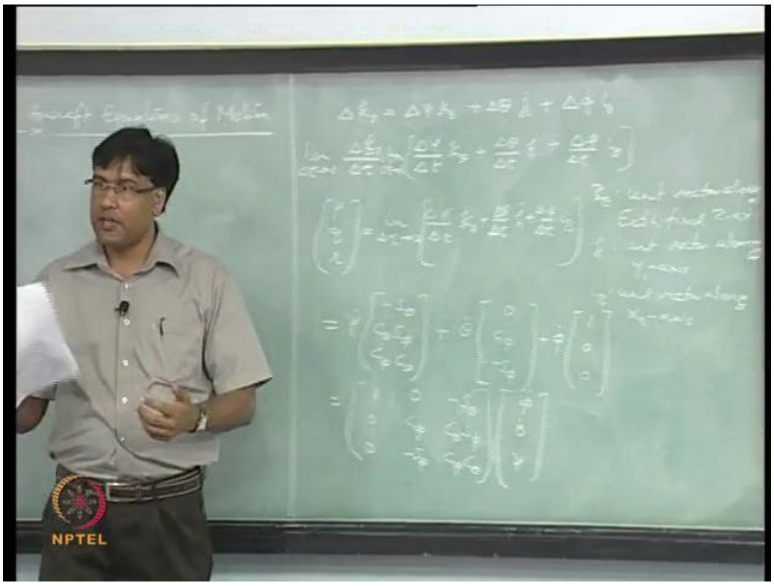

So, I can quickly also give you an expression for the attitude dynamics. So, let us say, we have a vector, you know, of angles. So, there is a small change in all the angles of the aircraft about the Earth fixed axis. So, the way we have used this here 3 - 2 - 1 rule, you can write this small change in attitude of the aircraft from the initial Earth fixed condition. So, let us call this angle, some angle, and this is the vector. So, aircraft has just changed its attitude by small angle. So, I will write this B for the body and it is a vector. So, this is equal to the first rotation about the k or the unit vector k attached to the Earth fixed Z axis plus delta theta attached to the new j_1. j_1 is the unit vector along new Y_1 axis.

Finally, delta phi which is about the X axis of the aircraft itself. So, we can write a new X_2 or X_B. B is for body of the aircraft. So, unit vector along X axis of the aircraft.

$$\Delta\bar{\sigma} = \Delta\psi\,\hat{k}_E + \Delta\theta\,\hat{j}_1 + \Delta\phi\,\hat{i}_2; \quad \lim_{\Delta t \to 0} \frac{\Delta\bar{\sigma}}{\Delta t} = \bar{\omega}_B = \dot{\psi}\,\hat{k}_E + \dot{\theta}\,\hat{j}_1 + \dot{\phi}\,\hat{i}_2$$

$$\bar{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \tan\theta\sin\phi & \tan\theta\cos\phi \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sec\theta\sin\phi & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad (8)$$

Now, you take the derivative of this with respect to time, or divide it by time delta t. This is going to give me a rate of change in the attitude, ... and take the limit when this delta t becomes small, and it will give you the rate, rate of change of angle. You take the limit. So, ... What is this quantity? You know, rate of change of angle of the body, in the body fixed axis system. So, this is related to the angular rate or the angular velocity of the body, or the aircraft. So, this is p q r equal to …(Refer Eq(8)) (Refer Slide Time: 44:57)

One thing you have to remember that these unit vectors are not along any one fixed axis system. They are three different axis systems. This k_E is the unit vector along Earth fixed Z axis. j_1 is unit vector along Y_1 axis, which you obtain after the first rotation. So, now I have to find, you know, I have to write, write them all in one axis system. That is the body axis system, because I am trying to balance out this p q r with the rate of change of theta θ, psi ψ and phi ϕ, you get this?

Now, I have given you the transformation matrices? If you have followed what I have done, then it will not be difficult for you to figure out that, this is equal to …(Refer Eq(8)). (Refer Slide Time: 46:59) So, finally, we find a relation between the rate of change of attitude angles and the angular velocity of the body. We want the differential equations for these angles? So, I have to invert this matrix. p q r the variables, state variables of the aircraft equations of motion.

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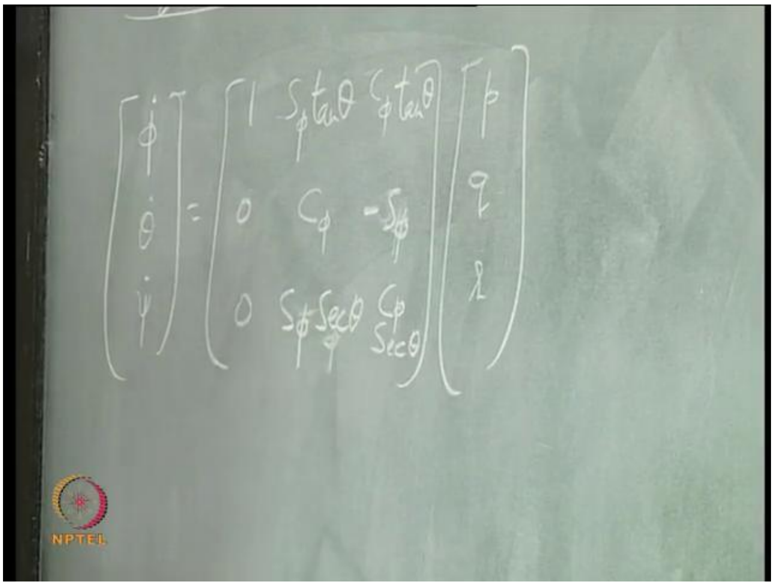

So, I want this (all derivatives with respect to time, dot terms) to be on the left hand side. So, we have completed writing down the equations of motion. So, how many now? Three plus three – six; three force equations and three moment equations. So, we have in all twelve equations, and what we are intending to do is, we are trying to study the aircraft motion and the dynamic stability of aircraft. So, we will continue from here in the next class.