Flight Dynamics II (Stability)

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Module No. # 08

Equations of Rigid Aircraft Six-Degree-of-Freedom Motion

Lecture No. # 23

Derivation of Translational Motion Equations

So, in the last class, we discussed that static stability analysis is not really enough to comment on the stability of aircraft in motion. And thereby, we have to look at the dynamic response of the perturbed variables from a particular flying condition. So, we have to look at the time response. To look at that, we need to first write down the equations of motion of aircraft. So, remember that, we, while doing the static stability analysis, we have not really taken care of the mass inertia and damping properties of the aircraft. So, all those will come into the equations of motion.

1:31

How many equations? How many degrees of freedom an aircraft will have? Aircraft is an elastic body. So, if you include all the elastic modes and elastic vibrations, then there will be some infinite degrees of freedom, but do you think it will be important everywhere? For big aircraft with large wing span where the wing can actually display a motion itself, because of the aerodynamic loads, there, it may become important to include those effects, but for our proposes, we will stick to rigid body motion of aircraft. So, we will assume the aircraft to be a rigid body.

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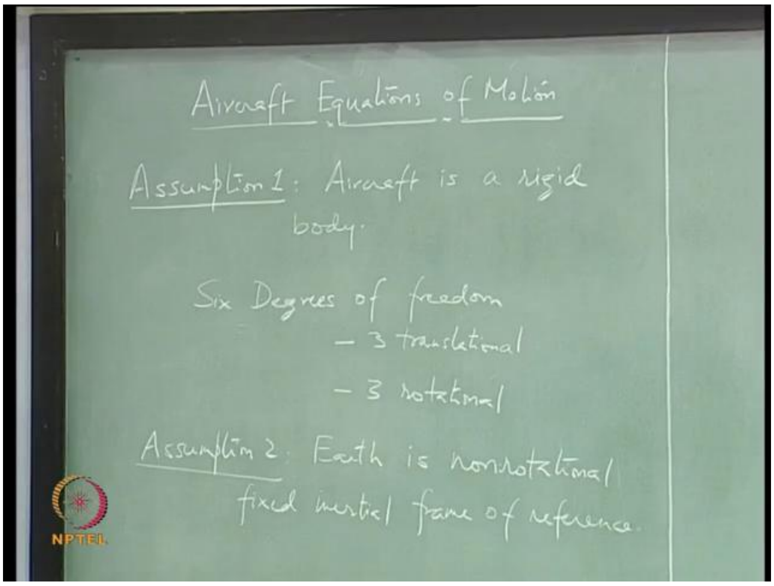

So, first assumption that I am making here is that aircraft is a rigid body, and a rigid body can have, how many degrees of freedom? Six degrees of freedom – three translational degrees of freedom and three rotational. So, what else you have to define before you start writing equations of motion? We can use Newton's law, and Newton's law requires that you should have a inertial frame of reference. So, we will assume Earth to be a non-rotating inertial frame of reference.

So, I am writing non-rotating specifically because we do not want to include the effect of rotation of Earth on the aircraft dynamics. That will be negligible because we are working at lower heights. And for our engineering purposes, you know, this will be sort of a valid assumption. Once we have these two assumptions written down, I think we can start writing our equations of motion.

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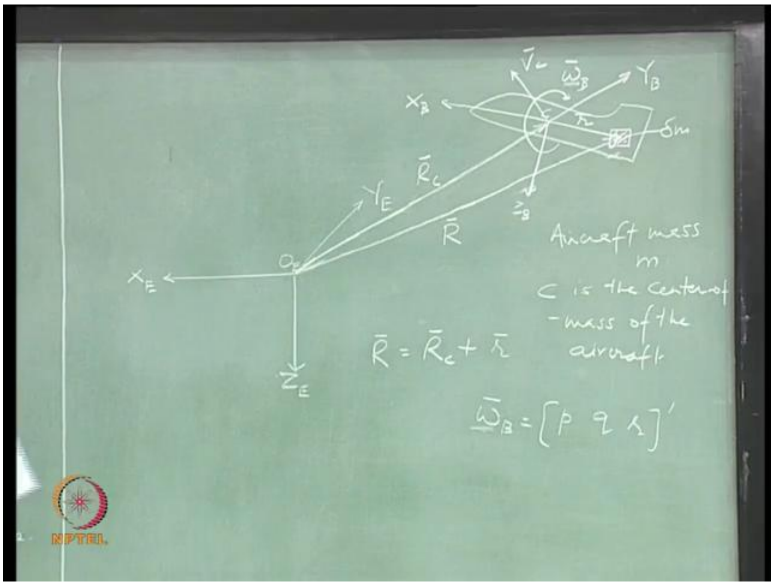

Let us say this coordinate system is fixed to a point on Earth and this is our aircraft which has mass M. So, this assumption automatically means that there is no relative motion between any two points on the aircraft. Let us call this point C as the center of mass of the aircraft. ....

8:15

So, what we will do first is - we will write the force balance or momentum balance equation for an elemental mass and then integrate it all over the aircraft. So, we will look at this elemental mass delta m and this delta m is at a distance small r from the center of mass. Distance between the center of mass of aircraft and the origin of Earth fixed axis system is rc, and this bar indicates it is a vector; we have to talk about X Y Z coordinates, is it not?

Let us say this is R. So, we can use the vector notation and write what this R is, R is .... is it not? And let say this center of mass of the aircraft is having a velocity Vc. So, this could be .... you know, ... all right. And we will also assume that the aircraft is rotating. So, it has an angular velocity which is denoted by this .... I can write a subscript B attached to this, so that, it is the rotation of the aircraft in the body fixed axis system, is not it? So, it has three components p q r, we know that, roll rate, pitch rate, yaw rate.

So, what would be the velocity of this elemental mass? If I say that aircraft is a rigid body, then this mass is not going away or coming closer to the center of mass, is not it? So, what other velocity it has? Angular velocity right. So, this omega B into this distance r is going to give a tangential velocity and the center of mass is having this velocity V. So, we can write the velocity of this elemental mass which is ... plus omega into r, omega cross r. Let me put this bar also on top. Sum of all the forces acting on this elemental mass is what is going to set this in motion. That is how we balance out the force.

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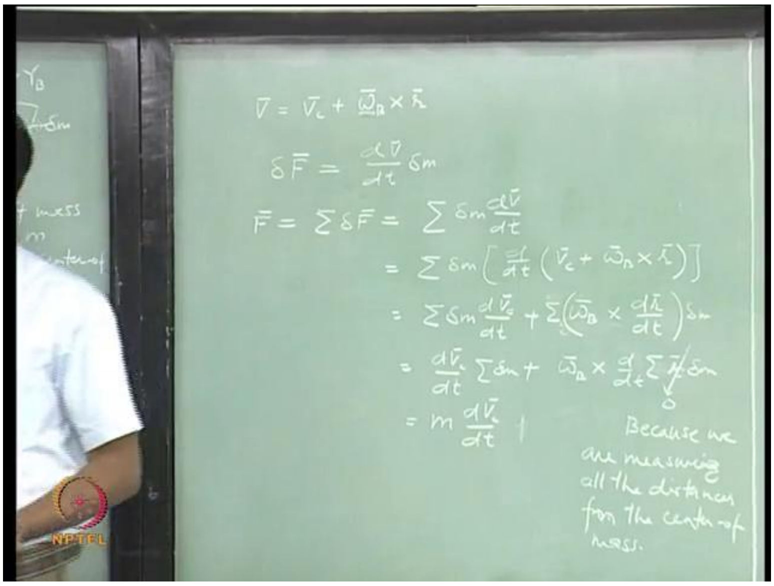

$$\bar{V} = \bar{V}_C + \bar{\omega}_B \times \bar{r}; \delta\bar{F} = \frac{d\bar{V}}{dt}\delta m$$

$$\bar{F} = \sum \delta\bar{F} = \sum \frac{d\bar{V}}{dt}\delta m = \sum \delta m\left[\frac{d}{dt}(\bar{V}_C + \bar{\omega}_B \times \bar{r})\right] = \sum \delta m \frac{d\bar{V}_C}{dt} + \sum(\bar{\omega}_B \times \frac{d\bar{r}}{dt})\delta m \quad (1)$$

$$= \frac{d\bar{V}_C}{dt}\sum \delta m + \bar{\omega}_B \times \frac{d}{dt}\underbrace{\sum \bar{r}\delta m}_{0} = m\frac{d\bar{V}_C}{dt}$$

So, if this is the force acting on the elemental mass delta m, then the equation of motion can be written in the simple form using Newton's law - F equal to m into a, and acceleration is the rate of change of velocity, or, time rate of change of linear momentum is going to give you the force. .....(Refer Eq(1))

15:51

Now, use this expression for V and you can further expand this. This omega B is actually independent of this elemental mass, it is the fixed angular velocity vector; it does not depend upon the mass. So, you can also write .... in the further step, you know, this equal to this. If you sum this up, then what you get is the total mass of the aircraft ..... What about this quantity? This is, time rate of change of this is missing. We just, you are right; we will have to rewrite this.

dr by dt is 0. So, we can also look at another representation, but right now, it is all right? What you said is correct, you know, but we can also write it in, in, a different form and see what that gives you. What this quantity is? So, one thing is that dr over dt is 0 because you are saying that the aircraft is a rigid body. So, there is no relative motion between the two points.

And so, that, that is also correct. Second representation is this, and what does, what is this quantity? This is 0, because we are measuring all the distances from the center of mass of the aircraft, is that clear? Because I am measuring all the distances, this small r from the center of mass, so, this summation is 0, because this is the definition of the center of mass. Any question?

22:24

In mechanics, you have also studied that, if the body is a rotating body, then we will also have to incorporate other accelerations. So, our aircraft is actually rotating; you know it is having some angular velocity. So, this term, will have two components, is not it? You know that, if you have any vector A, rate of change of that vector as seen from the inertial frame is going to be equal to rate of change of that vector as seen in the body plus omega B into V bar c.

$$\left.\frac{d\overline{V}_C}{dt}\right|_I = \left.\frac{d\overline{V}_C}{dt}\right|_B + (\overline{\omega}_B \times \overline{V}_C); I: \text{Inertial axis system}, B: \text{Body fixed axis system} \qquad (2)$$

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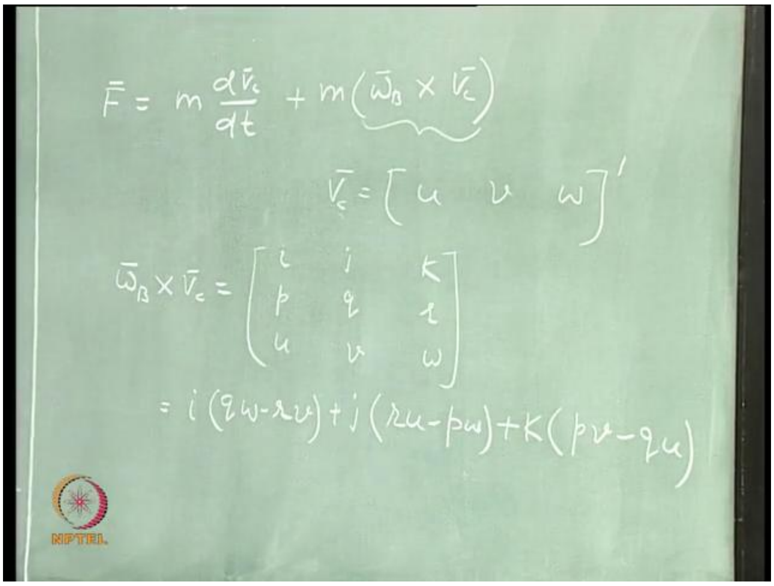

$$\bar{F} = m\frac{d\bar{V}_C}{dt} + m(\bar{\omega}_B \times \bar{V}_C); \bar{V}_C = [u\ v\ w]';$$
$$\bar{\omega}_B \times \bar{V}_C = \begin{bmatrix} i & j & k \\ p & q & r \\ u & v & w \end{bmatrix} = i(qw - rv) + j(ru - pw) + k(pv - qu) \tag{3}$$

So, this you can, you must have seen this somewhere and we are going to use this, because it is not important for me to explain you this in this course. So, I will just use this, and write down the force balance equation. ....

What is my V c? V c is the velocity of the center of mass of the aircraft; it will have three components, along aircraft fixed axes. So, aircraft is having velocity component along its axes, u v and w. So, now, I am ready to expand this. So, let us see what we get after expanding this. ....

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i j and k are unit vectors along aircraft body fixed axes.

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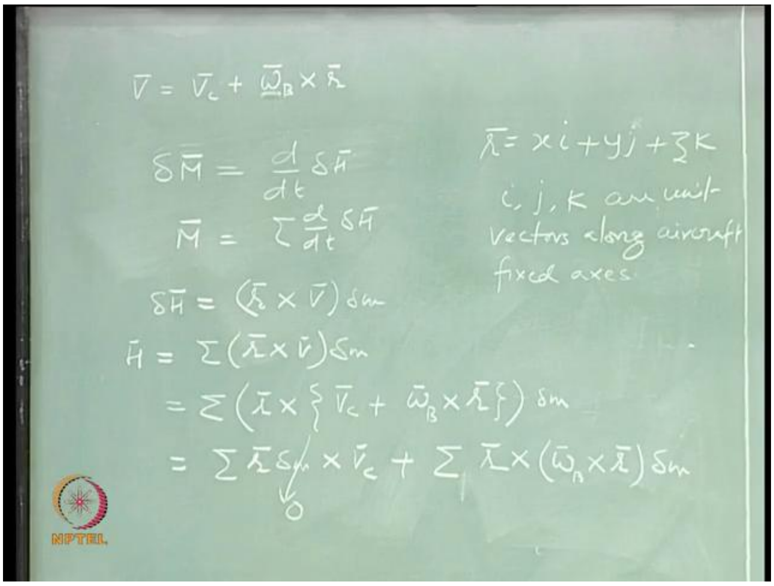

$$\delta\overline{M} = \frac{d}{dt}\delta\overline{H};\overline{M} = \sum\frac{d}{dt}\delta\overline{H};\overline{V} = \overline{V}_C + \overline{\omega}_B \times \overline{r};\overline{r} = xi + yj + zk;\delta\overline{H} = (\overline{r}\times\overline{V})\delta m;$$
$$\overline{H} = \sum(\overline{r}\times\overline{V})\delta m = \sum(\overline{r}\times\{\overline{V}_C + \overline{\omega}_B \times \overline{r}\})\delta m = \underbrace{\sum\overline{r}\,\delta m}_{0}\times\overline{V}_C + \sum\overline{r}\times(\overline{\omega}_B \times \overline{r})\delta m \qquad (4)$$

So, now, I can find out the equations of motion along X Y Z axes of the aircraft .... And this force is going to be sum of all the forces acting along the axis of, X axis of the aircraft. This is what is going to give you this. So, I will put a subscript B here to indicate that these forces are along aircraft fixed axes.

29:06

So, we have three force equations, and now, what else I need to write. We have to write three equations for the rotational motions. I will assume that there is an external moment acting on this elemental mass, delta m. So, the moment is, external, sum of all external moments acting on the elemental mass m and this is going to be equal to… How do you write moment? Like force is the rate of change of linear momentum. So, moment is rate of change of angular momentum. Let us say delta H is the angular momentum of this elemental mass delta M. So, M is ....(Refer Eq(4)) this.

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What is delta H? r cross V, V is the velocity of this elemental mass delta m, into delta m right. That is how we write the angular momentum.

32:37

So, r is depending upon this mass m. So, I am clubbing these two together and, V c is actually independent of this, velocity of the center of mass. So, we can keep it separate.

33:26

You know as long as you are following this much, its all right for this course. For any detailed explanation on this or mathematical description, you can go back to your classical mechanics, you know, course, you would have done a course in mechanics.

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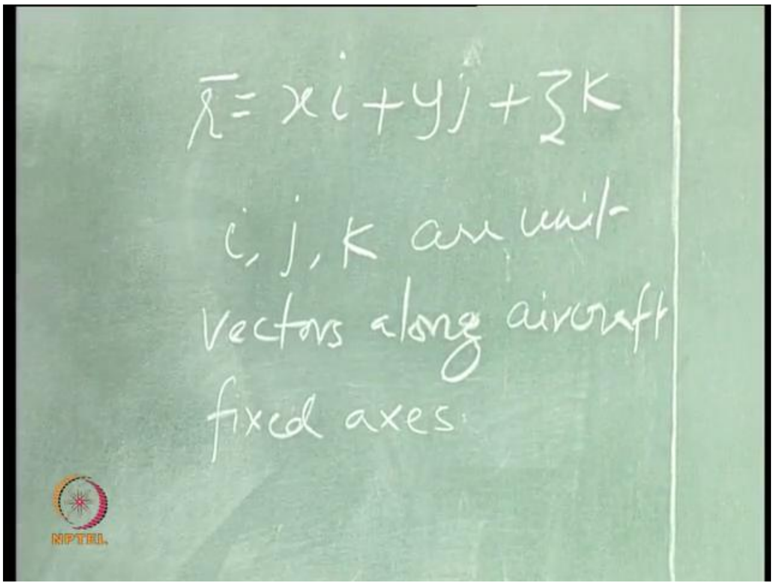

So, this (Refer Eq(4)) becomes 0 again, you know, for the same reason that we are measuring all the distances from the center of mass of the aircraft. So, this term goes to 0, and let us expand this; we will try to expand this. So, we will assume the coordinate of this elemental mass delta m to be x y z. So, r is .... , and i j k are unit vectors along aircraft fixed axes. Let us see what we get when we expand this. .... (Refer Eq(4))

37:15

Is this correct? No problem here, right.

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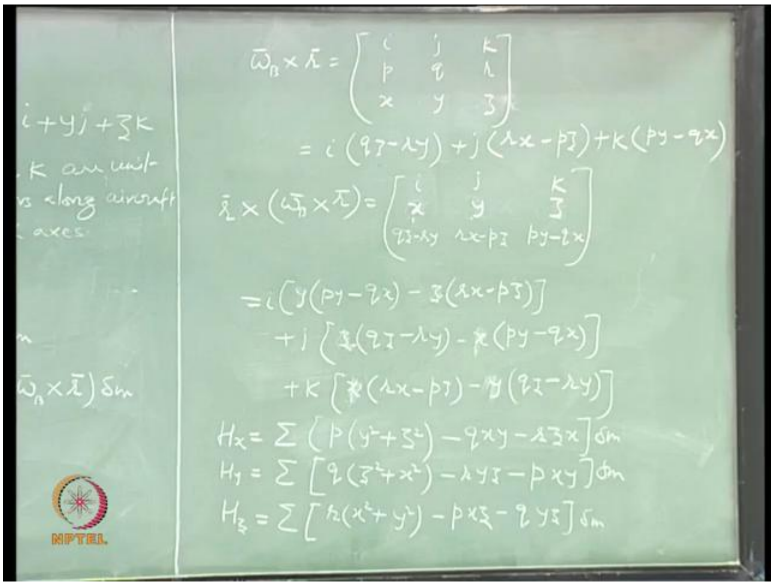

$$\overline{\omega}_B \times \overline{r} = \begin{bmatrix} i & j & k \\ p & q & r \\ x & y & z \end{bmatrix} = i(qz - ry) + j(rx - pz) + k(py - qx)$$

$$\overline{r} \times (\overline{\omega}_B \times \overline{r}) = \begin{bmatrix} i & j & k \\ x & y & z \\ (qz - ry) & (rx - pz) & (py - qx) \end{bmatrix}$$

$$= i[y(py - qx) - z(rx - pz)] + j[z(qz - ry) - x(py - qx)] + k[x(rx - pz) - y(qz - ry)]$$

$$\begin{aligned} H_x &= \sum\left[p(y^2 + z^2) - qxy - rzx\right]\delta m \\ H_y &= \sum\left[- pyx + q(x^2 + z^2) - ryz\right]\delta m \\ H_y &= \sum\left[- pxz - qzy + r(x^2 + y^2)\right]\delta m \end{aligned} \tag{5}$$

38:15

So, I have found an expression for the components of this vector h along body fixed axes. So, what is the x component? (Refer Slide Time: 38:43) This into delta m and we have to sum it over the whole aircraft.

Then see any mistake anywhere.

Sir, second row of the… Sir, on the first above, we had a r s x, y, z; here is the minus t q r.

So, this, yeah, thanks. You should keep doing this. When I am making mistakes, you should tell me; otherwise we will have to...

40:03

So, now, I am ready to write the x component of this vector, along body fixed x axis. .... Is this all right? So, you have components of angular momentum vector along the body fixed x, y and z axes (Refer Eq(5)). So, if you want, so, it is written for, this b for body - the aircraft body. I think we will, we will, stop at this point.