Acoustic Instabilities in Aerospace Propulsion

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Lecture - 5
Standing Waves

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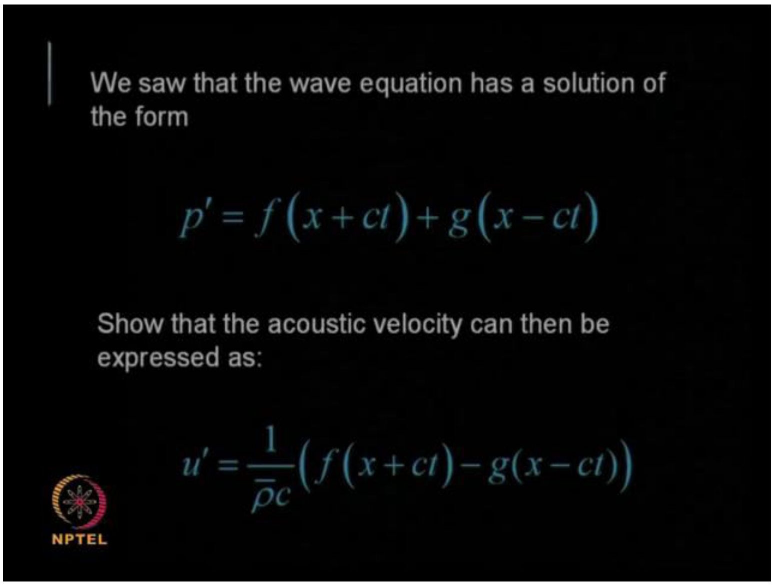

Good morning everybody. Today we will start with some brief exercises based on whatever we have done, so far. So, the first question is we derived that the pressure can be written ((Refer Time: 00:24)), in terms of a function f, which is this f of x plus c t, and another function g, which is ah function of x minus c t. And given this derived the following expression that acoustic velocity can also be written in terms of f and g, but it is on over row sheet and f minus g.

Please work this out we have almost done this exercise in the last class in another form. So, this is the first exercise and I wish to explain what is meant by acoustic velocity, there was a question in last class as to what is the difference between acoustic velocity and speed of the wave, speed of sound which we denote as c square root ((Refer Time:01:13)).

So, when you think of a fluid and if you think of a fluid element, the fluid element get displaced and in the displacement it also has the velocity. And so you think of little fluid particle moving and it is the velocity of small particle that is to be referred as acoustic velocity and also called particle velocity. The speed of sound see this square root of dou p by dou rho at constant rho p, that is the speed with which the wave itself travels.

So, that is the pattern is travelling with that speed it is also called phase speed, now if you give analogy. Let us say, we are standing in a line to take a movie ticket or something and somebody pushes from the back and you move a little bit. But, you push the guy in front little bit and he pushes the front little bit and eventually the person at the front line gets push.

And so although you are not physically moving you moved only few centimeters, but the wave itself moved several meters and it is the speed at which the push travels not the person travels, it speed at which push travels. Push is some kind of stress behave or something. So, that speed would be analogues to our speed of sound and he the velocity at which the velocity at which you move forward when you get a push that would be analogues to the particle velocity, acoustic velocity presume this is clear.

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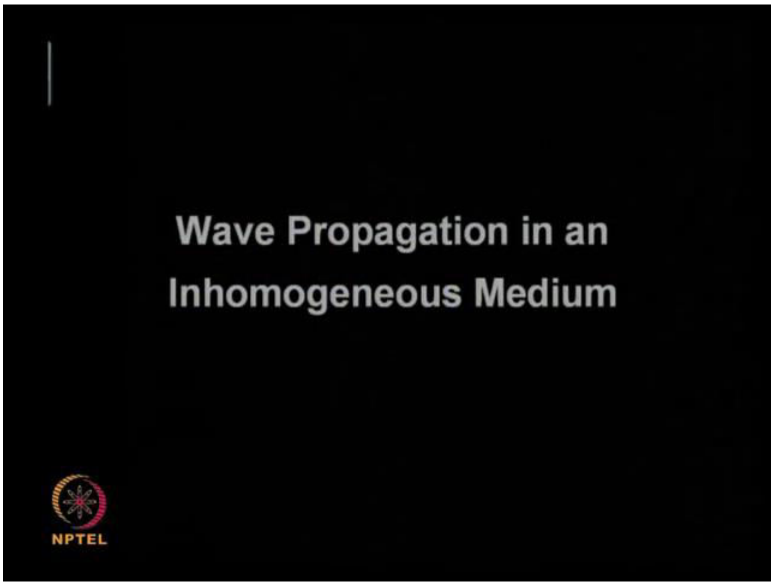

So, there are questions about what happens, when the properties of the medium or non uniform.

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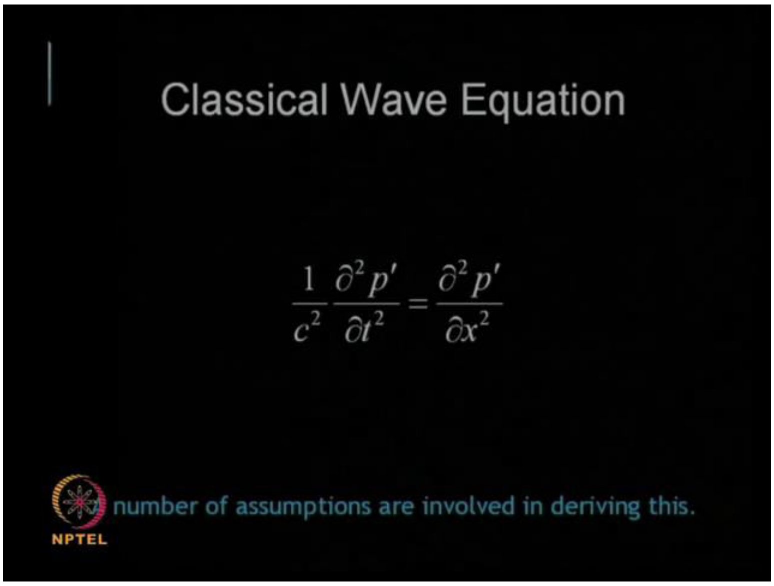

And we have derived this classic wave equation under the lots of assumption involved in deriving, we said that the flow was non x not there, it was a quasar medium and temperature density pressure. The mean values of temperature density and pressure where constant and so on. We had to make a lot of assumptions, because the equations gets very complex and also as I mention in the presence of flow, separating the acoustic wave from a vorticity wave and entropy wave is not a an the wave from the base flow itself is not a trivial thing. To keep thing simple, we made this assumptions and we derived this wave equation on over c square dou square p prime over dou t square equal to dou square p prime by dou x square.

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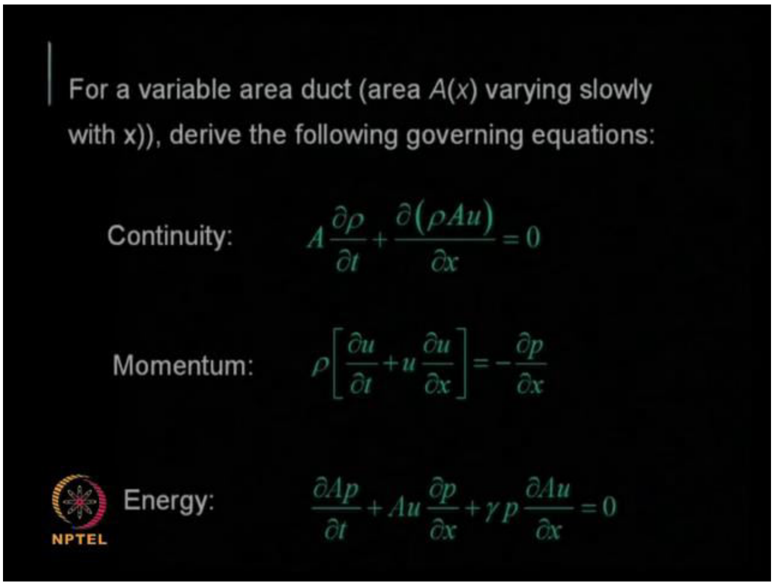

Now, let us try relax this assumptions. So, we had done this homework already that we derive the equations for motion or I previous we have done this already, the quasi 1D equations for continuity momentum and energy.

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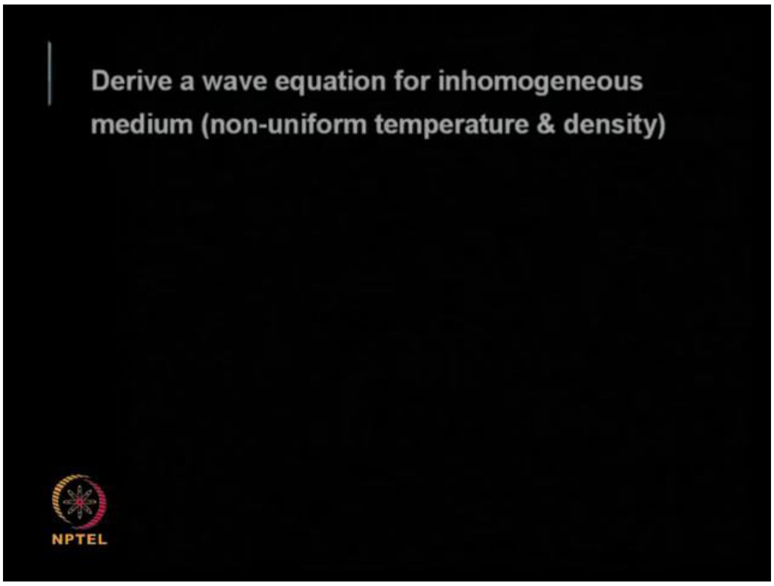

You can use this these expressions, these partial different equations and derive a wave equation for homogenous medium. Homogenous medium here would imply a non uniform temperature or non uniform density let us, still keep the assumption that there is no base flow that is it is a caisson medium.

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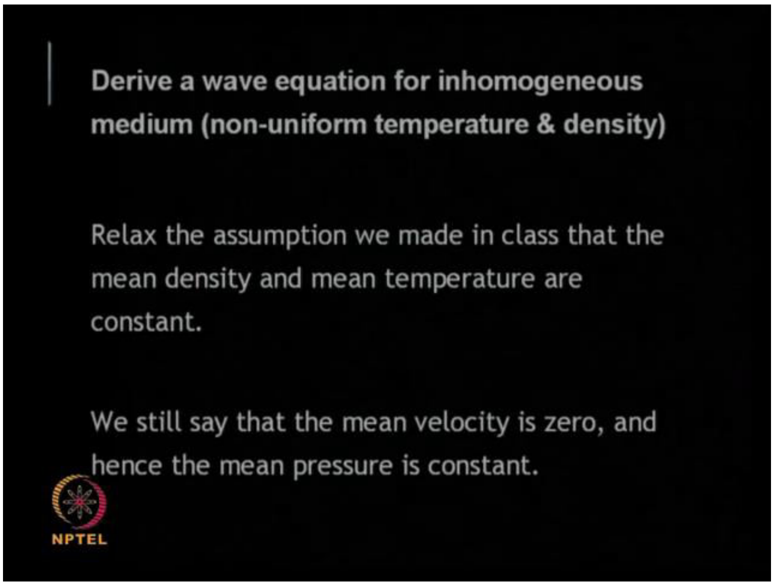

And relaxing this assumption that temperature is constant and density is constant, we can actually derive a wave equation, we still say that the mean velocity is zero and hence mean pressure constant this simplifies the matter is a lot.

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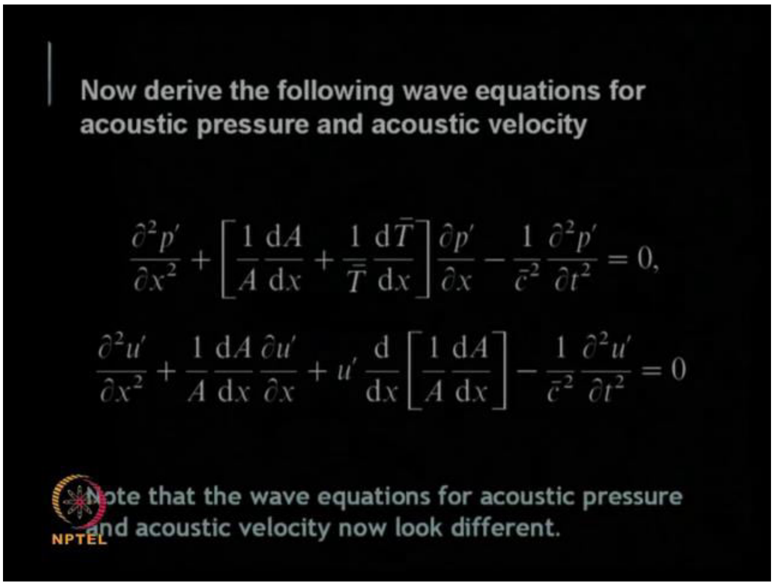

And now you can manipulate the governing equations and derive these 2 equations of the atopic equation is the wave equation in terms of acoustic pressure and the bottom equation is the wave equation in terms of the acoustic velocity. Please, derive these two things and I wish to note that wave equation for acoustic pressure. And the acoustic velocity now looks different from each other.

If you think of the classical wave equation, we saw that the wave equation for pressure was exact identical to the wave equation for the velocity accept that, we had same operator you can replace the pressure p prime with the velocity u prime. But, now you can see that the although, the second derivative terms are same the first derivative terms are not identical in the operator for either equation. So, please do this problem at home and it will be very interesting exercise, it is quite simple to do this.

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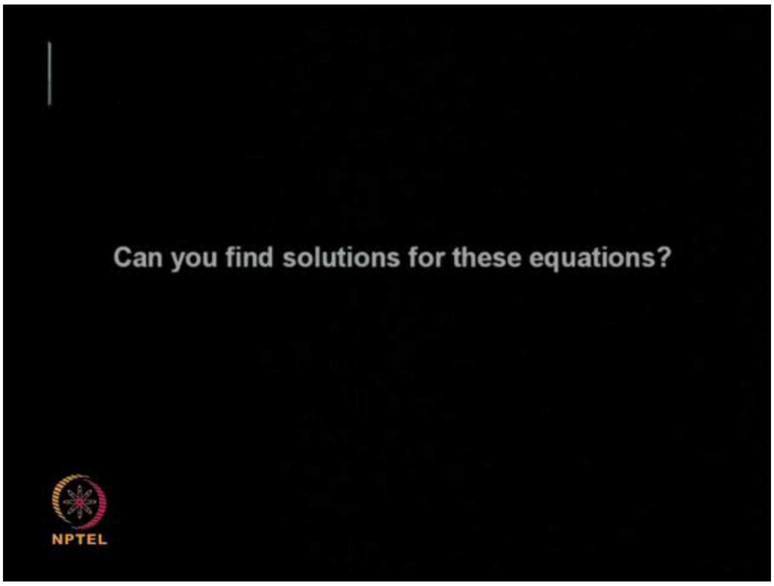

Now, the last part is can you find solution to this equations, now if you go back and look at this equations.

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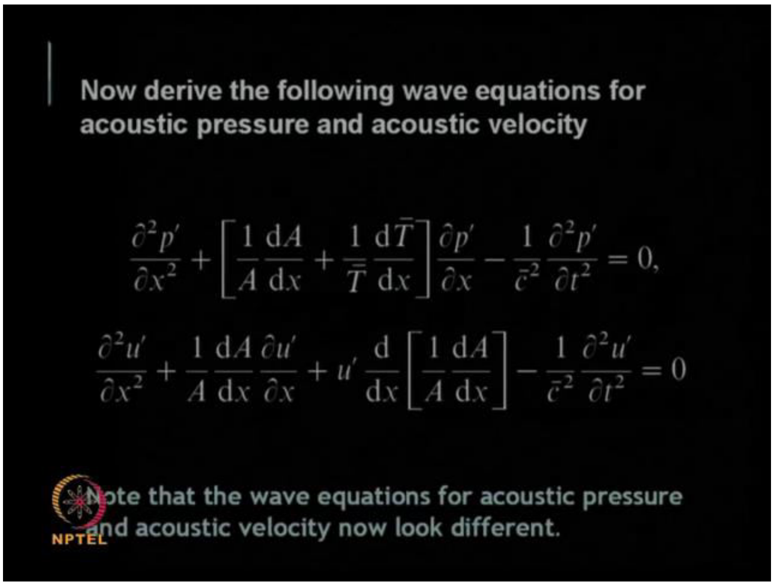

They are partial differential equations to get the solution, generalized solution of kind of the form f of x plus c t, g f minus c t may not be, so easy it may be possible. And if you do it you can surly get famous, I have derived solutions for this equation, in harmonic domain that you say p prime goes like p hat e power i omega t.

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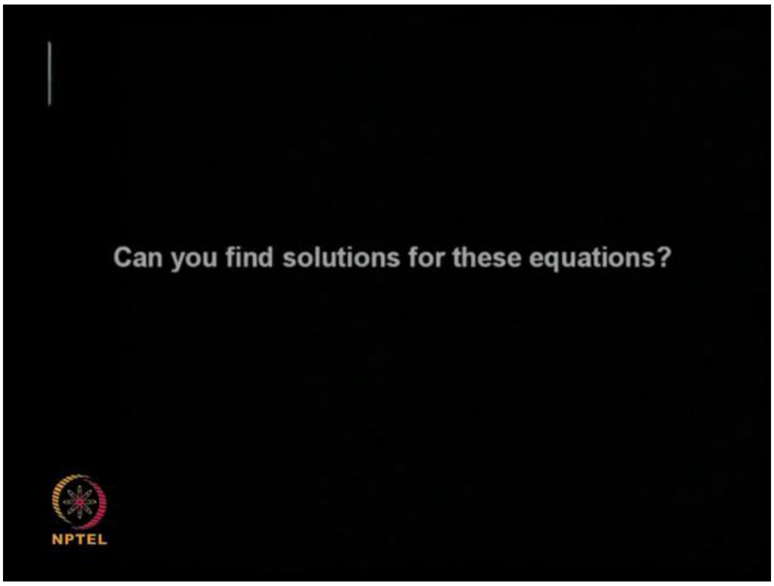

And then you can and then you can try to reduce the equation, the partial differential equation into ordinary differential equations and then you can see this ordinary equation is of the class of some of the existing differential equations. For example, to give a hint if the temperature is varying as a leaner function sort of the form t not plus m x then you would get the you can reduce the ordinarily differential equation to a Bessel differential equation. And you can actually get the solutions in terms of the Bessel function.

So, please work out these problems. And now, so now let us try to look at standing waves, in the last class what we did was we looked at a travelling waves and we wrote the solutions for the travelling waves, now we will try to derive solutions for the for standing waves. So, let us start with what is called a water wave tube.

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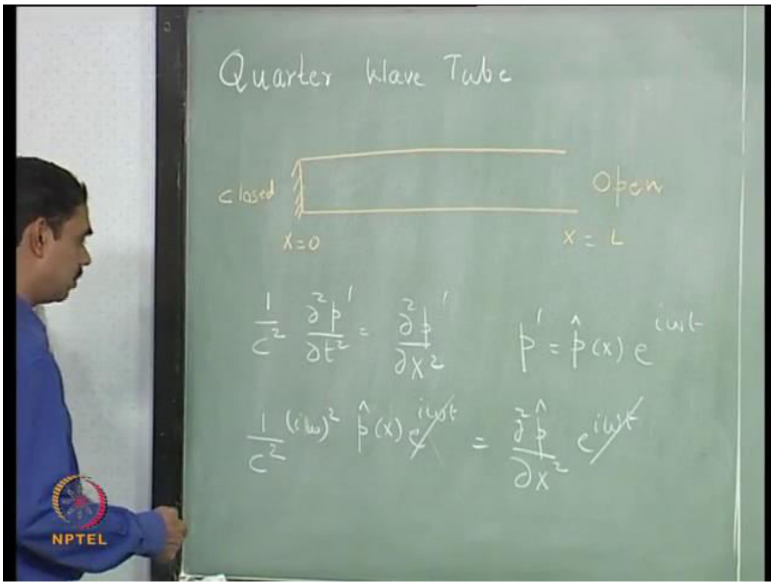

So, this is a tube which has all end open and another end closed. So, let us say this is x equal to 0 and this is x equal to L and this side is closed and his side opened. And we wish to solve for the wave fail inside this track that is there. You can actually see that this is similar to the some of the exercises, you would have done in high school laboratories, where you took a tube and immersed it in a in water and then you try to measure the length of the tube and then you used a tuning fork to determine the resonance frequency and so on.

It has more profound imply a profound uses and implications for example, we can model even with this problem, a geometry has complicated as a solid rocket motor, because there you have both sides closed. So, it is this is open on one side closed to another side, if you keep both side closed then we actually get the geometry, where acoustic configuration very similar to that of the solid rocket motor accept that there is a nozzle, which is very complicated. But, the nozzle surprisingly behaves very close to a closed.

So, solving this problem will certainly help in getting understanding of more difficult problems. So, what we first need to do is to or the way you solve any differential equations, you used to write the boundary conditions and then we also with the boundary conditions we try to solve for the equations. So, let us start solving this problem we start with the wave equation itself. So, make this n sets that p prime equal to p hat of x e power i omega t. So, if you substitute this over here, what I would get is 1 over c square i omega square this would be, so we can cancel e power i omega t on both sides and i omega square would be i squared is minus 1. So, we can enough to write this is least one, because now it is a full derivative.

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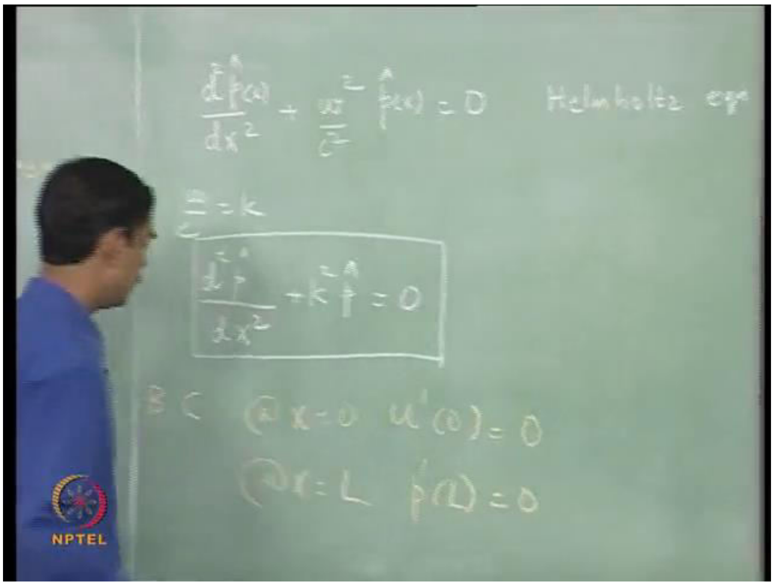

Is this any mistake, if there is any mistake please find out this is the Helmholtz equation, we can rewrite omega over c as k. So, we can indeed rewrite this equation as d square, I think from now on, I would not write this dependence explicitly, I guess by now we are clear that p hat depends on x. So, this is the equation we need to solve. And let us now think about what are the boundary conditions. So, the boundary conditions are we said that this end is closed and the other end is opened. So, what would closed end in terms of boundary conditions.

Student: ((Refer Time: 11:57))

When you have closed end or a rigid end, rigid end mean something is rigid that means, that does not move. So, we said that velocity is zero, that is the boundary condition and if you have. So, let us write this now at x equal to 0, u prime of 0 equal to 0, now at what is the boundary condition at x equal to l the ((Refer Time: 12:28)). Why is p prime is equal to 0, because the open end the end is opened to atmosphere. So, the pressure is equal to the atmospheric pressure, so p prime is the pressure minus the base pressure. So, p prime minus p bar sorry p prime is p minus p bar and p is now equal to p bar. So, c prime is 0. So, these allow to boundary conditions, now we need to write an expression for acoustic velocity before we proceed to evaluate this.

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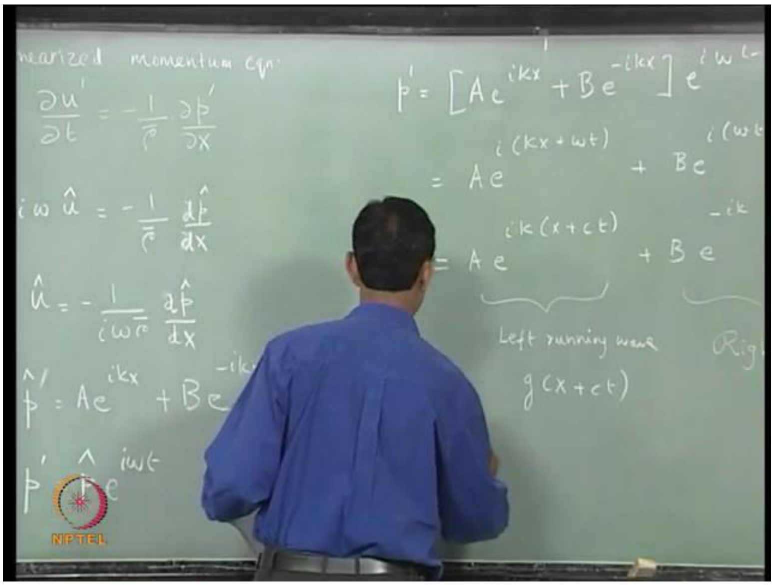

So, we can use the linearized momentum equation d u primed d t is equal to minus 1 over rho bar, d p prime by d x and if you now substitute this n sets, we said that p prime is p hat of x e power i omega t. And also we make an answer u prime equal to u hat of x e power i omega t, box this here and. So, when you differentiate e power i omega t, you may get i omega out, e power i omega t both sides will cancel. So, you have equal to minus 1 over i omega rho bar, so let us now try to write this solutions for this differential equation, what would be the solution for this differential equation.

Students: A to the power of 5 k x plus d 2 d power of minus i k.

So, let us write that, so this is the solution and before we proceed to actually solve the problem, lets a take a look at this equation until tight interpretance of s n g. So, we understand what is going on, so p prime would be p hat e power i omega t. So, I multiply this by i omega t. So, A e power i k x plus B e power i k x, times e power i omega t, which i can rewrite it as A e power i times k x plus omega t plus B e power i times omega t minus k x at this instant it is probably clear, which is the left running wave and which correspond to right running wave, which is the left running wave ((Refer Time: 16:49)).

Let us take a look at it. So, this would be this can this is the left running wave, let us say of the form g of x plus c t and this would be the right running wave of the form f of x minus c t. So, what would be velocity. So, velocity would be you use the expression for acoustic pressure on this equation for acoustic velocity. So, let us do that, so you have equal minus 1 over i omega rho time d p by d x. So, that will be i k will come will out, i k and it will be A u power i k x plus B e power minus i k x.

So, we can replace omega as k over c k over c. So, this in term becomes, so the k can cancel rho c comes to the top. So, these will indeed the minus 1 over rho bar c, I made a mistake minus sign here this. Now if you if you multiplied this u hat by e power i omega t we can get expression for the velocity in the time domain and then we can see how the waves left running light, right running waves are moving. So, I am going to erase here.

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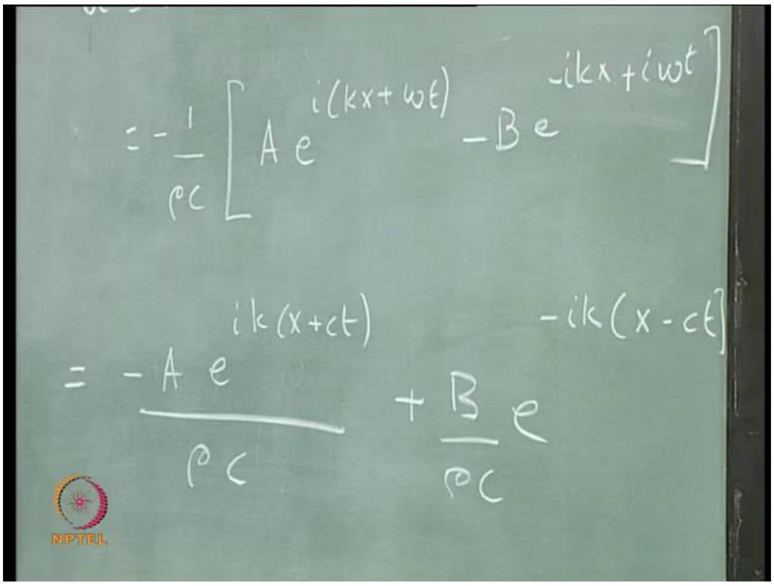

So, this is the what wave is this, left running wave and this is the right running wave and we can see that the right running wave has the positive sign in the front that is understandable. Because, if you think of a combustion wave that is the pressure is going up and moving to your right, the gas the particle the fluid particle will move behind the wave, but to a along with these waves. So that means, there should be a plus sign because your particle is going toward the possible x axes.

But, you are looking at the left running wave, when if you think of combustion wave moving to the left, the particle also move along with it behind it to the left. So, negative x axis is the direction of the particle. So, that is why minus sign is coming. So, this is in congruence with our understanding or our intuitive understanding. So, this having done this we can actually proceed to implement our boundary condition. So, this harmonic domain and time domain approaches are very consistence with each other as you saw, it just at harmonic domain it is quite easy to solve for the equations.

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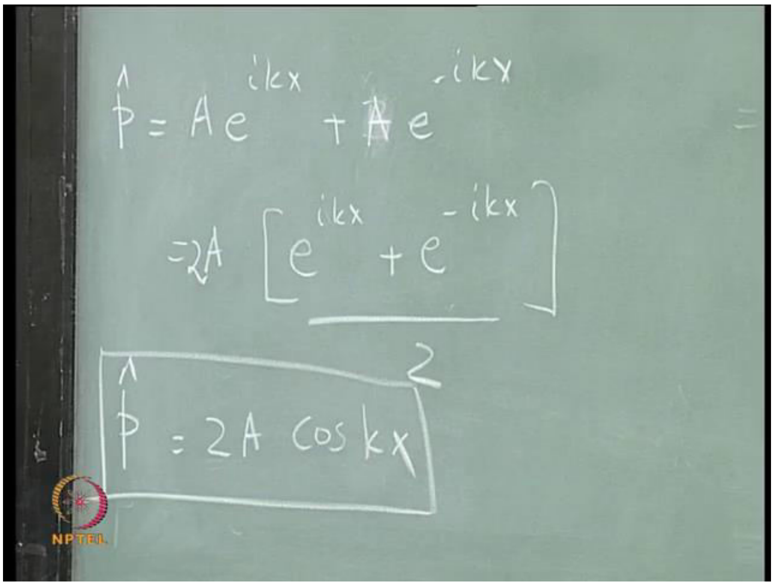

So, we that velocity is 0, so that would mean that u at 0 equal to 0 and or expression for velocity was here is the expression velocity, so we can say that e power i times 0 is e power 0, which is 1. So, you will get A minus B equal to 0, so this would mean that A equal to B. So, now, we can rewrite the expression for pressure amplitude as A e power i K x plus B e power minus i K x, we and can replace this B with A. So, this can be written as A times e power i K x plus e power minus i K x, now what I will do is multiply and divide by 2 for convenes. So, what is this expression?

Students: Cos x

Cos K x, so this would be 2 A cos K x here, so got a nice simple formula for p hat, so now the second boundary, I will wait any questions here.

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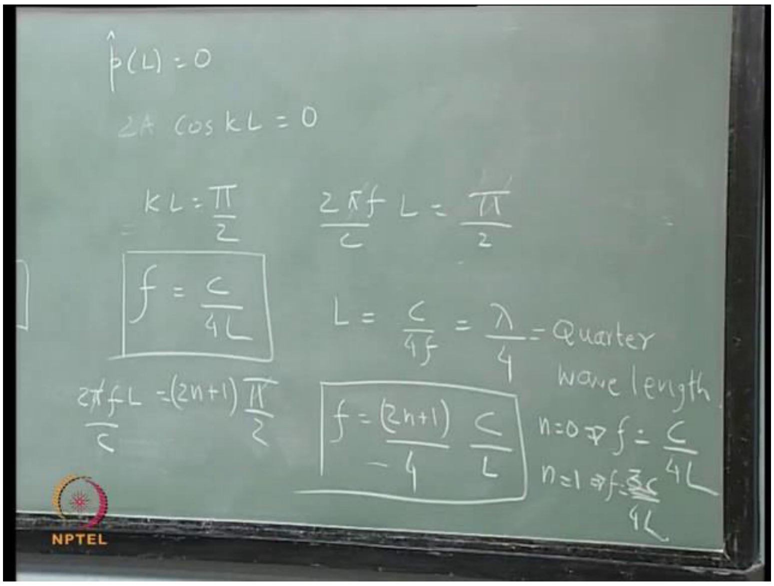

So, the second boundary condition says that p hat of L equal to 0. So, p hat of L would be B 2 A cos K L equal to 0 or alternately, if A is not 0 then you should get cos K L is 0 now this is now Eigen value problem. Because, there are only certain values of k which will actually, make this relation satisfied and for example, when this cos of some function equal to 0 ((Refer Time: 25:01)).

So, ninety degrees two seventy degree and so on, so K L equal to pi over 2 would be the first solution. So, K is 2 pi f over C else L equal to pi over 2 or f equal to C over 4 L this is. So, now, it is properly clear, why this tube was called quite quarter wave tube, because you have one quarter of the wave length with in the within the tube. So, we can see that L between 2 length is C by 4 f which is c by f is lambda, so it is lambda by 4. So, one length with you has one quarter of the wave length, hence it is called the quarter wave length.

Now we can get a general solution. So, we can say that K L equal to 2 n plus 1 into pi over 2 right. So, K is a 2 pi f over C. So, therefore, we can get a general expression for f as 2 n plus 1 over 4 times of C over L, I hope this is right please check yourself. So, this is the general expression for frequency. So, n equal to 1, n equal to 0 gives f equal to C over 4 L, n equal to 1 give f equal to C, 3 C over 4 L and then we have 5 C over 4 L and so on, so the natural frequency of this tube given by C over 4 L 3 C over 4 L 5 C over 4 L and so on.

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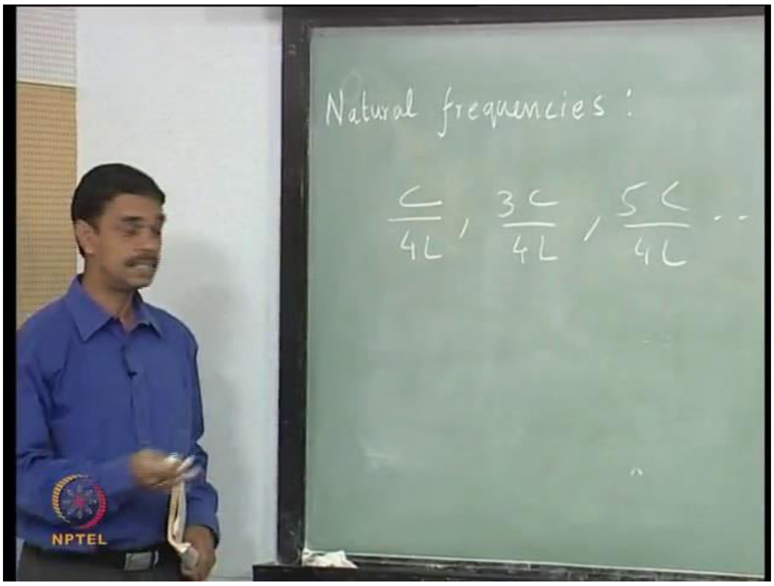

These such a problem is called Eigen value, problems because only for certain unique values of this parameter k, you get solutions for the differential equation. So, Eigen value Eigen is a term from German, which means once all on or unique. So, only for these unique frequencies you get solution. So, that would be general mathematical term of this natural frequency they are called Eigen value. It is one word Eigen values of Eigen frequencies.

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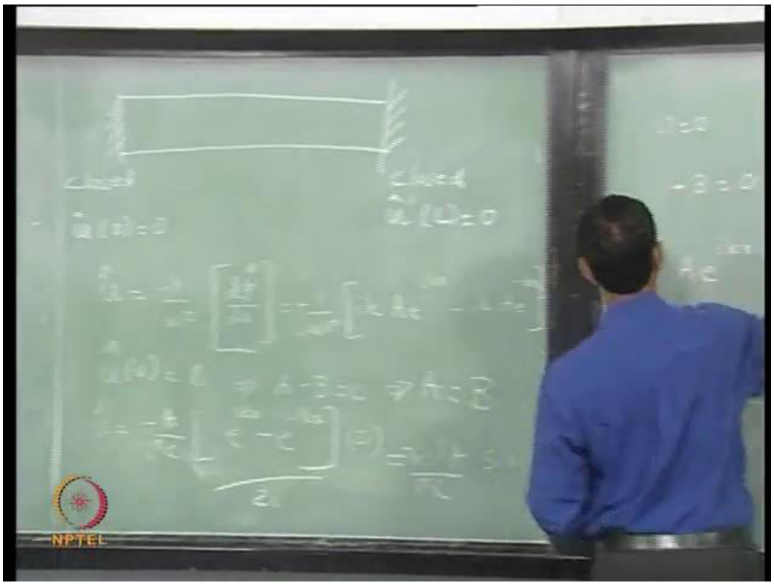

Now let us consider a tube, which is closed at both the ends closed. So, the boundary condition would be u of 0 equal to 0 and here will be. So, can you try to work this out pass for two minutes to see if you can work it out then I will do it. So, expression for acoustic velocity was u hat equal to minus 1 over i omega rho. So, u hat of 0 would be e power 0 would be 1. So, that will go like equal to 0, that means this A equal B. So, we can write u hat equal to minus 1 over rho C times, A e power i K x, I will take the A out minus e power minus i K x. And if I multiply and divide by 2 i here, I would get this would be 2 i minus A over rho C into what is this formula.

Students: Sign K x

Sign K x, so can we now proceed to determine, what would be the Eigen frequencies.

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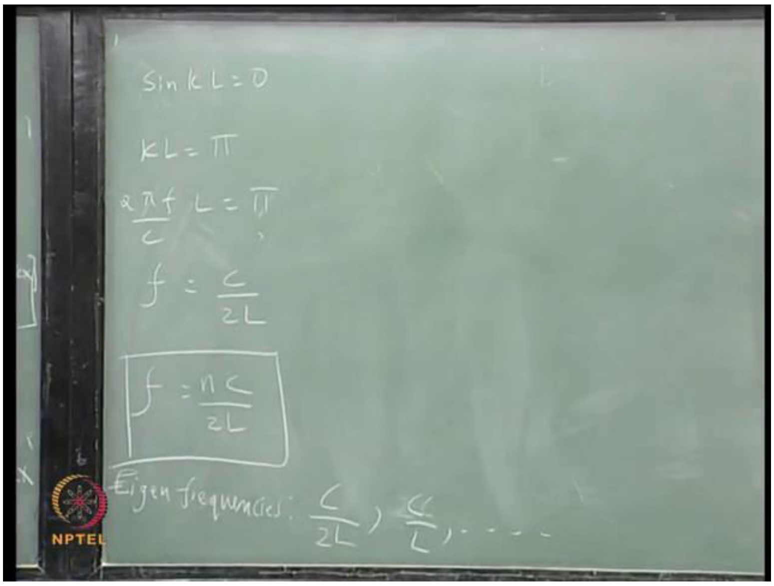

So, we should get sin K L equal to 0 and let us look when sign K L will be 0 first and when is the first root of sin K L equal to 0 comes K L equal to, no the next one, that is 2 trivial. So, K L equal to pi K is 2 pi f over C L equal to pi or f is equal to C over 2 L and general formula would be ((Refer Time: 33:25)). So, our Eigen frequencies or the natural frequencies are C by 2 L, C by 2 C by 2 L, which is C by L this.

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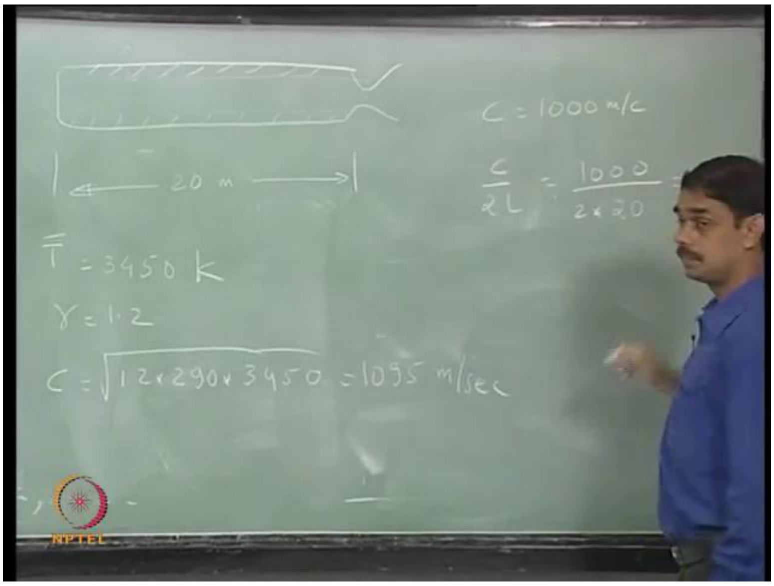

So, let us try to work out a problem consider a solid rocket motor like it buster lets think about the p s l v. So, the typical length of p s l v is the order of s s p s one motor. It is the order of 20 meter approximately, so let us a round number 20 meter long and what would be the speed of sound, temperature what is the temperature, 3458 meters. T bar is almost constant most of the tube as it is 2500 would be the temperature you get, if you burn methane or something.

But, this would be enough of energetic material, I think it should be abused gamma equal to 1.2 anybody has a calculator, can you find out what is the speed of sound what would be this 1095. So, we can round it of to 1000 meter per second. So, let us say C is 1000. So, what would be the natural frequency, C over 2 L right. This is 1000 divided by 2 into 20. So, this would be 25 actual this motor is a fairly stable motor, any motor will have some level of oscillations, but this motor has been working nicely therefore, pressure transfuses and found that the frequency is C the amplitude is small.

But still we see natural frequency in terms of small oscillations; the frequency is order of 28 hertz. So, we are pretty close with or really proved calculation, any questions yes please.

Students: End socket means it work on pressure with roots. So, is there any reason that I will distinguish using a boundary condition and pressure.

So, there is a in general you have to write a boundary condition in terms of the admittance.

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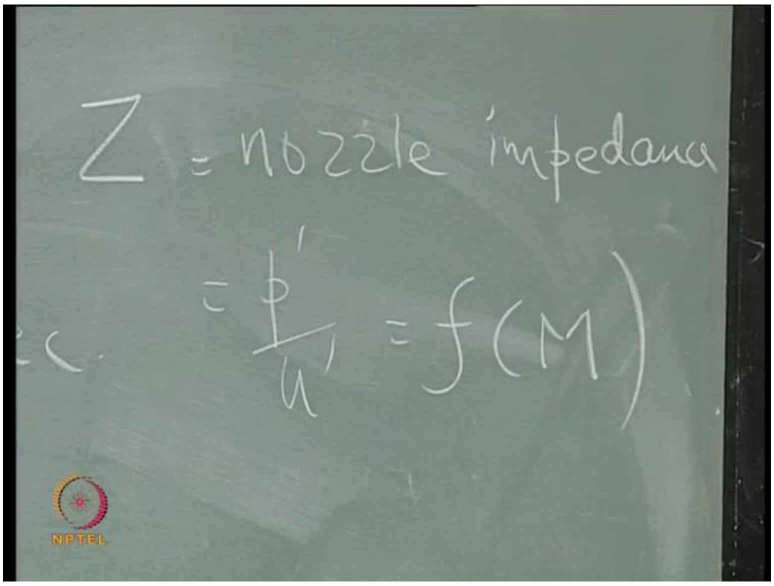

The impedance which is given by p prime over u prime and there are a formulas of for this. In terms of the, which you say a nozzle exit mach number. So, if you can use, there is a formula given by is not on this and if you can use that and you can get the better estimate for the natural frequency so.... But, it is very close to a hard end now as well Nozzle is open you think it is what a mean, by closed end is most of the acoustic wave gets reflected and come back and it behaves almost as if a open as if a acoustic is closed is very close well. And that bring us to the concept of acoustic impedance, thank you for the question and we will get to it, but before that I want to give a homework or class work. Can you try to derive the natural frequency of a tube, which is open and opened, you can do it now.

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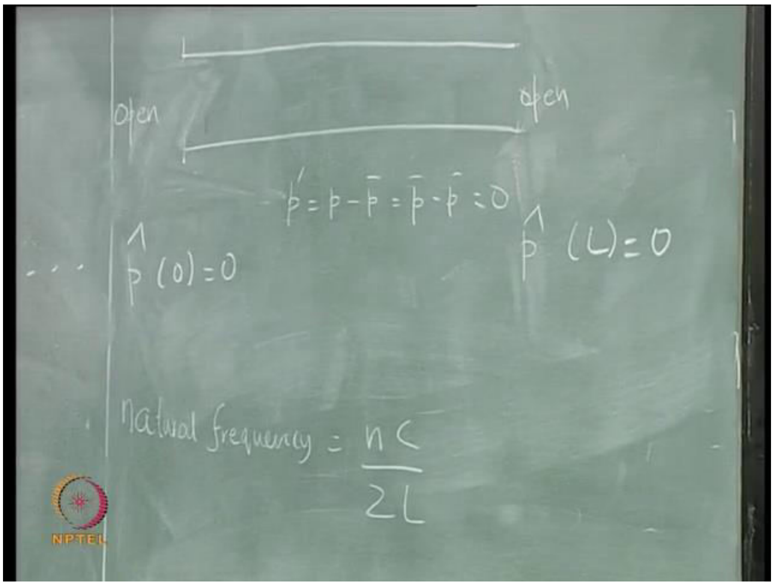

So, for a open condition p equal to p prime is p minus p bar, which is p bar minus p bar equal to 0. So, you can say p hat of 0 is 0 and p hat of L is 0. So, what would be the natural frequency, you get ergative for example, we will work very close to open and opened. So, in this, we saw that the closed open end that kind of tube was called quarter wave length.

So, open, open; and close, close, they are like half wave tubes. So, I wish to aphesis that for any solution to a any ordinary differential equation you need boundary conditions. So, the type of boundary conditions, we used are in terms of pressure is equal to 0 or d p by d x is 0. So, it is these are called distil environment boundary conditions.

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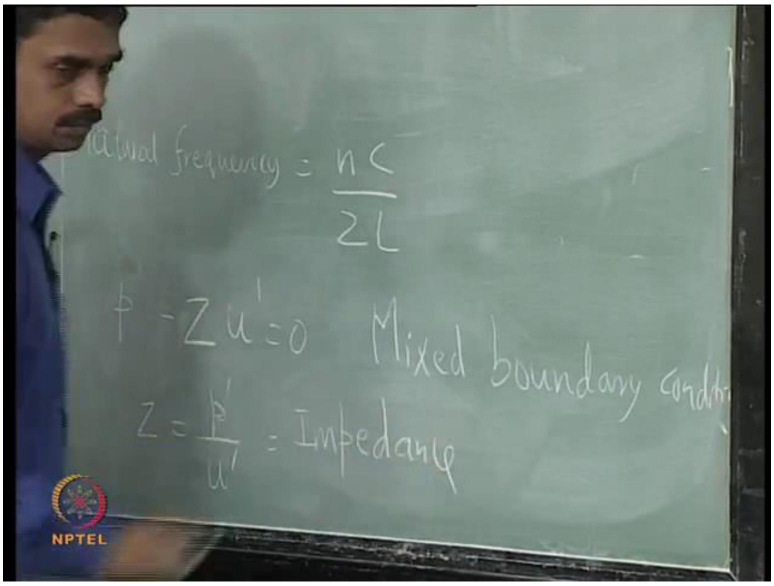

We also have can have mixed boundary conditions of the form p prime plus alpha times, this would be a mixed boundary condition. So, this would mean that alpha would be equal to minus p prime by u prime. So, we can rewrite alpha with a minus z , we can say z equal to p prime over u prime, so this z is called impedance. So, for a general boundary condition, this is what we would be using and we will see later that the either, if you have pressure as 0, p prime as 0 or u prime as 0 then you do not do any work at the boundaries, if you have either p prime as 0 or u prime as 0.

The quantity p prime u prime, which is like the work done the acoustic intensity, would be 0 at the boundaries. So, that means no work is done or taken out and therefore if you have a measure of the acoustic energy in it and it would stay as constant. But, in reality it could be possible that energy is radiated out of the duct or coming into a duct and then you would need such a boundary condition.

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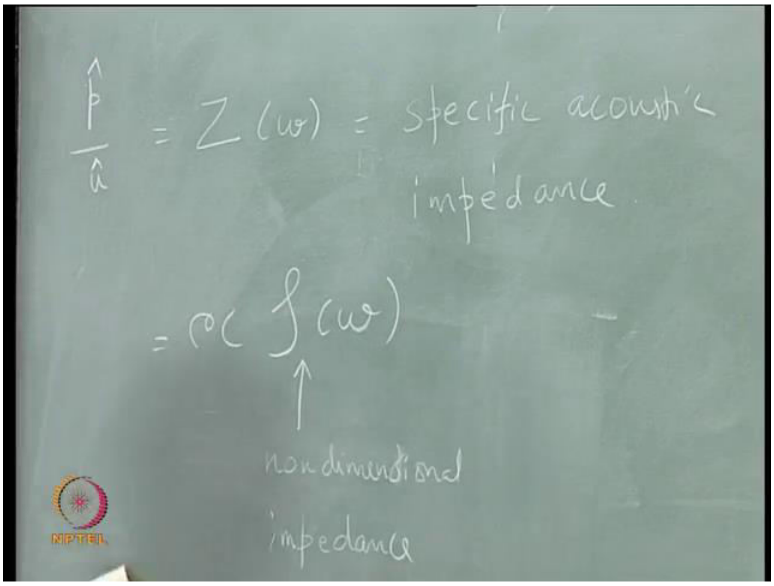

So, this quantity p hat over u hat is equal to z and then in general it is a function of frequency and this is called specific acoustic impedance and it is written at a interface. So, in case you are talking about interface, this if the velocity is not one dimensional and this actually amount to the velocity, which is normal to the interface. And in general frequencies of different, waves are different frequencies behave differently at the interface. So, which is why this question is a z is a function of omega.

And I wish to emphasize that, this is ah different from the characteristic impedance. Characteristic impedance is a characteristic of the medium and there it is defined as rho C and that is property of the medium. That this actually, depends how the wave behave the wave may be getting reflected at some boundary and coming back in soon and therefore, you can have pressure going as f plus f plus g.

But, velocity may be f minus g, so depends on the boundary values how the reflections take place at the boundaries and so on. It will effect what the impedance value is where as the specific acoustic impedance does not really it depends sorry, the characteristic impedance of the medium depends on the geometry or any such thing it just depends on the fluid, whereas here this can depend on lots of things.

Now, you can, so this is like in electricity, we have impedance which is the ratio voltage to current. So, this is very analogs to that, I think this kind of approaches came from an electric engineering and now we can non dimensional it as. So, this would be the need of omega would be the non dimensional impedance and rho C is the characteristic impedance.

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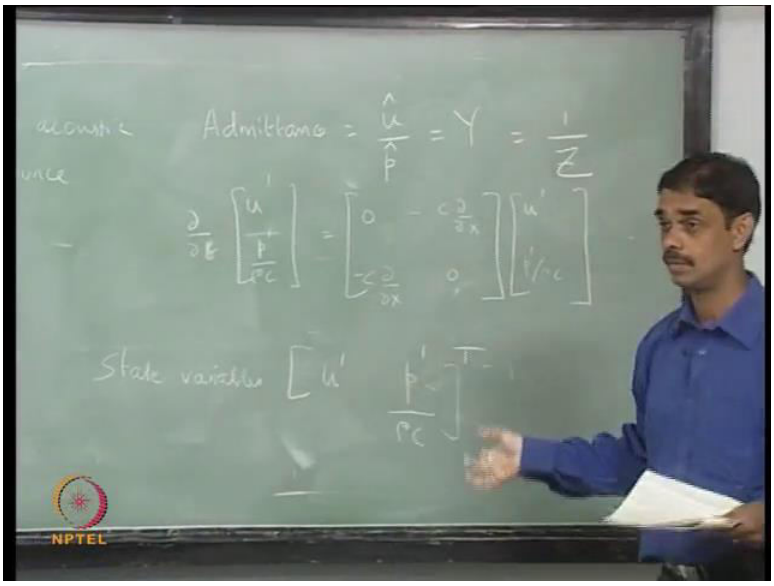

We can also define a quantity called admittance, which is u hat over p hat, so this is like the reciprocal of impedance. So, admittance is usually given by the symbol Y, which is goes like 1 over z and this is similar to conductance. So, it current divided by voltage is conductance and admittance is indeed a familiar quantity, you can actually express the energy, that is coming in acoustic intensity, that is coming into a tube or going out of a tube in terms of the acoustic admittance and we will see that later.

So, we need this is qualify, what is the acoustic power that is coming into a medium or going out of the medium. So, if you have nice boundaries like velocity 0 or pressure is 0, then I mentioned already that the acoustic energy in the tube stay constant. But, if you have something coming in or going out, let us say there is acoustic power coming into the tube then what happens to the acoustic energy, it will keep growing.

So, there will be kind of growth rate of course, the linear will keep on growing exponentially to infinity, but in reality non-linearity will come in and this growth may not happen forever. But, let us say as we have a growth and we have to determine the growth right and this growth rate can be determined in terms of the impedance or the admittance.

Similarly, if you have acoustic energy in a duct and the energy is going out of the duct then your acoustic field will decay and the decay rate can be expressed in terms of the impedance or the admittance, we will do that in the coming classes. We also need to have a measure of the acoustic energy, so we can write the equations in a dynamiter six term language and now I have written in this manner. So, this operator gets normal and it becomes a nice form. If I do not normalize by rho C then I would not have this operator here being normal and having normal operator has lot of good properties. So, now, this is the state variables. So, we have state variables as u prime and p prime over rho T.

So, need some kind of measure for characters in this system, which characters the energy of the system. So, that can be obtained by squaring this and squaring this. So, it will go something like u square plus p square accept there is a factor in front of these things. So, we see, if we can get such a measure of acoustic energy of the form of velocity square plus pressure square weight appropriately. So, the velocity square would correspond to the kinetic energy associated to the wave, the pressure square will correspond to the energy in the wave it is like a spring when it is compressed it has some energy stored in the spring. So, pressure square would be some term of the sort.

So, we will do it in next class. So, somebody we have take an acute look at standing waves in a duct and now, we have now know how to calculate the natural frequencies of ducts, which are open and open ends and closed ends. And we even thought of how this can be used to calculate acute estimates of natural frequencies of even such a complex thing as solid rocket motors. And then we ended definition impedance admittance and we will see how that can be used to characterize the growth and decay of acoustic energy in this system, I will stop here see you tomorrow have a good day.