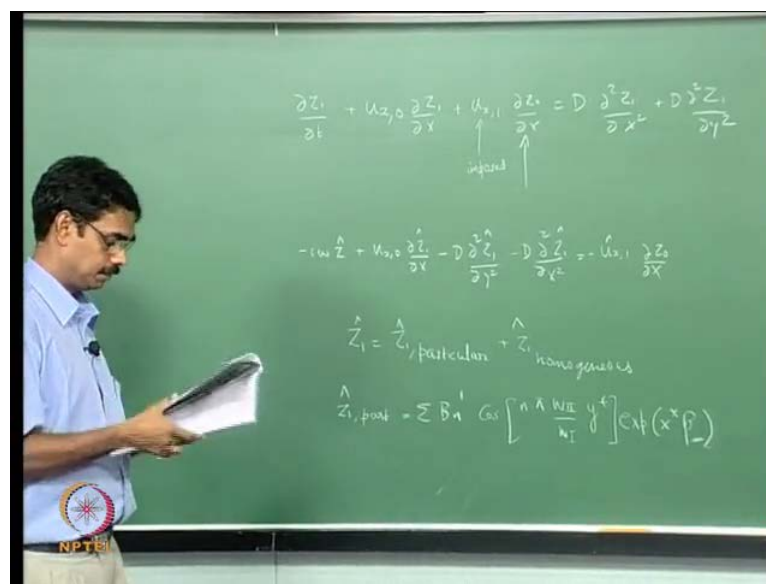


Acoustic Instabilities in Aerospace Propulsion
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Lecture - 41
Response of a Diffusion Flame to Acoustic Oscillation-3

Good morning everybody. We will now try to work out the unsteady solution, for the unsteady Burke Schumann problem.

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We are trying to split z as z naught plus z 1. When you linearize, you have the term u d z by d x , so that you will have u x comma 0 d z 1 by d x plus there is u x comma 1 d z naught by d x . Those are both first order terms and then you have the diffusion terms here. So, this is imposed. We will say u had e power minus i ω t following the convention of Lieuwen. We are following the analyses from the text book unsteady combustor physics of T Lieuwen. This term we know because we have now derived a solution for z naught. So, we can actually find an expression for d z naught by d x .

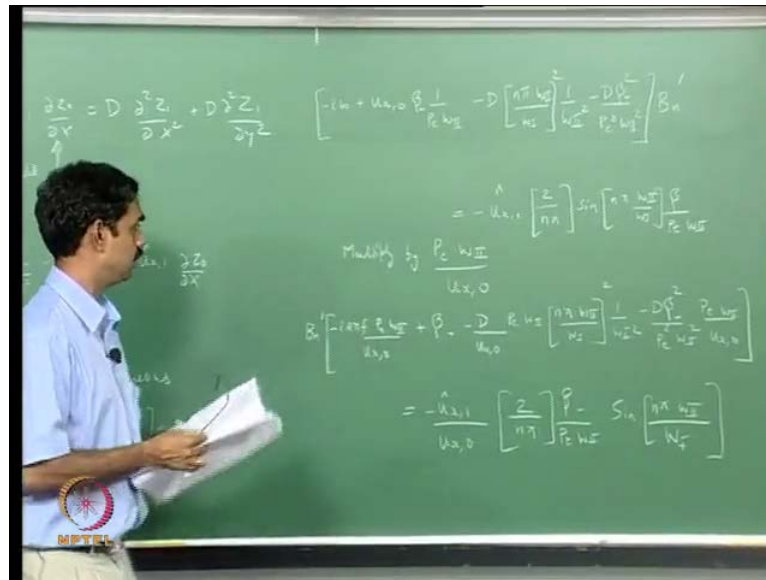
This part is kind of known. So, you can bring these terms to this side, and bring it there, so that you can think of that as like a forcing or something rather. If you use you had this e power minus \sin ω t , when you differentiate e power minus \sin ω t , you get i ω times z hat. So, this term would be minus i ω z hat, z hat would be like the

four year transform plus $u \times \text{comma } 0 \text{ dau is dau hat by dau x}$. I will bring this term here for convenience equal to minus $u \times \text{comma hat}$.

This is what we need to solve. So we can assume the solution to be of certain form, and then we can try to derive what the solution is. The solution will have two parts, there will be a particular integral and homogeneous integral. We will have to solve for that. We will first solve for the particular solution and then we will solve for the homogeneous solution, which is a standard technique in differential equation. In several methods we can obtain.

So, you can make a clever guess. It makes sense because this is a form of z naught that we got. So, it is like a guess and then we will substitute this in to this equation, and then if you can find the coefficients B_n prime. You put prime to make clear that is unsteady part, earlier we use the same B_n for steady part, so we solve for the B_n prime and then we will have to find the homogeneous solution. So, that is the method. I hope it is clear. If you do this, and substitute this thing here, you will get the following.

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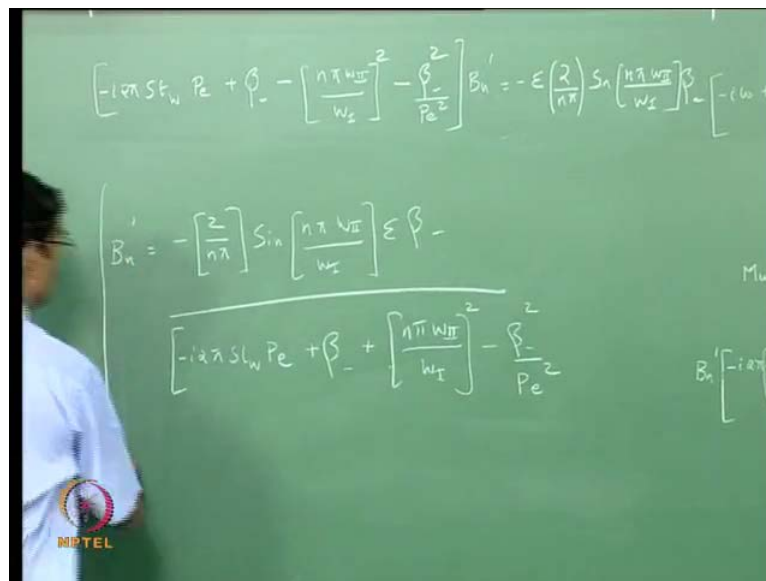


When you differentiate this term with x , you will get a β out. If you are differentiating with y you do not get. So, when you differentiate with x , that is wherever this x terms are there, this and this, you will have these β coming. You differentiate twice you will get β squared, and when we differentiate with y this is actually y star, so you have to put

this term 1 over w^2 squared. When we differentiate with x^* , x^* is x divided by this, so that extra term comes. I hope that is clear. It is a straight forward algebra.

Now, we just need to multiply this expression by Peclet w^2 by $u x$ naught that will simplify things, that is this term we just take out. So, you will get ω is 2π here, Pe w^2 $u x$ naught plus β minus d over $u x$ naught Pe w^2 . All I have done is to multiply this expression by this, and this will actually dramatically simplify things. If you club this term, this is Strouhal number.

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Next term is just β , minus you just take one of these w^2 , so we can clearly write this as w^2 w^2 , and this will cross with this, and what is this? 1 by Peclet number. 1 by Peclet number cancels with Peclet number, so that is 1 . We have the β minus squared and let us again club terms. If we take d over, we cancel one of these w^2 , and then we have this. What is this? 1 over Peclet number. So, Peclet number cancel with Peclet number, there is a 1 over Pe squared left. $u x$, 1 over $u x$ naught, we call that ϵ , that is the ratio of the amplitude of the fluctuating flow to the base flow or the mean flow.

So, we can just cross multiply and get B_n prime equal to minus 2 over $n\pi$ $n\pi$ w^2 over w^2 into ϵ β minus whole divided by minus i 2π $s t$ w Pe plus β minus plus $n\pi$ w^2 w^2 i square minus β minus square divided by Pe squared. So, this is the expression for B_n . I hope it is clear. Actually I multiplied throughout by Pe w^2 $u x$

naught, so that removed that term. Now, we solve for the homogeneous solution. We remove the terms from the right hand side. We just solved for the z on hat terms.

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$$-i2\pi f z_1 + u_{x,0} \frac{\partial z_1}{\partial x^*} \frac{D}{u_{x,0} w_{II}^2} = D \frac{\partial^2 z_1}{\partial y^{*2}} \frac{1}{w_{II}^2} + D \frac{\partial^2 z_1}{\partial x^{*2}} \frac{D^2}{(u_{x,0} w_{II}^2)^2}$$

$$y^* = y/w_{II} \quad x^* = \frac{x D}{u_{x,0} w_{II}^2}$$

Multiply by $\frac{w_{II}^2}{D}$

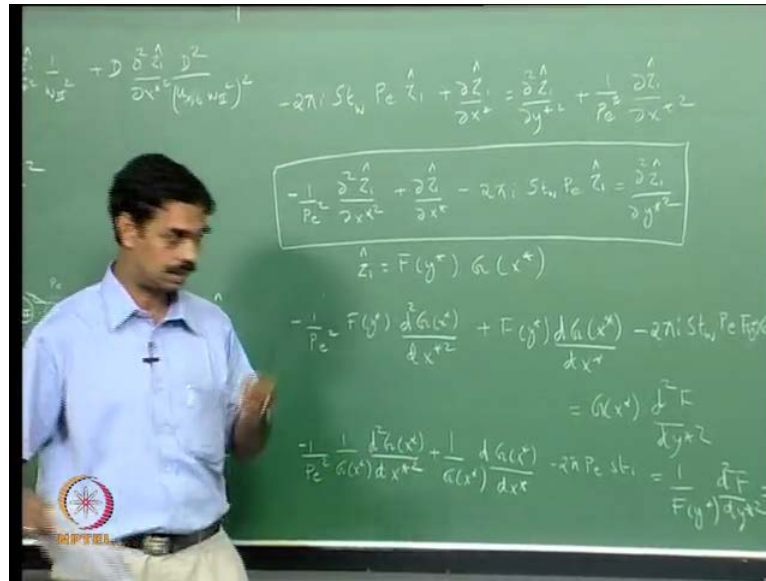
$$-i2\pi f z_1 \frac{w_{II}^2}{D} + \frac{u_{x,0}}{D} \frac{\partial z_1}{\partial x^{*2}} \frac{w_{II}^2}{u_{x,0} w_{II}^2} = \frac{\partial^2 z_1}{\partial y^{*2}} + \frac{\partial^2 z_1}{\partial x^{*2}} \frac{D^2}{(u_{x,0} w_{II}^2)^2}$$

$$-i2\pi f \left(\frac{w_{II}^2}{u_{x,0}} \right) \frac{1}{D} z_1 + \frac{\partial z_1}{\partial x^*} = \frac{\partial^2 z_1}{\partial y^{*2}} + \frac{1}{Pe^2} \frac{\partial^2 z_1}{\partial x^{*2}}$$

We have to remember that y star is y over w 2 that is why when we differentiate twice we get this term, and x star is x d over u x naught w 2 squared. So, what I will do is to multiply by w 2 squared over d just to simplify this thing. So, I will get minus i 2 pi f z on hat w 2 squared over d plus u x comma 0 over d. This term cleans up because I have taken this coefficient, cross multiplied by that.

Now, looks like a mess but, lot of things will cancel. So what is this? This is Peclet number and d by w 2 squared, this is also Peclet number, this 1 over Peclet number, so they will go. Multiplying and dividing by u naught, this is Strouhal number and this is Peclet number that has come out neatly. Plus the mess cleared up because we have 1 over Peclet number multiplying Peclet number is equal to dau square z 1 hat dau y plus, now the last term this goes out, if I take 2 out, now I have 1 over Peclet squared.

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I can say it as minus $i 2 \pi i s t w P_e z$ on hat plus $d^2 z$ on hat over $d^2 x$ square equal to this. We have got a neat equation. We need to solve this. How would you solve this? If you club all the x star terms one side you will get this equation. What we can do is just solve it by separation of variables. So we say z_1 hat equal to f of y star g of x star substituting here. I think now you must be experts in this. Come on help me. f of y star into $d^2 g$ of x star divided by $d^2 x$ star squared plus f of y star $d^2 g$ of x star divided by $d^2 x$ star minus $2 \pi i s t w P_e f g$ equal to g of x star, here also I should have arguments, f of π star $d^2 f$ by $d^2 y$ star squared.

If you divide throughout by $f g$, then you can simplify it. So, you will get minus 1 over p e squared 1 over g of x star $d^2 g$ of x star over $d^2 x$ star squared plus 1 over g of x star $t g$ of x star over $d^2 x$ star minus $2 \pi i p s t i$ equal to 1 over f of π star $d^2 f$ over $d^2 y$ star squared. Now we have a function of x equal to function of y , so they must be equal to a constant.

Minus $k n$ squared, minus is just for convenience, you can do this without the minus also; there should not be any problem. So, we have two differential equations, one is of this form, $d^2 f$ by $d^2 y$ squared plus $k n$ squared f equal to 0 and that would give sin and cos and then the other one would give e power some βx or something other like that. So let us work it out.

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$$F(y^*) = C_3 \sin(k_n y^*) + C_4 \cos(k_n y^*)$$

$$G(x^*) = C_1 \exp(\beta_{hom} x^*) + C_2 \exp(-\beta_{hom} x^*)$$

$$-\frac{1}{Pe^2} \beta_{hom}^2 + \beta_{hom} - 2\pi Pe St_w i + k_n^2 = 0$$

$$\beta_{hom}^2 - Pe^2 \beta_{hom} + (2\pi Pe^3 St_w i - k_n^2 Pe^2) = 0$$

$$\beta_{hom} = \frac{Pe^2 \pm \sqrt{Pe^4 + 4k_n^2 Pe^2 - 8\pi Pe^3 St_w}}{2}$$

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This is the solution for f, and g of x star would of the form c 1 exp beta homogeneous x star plus, beta will have two roots, one will be positive one will be negative. So, if you substitute this equation here, you can write it nicely as a quadratic equation. This is a quadratic equation. You can find its roots straight forward. We will now apply the boundary conditions. We have a symmetric condition about y equal to 0, so we will apply that first.

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Symmetry about $y=0 \Rightarrow C_3 = 0$

$$F = C_4 \cos(k_n y^*)$$

$\frac{\partial T}{\partial y} = 0$ at $(x, y) = (0, \pm \pi)$ } wall B.C

$$\sin(k_n \frac{W_I}{W_I}) = 0$$

$$k_n \frac{W_I}{W_I} = n\pi \Rightarrow k_n = n\pi \frac{W_I}{W_I}$$

Solution should be finite at large x^*

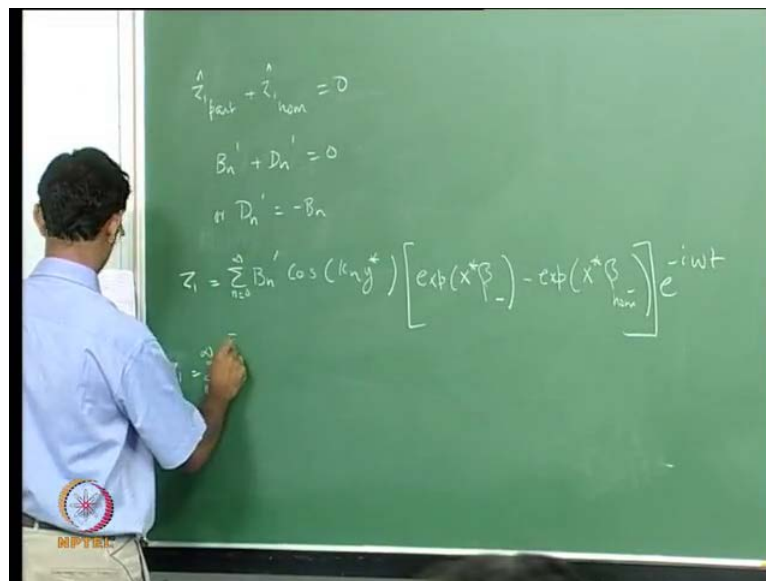
$$\Rightarrow C_1 = 0$$

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This will give c_3 equal to 0, \cos is a symmetric function, \sin is not. You can calculate a derivative, put it 0, you will get the same thing. Your f would be $c_4 \cos k_n y$. We know that $\frac{dz}{dy} = 0$, at the wall x, y . If you differentiate this you will get \sin . So, $\sin k_n \frac{w_1}{w_2} = 0$. When is $\sin 0$? For $n\pi$, $k_n \frac{w_1}{w_2} = n\pi$, which gives k_n equal to $n\pi \frac{w_2}{w_1}$.

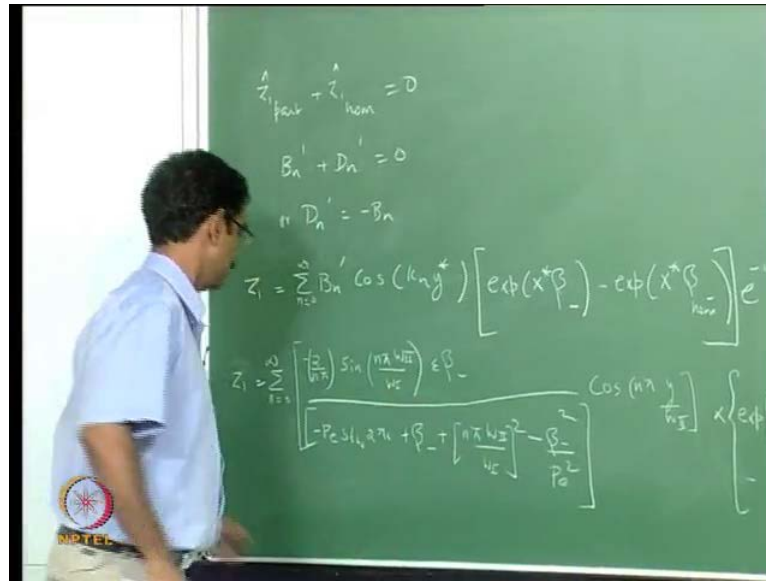
This is from the wall boundary condition. We have to make sure that the solution remains bounded when x tends to infinity. So we have to remove this function, c_1 is 0. We have done the problem. Now all we need is to find the coefficients that we can get from the boundary condition at the inlet play. Since the mixture fractions are fixed, we said that at the fuels slot z is 1, at the air slot or the oxidation slot z is 0. So, there is no fluctuation.

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z_1 particular hat plus z_1 homogenous hat is equal to 0. So, this could give B_n prime plus D_n prime equal 0 or D_n prime equal to minus B_n prime. I mean D_n would be this coefficient of c_2 times c_4 for each of the Eigen values, so that is why subscript n , because we are summing our Eigen values. So this is the final solution.

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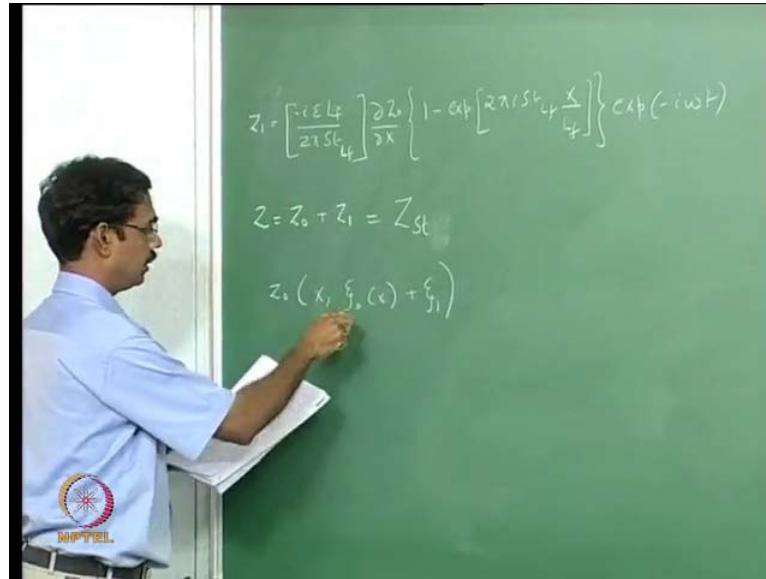
We know the value of B_n . We have calculated it just a while by. We are summing our all Eigen values. So this much is the B_n times $\cos n$ by y over w^2 times e power minus ω t . This is a long formula, but this is B_n primes times our function f which is $\cos n$ by y over w^2 times. We have the two solutions, the particular solution and the homogeneous solution, and this is the periodic part. So, this is the solution. Now, last thing I just want to see what happens at limit of Peclet number being large. That is quite simple.

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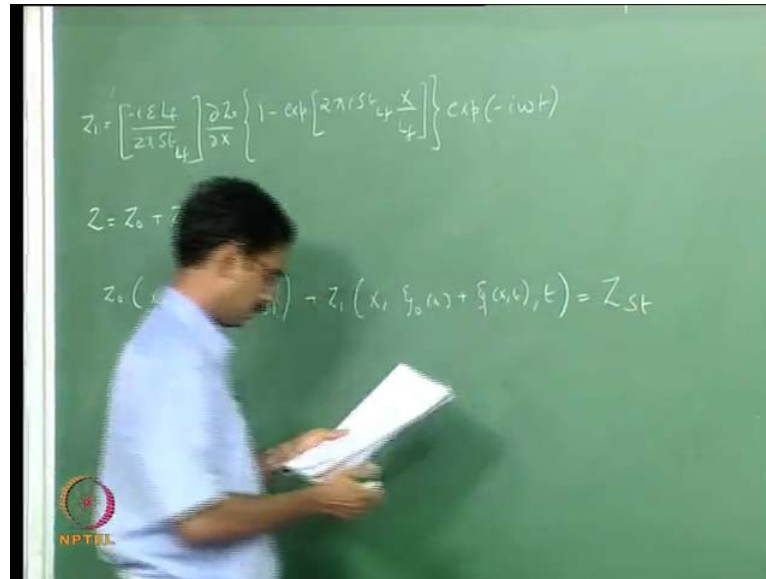
Minus 1 plus 2 k n squared by Pe powered 4. Peclet number is large, this term will drop, and this first term will dominate. So that is the solution we have written in the two classes back. The derivation is over. Now, we just have to see what it means. We have written the solution with the Peclet number being large, that is when we have this as the arguments. So, we can simplify that and write it very nicely.

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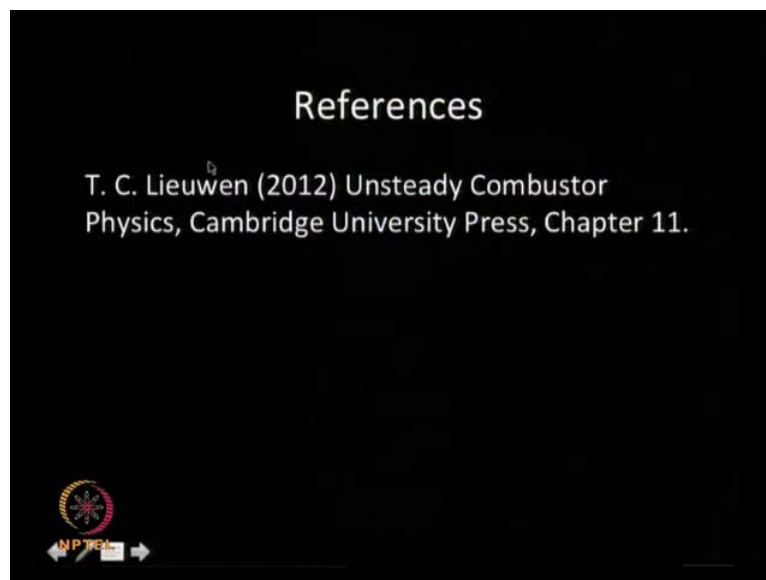
You can do this Strouhal number i in terms of $L f$ by multiplying top and bottom by $L f$ and then you can write Strouhal number in terms of flame length. Many people preferred to do that, because flame length is something which we can measure very easily and when a flame is oscillating, we can see the flame length is oscillating and so and so. If you write, most of these terms can club into $d z$ naught by $d x$ because this is coming from that particular integral which is coming from this term being on the right side forcing the equation. How do you look for the flame sheet or flame surface? We look for the stoichiometric surface and that is where, we have the flame. So, Z is Z naught plus Z_1 . That is the instantaneous value. Z is Z hat times e power minus $\sin \omega t$, but you take the value at that particular instant.

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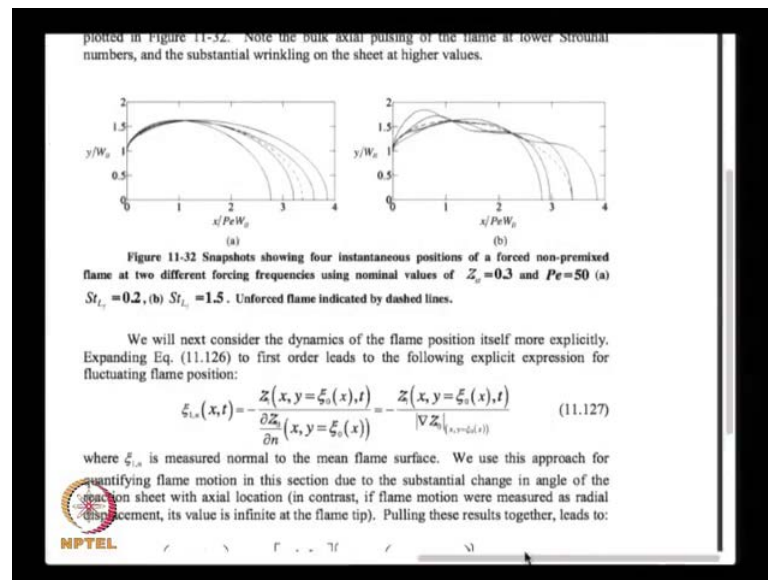
So, this can be recast as this. Psi naught is the flame position of the steady flame, and when you perturb it, it is psi. We will have to implicitly solve for this equation. I will show you how the solution looks like.

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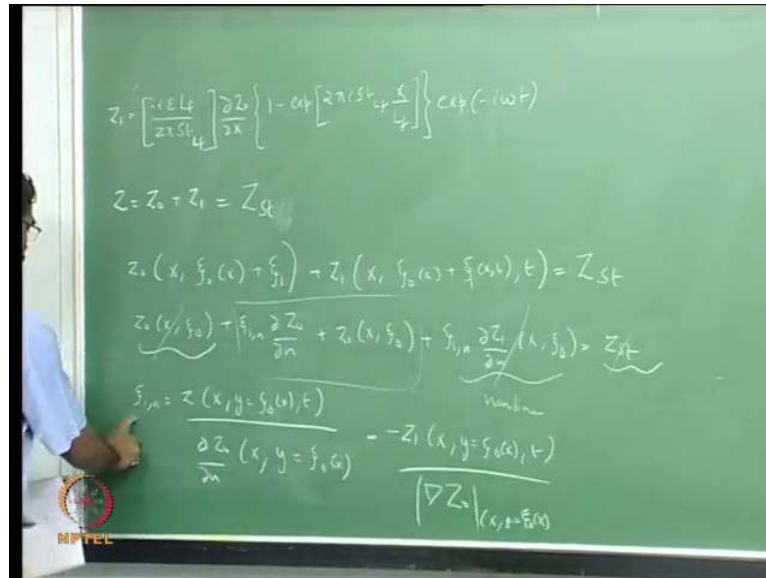
We will see the results from Lieuwen. I am showing this with his permission.

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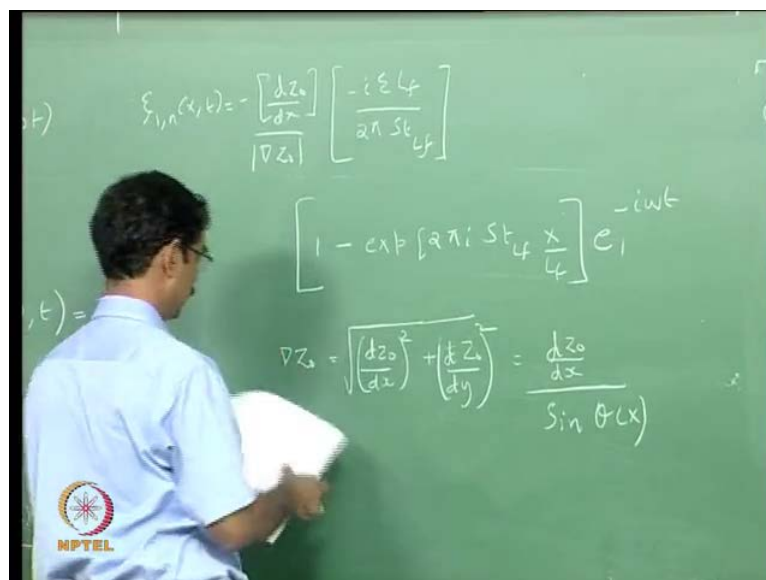
Here we have snap shots of 4 instantaneous positions of a post non-premixed flame. Results produced from Lieuwen with permission. It is forcing at two different frequencies, and the one on the left corresponds to St_L , Strouhal number corresponding to flame length 0.2, the figure on the right is for high frequency, Strouhal number 1.5. In the low frequency limit the flame just goes back and forth, whereas in the higher frequency you actually see the flame starting to have wrinkles. At the low frequency oscillation the diffusion flame just goes up and down gently, but for the other one you see is a fancier wrinkled pattern. This is what you can calculate by calculating the flame position. I hope this is clear.

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We Taylor expand this, psi we are measuring the flame position as a normal from steady flame that is actually quite convenient because otherwise the flame angle will change dramatically towards the flame tip. This is same as this, so you can cancel because this comes from steady equation also. This term is non-linear, so we can drop that. So, this equal to 0 is the equation. So, you will get this. Combining this, we can write the formula for psi 1, n as follows.

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We can write this term in terms of flame angle. We will talk about flame angle as supposed to be the premixed flame, where flame angle was constant; we had like a tent flame or conical flame. Here, as you saw, the flame angle was a function of x.

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Non-premixed

$$g_{i,n}(x,t) = \frac{i \Sigma u_{x,0} \sin(\theta(x))}{2\pi f} \left\{ 1 - \exp\left[-\frac{(2\pi f)^2 x^2}{u_{x,0}^2}\right] \right\} \exp[-i2\pi f t]$$

Premixed

$$g_{i,n}(x,t) = \frac{i \Sigma u_{x,0} \sin\theta}{2\pi f} \left\{ 1 - \exp\left[-\frac{(2\pi f)^2 x^2}{u_{x,0}^2 \cos^2\theta}\right] \right\} \exp[-i2\pi f t]$$

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Using this, we can substitute here, and we will get the final expression which you want. So, this corresponds to the flame oscillation. Now, let us write the corresponding formula for premixed flame and see whether there is anything in common. So this is for non-premixed. Let us write for premixed. You see, the formula is very similar. The flame angle is constant as long as you have a uniform velocity profile. If theta has to vary, either the flow speed has to vary with space, or flame speed has to vary with space. If those things are not happening as we had assumed in our earlier derivation, then theta should be a constant, that is why I emphasize theta x here, whereas this is just theta.

You will find another difference, here it is just the convection, there is a cos squared theta sitting here, times exp minus i 2 pi f t. You can see the similarities, this term is same, and this term is same, except for the denominator difference. This is also same. The diffusion flame or the non-premixed flame does not propagate, so you have the speed as the convection speed itself.

For the non-premixed flame, wrinkles will be propagating only with this convection speed, whereas for the premixed flame, the flame is trying to propagate and then flow is trying to come in. So, you would not get exactly u x naught but, something that depends on u x naught, and the flame angle because flame angle will be reflecting the flame speed. That is why you have this difference.

So, in both cases you have local minima and maxima created through this term, here and here, but the difference is because of the nature of the propagation of wrinkles. In premixed flame, the flame itself is propagating, whereas diffusion flame just stands. Now, due to the interference between the wrinkles generated at x equal to 0 boundary, and the disturbance locally, you will have this pattern and that is being reflected in this term, spatial pattern.

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non-premixed

$$\xi_{1,n}(x,t) = \frac{i \xi u_{x,0}}{2\pi f} \sin(\theta(x)) \left\{ 1 - \exp\left[\frac{i2\pi f x}{u_{x,0}}\right] \right\} \exp[-i2\pi f t]$$

Premixed

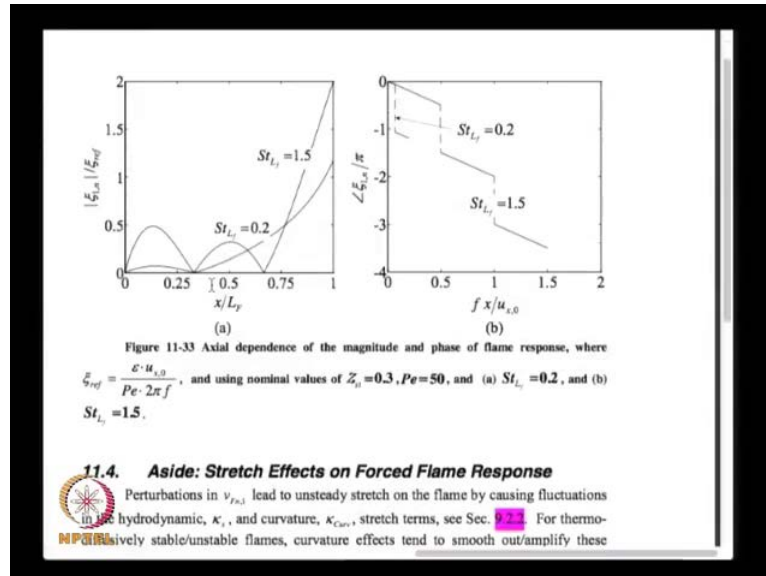
$$\xi_{1,n}(x,t) = \frac{i \xi u_{x,0}}{2\pi f} \sin \theta \left\{ 1 - \exp\left[\frac{i2\pi f x}{u_{x,0} \cos \theta}\right] \right\} \exp[-i2\pi f t]$$

$$1 - \exp\left[\frac{i2\pi f x}{u_{x,0}}\right] = 2 \sin\left(\frac{\pi f x}{u_{x,0}}\right) e^{i\left(\frac{\pi f x}{u_{x,0}} - \pi/2\right)}$$

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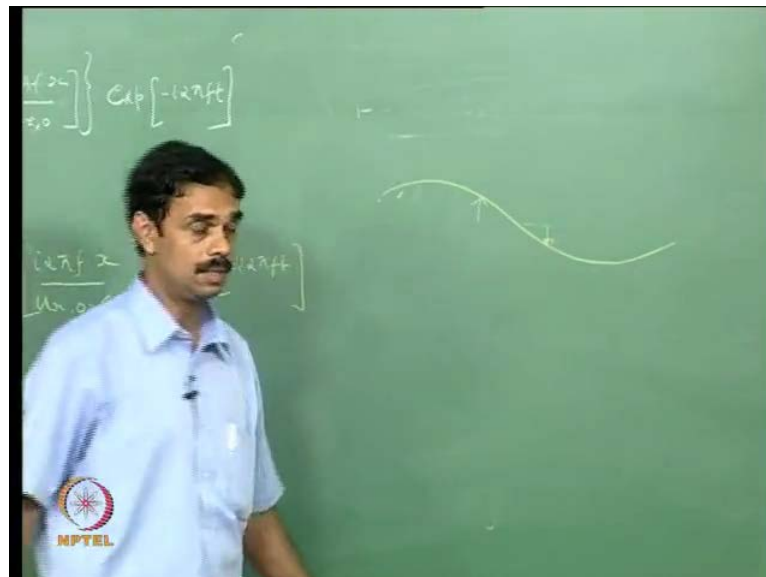
You have this term $1 - \exp\left[\frac{i2\pi f x}{u_{x,0}}\right]$ would be nothing but $2 \sin\left(\frac{\pi f x}{u_{x,0}}\right) e^{i\left(\frac{\pi f x}{u_{x,0}} - \pi/2\right)}$. So this is the sinusoidal pattern that is propagating and this you see is the speed. So, we have studied two different physical phenomena, one is with premixed flame and one is with diffusion flame and we are able to see the commonalities that is both can have wrinkles and both can move, but here the wrinkles move with the convection speed whereas in premixed flame there is of course, a balance between the flame propagation and the flow.

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This is again reproduced with permission from T Lieuwen from his book combustor dynamics. If you look at the axial dependence of the magnitude and phase of what we calculated here, on the left half is the amplitude. These are the maxima and the minima, and from this term you can see that, across every minimum, if one part of the flame is going up, the other part must be going down.

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If you have something like this, here it is going up; here the flame is coming down. That is why there is a shift of 180 degree at each of the nodes. So I will rub up this lecture here since we have studied diffusion flame and found how it behaves in an unsteady way. Of course, you can calculate heat release rate and so on from this and get a global

transfer function and so on. But what is really interesting is that it has a lot of similarity with the premixed flame, but there are some differences also.

First of all, in the analysis, the main difference was that, g really has meaning only at the flame, whereas z we have solved for the whole field. From there, we actually look for the stoichiometric surface, then calculated how the flame is responding, then got the flame response. The flame response also behaves in a very similar way with the premixed flame. That is, you have a spatial pattern which is moving. The difference is that, in here it is only the convection, because a flame itself is not propagating, whereas premixed flame is propagating. In these two expressions you can see the commonalities and the differences.

This is a good exercise to help you to solve things mathematically. Of course, this is simple problem; even you can try more complicated things. Everything may not be analytically tractable, but then you can actually solve things numerically, and this will be like a simple bench mark problem. In the first half of the class, we studied acoustics way of propagation and so on. In second half we studied thermo acoustic instabilities and we looked at a variety of different situations. We analyzed problems in the harmonic domain also in the time domain.

We looked at the model analyses non model stability analyses and so on. The key to all the analyses was, one is the stability part where you either solve the equation in time domain or you saw for growth rates and frequencies. Then we also looked at singular value decomposition, explained how that characterized energy growth and so on. But the centre of this calculation is the input to the model which is the flame response.

How the flame behaves in response to the fluctuations, and that we have made models for a several different systems. For example, just now we did diffusion flame, earlier we did a premixed flame, and we have studied a lean premixed flame. Like that, for each of the situation you can construct a model for the unsteady heat release rate. Then you plug it in to the corresponding wavy equation, and then you solve in whichever way you are solving. So, I hope you will be able to extent all these analyses to your thermo acoustic system and be able to analyze what happens to it, both the linear stability and the non-linear stability analyses.

Thank you.