Acoustic Instabilities in Aerospace Propulsion
Prof. R. I. Sujith
Department of Aerospace Engineering
Indian Institution of Technology, Madras

Lecture - 04
Harmonic Waves

In the last class, we derived the wave equation and please remember that we derived the wave equation from the fundamental equation of the fluid mechanics. And we just took the equations of fluid mechanism. We linearized this equations and then manipulated equation to get a nice linear equation which is its second order partial different equation and we actual actually solutions for this equation. Now, this solutions where if you remember of the form f of x plus c t and g f x minus c t.

So, this x minus c t when that is the argument corresponds to the right running wave and x plus c t corresponding corresponds to a left running wave. And then we also derived expression for acoustic velocity in terms of acoustic pressure we saw that they where related by the characteristic impedance rho c. For the right running wave the acoustic velocity was acoustic pressure divided by rho c. And for the left running wave those are minus sign it was negative of acoustic pressure divided by rho c that is the biggest difference in the direction of motion of the gas when difference in direction of the particle velocity that the sign is there.

Now, in this class we are going to try to look at harmonic waves. Now, this is very simple concept and we can understand a lot of things. Although we are often you could have many different frequencies but, now at least the situation which you have just 1 frequency is not all that diagram it can happen quit easily and you can set it up nicely in a laboratory.

So, physically you have harmonic waves you need a sound source that gives sound at a single frequency. And that is not too hard to imagine you can have a loud speaker or a piston which is vibrating at a single one particular frequency and so, this is a very simple and ideal case of wave propagation. And it is quote fundamental because even if you have transcend signals or signals with many different frequencies in it or waves that are not sinusoidal order. You could actually use for the transform and then now actually you can express this complex signal as a super position of various different harmonic waves and with so, if you can analyze one particular frequency you can analyze all particular frequency.

And since we are having a linear theory that is our differential equation linear and if our boundary condition also linear then you can actually solution which can be super post and you can construct solution for more complicated situations. So, let us get on with harmonic waves. So, for studying harmonic waves you would have guessed correctly that we would use sines or cosin and you can use either of them it is 1 wave is phase shifted to get another. So, there is no problem with that.

(Refer Slide Time: 03:14)

So, let us write. So, this is a typical expression for plane propagation of harmonic waves plane because we are talking about 1 dimensional wave. So, X is the only special dimension and you can also see that this is off the form f of t minus X over C. This is like you know we saw the solutions f of t minus X over C and this was the right running wave. And if you did have the other function g of t plus X over C. Then you could write the corresponding wave as a Cos 2 pi f, f is the frequency here times t plus X over C minus phi naught.

This is a oscillating at a single frequency and the frequency here is f. So, this is the f which correspond to solution which we had last time but, what have written here f is the, f is the frequency. Now, phi naught is the initial phase. This is because depending on your choice of coordinate system it depending on when you stop the clock you can change phase depends on that it is with reference to a reference time. So, depending on the starting of time t equal to 0 you need a phase because Sin and Cos depends on you know the phase depends on when you start your time.

So, there are 2 possibilities to visualize a harmonic wave. You can either, you know we have a function of time and X. So, we can one particular way to do this is, we can freeze the time and look at it has a function of X the other possibilities you can freeze X and look it has function of time. So, you freeze the space and look at it look at the wave as the function of time. And we can make out certain characteristic from it and we can do the opposite.

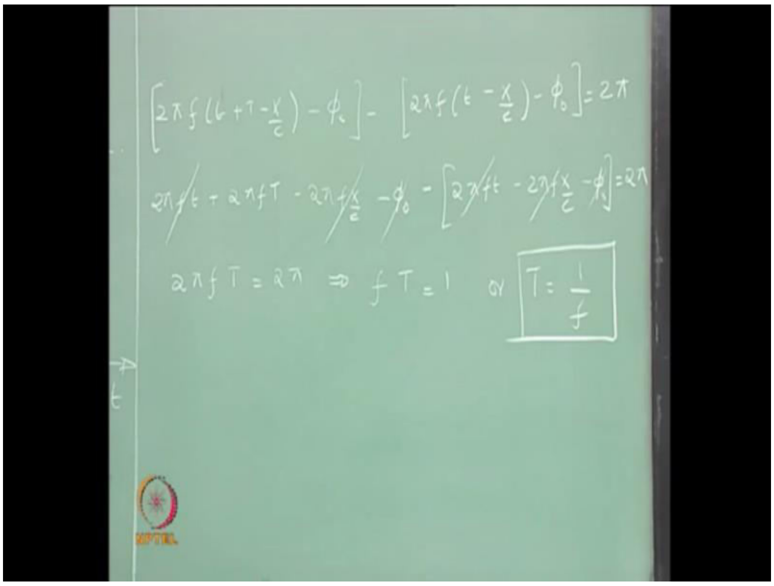
Let us be a first look at a freeze the space that is at 1 particular space I am sitting and looking at the wave. So, What I could do? Is to put a microphone at 1 particular location and see how the wave looks like and we can take the signal from the microphone to oscilloscope and visualize it. Let me draw how a typical signal would look like? So, this would be the time period T.

The period is defined as the time with the pattern repeats. So, the either the time its interval between 2 crust or 2 trough or any 2 points which are similar. And you have the amplitude here which is A so that, is the amplitude of the weave and here we are. Now, let me mark the axis its p of its some reference value p of t 0 comma t as a function of t. And you can easily see this particular value if I read of this would be A Cos phi naught so that, is because the reference phase.

So, this picture clearly defines what is A? What is phi naught and what is time period T?

And the idea is that the wave repeats after every time interval T. So, with that we can look at what happens to we can actually derive the relationship between time period T and frequency. So, if the wave repeats after every time interval T your argument will actually change by 2 phi you know Sin theta is same as Sin theta plus 2 pi.

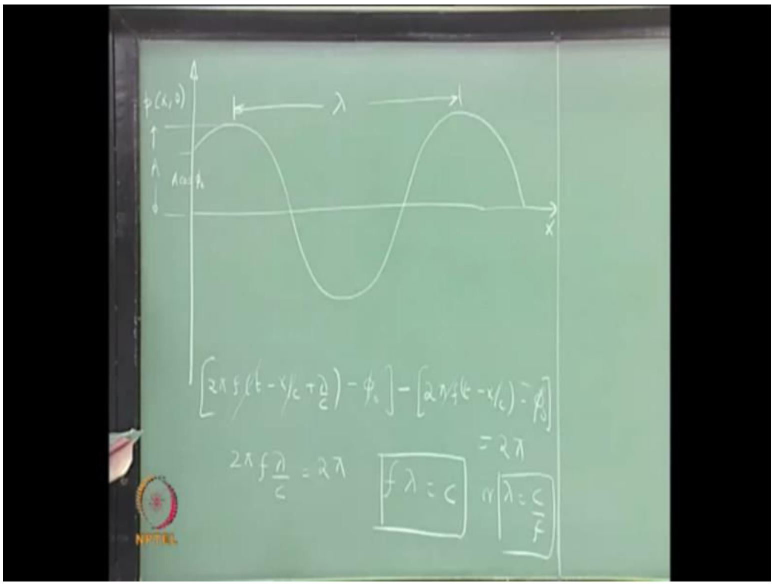
(Refer Slide Time: 07:36)



So, if you use that and say that. So, this is after a time interval T minus. What was there at original time? So, if you say this periodic we will get a phase difference of 2 pi. So, if you expand this you will see 2 pi f t plus 2 pi f t minus 2 pi f X over C minus phi naught minus 2 pi f t minus 2 pi f X over C minus phi naught equal to 2 pi not. And you can see this cancels, this cancels, this cancels. So, what you would get is? 2 pi f T equal to 2 pi which we can simplify and say f T equal to 1 or time period T equal to 1 over f. I think all of you knew this in advance but, this is like a reverse saying that the time period of a wave a goes like the reciprocal of the frequency. Now, we will do the other approach. What we can do is?

In this particular example, we froze the distance. So, we said we will stay at X equal to 0 and we looked at the variation in time. So, instead what we can do? Is we freeze at 1 particular time and then we see the variation next. How do you? How is it possible to freeze at 1 particular time? So, what we can do? Is we can put microphones at various locations along the duct. Whatever and you make measurements at 1 instant T one particular instant of time you make measurements with several different microphones along your duct where sound is propagating and plot them so that, would be the way how to get the distribution along space to what a you would get be something of this form.

(Refer Slide Time: 09:55)

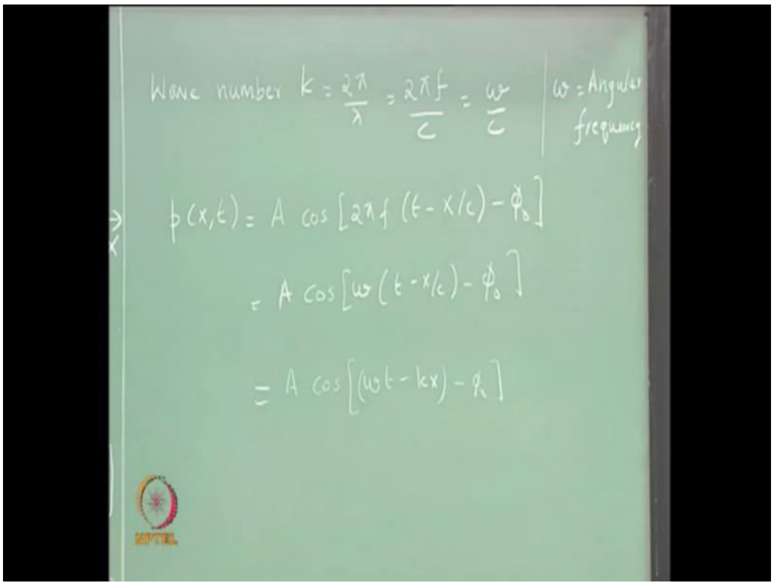


Now, what I am having is p of X coma 0 that is the time t equal to 0 as a function of X. and lets mark the important features. Now, the distance between 2 peaks would now be the wave length lambda. That is the length of the wave that is the distance between 2 different waves 2 difference distance between 2 different peaks you can also define it as 2 different crust and so on. And amplitude here is the same A and you also have the rough reference phase A Cos phi naught actually A Cos minus phi naught but, A Cos of minus phi naught is same as Cos of phi naught. Now, you can total get a relationship between the wave length and the frequency.

What we can do? Is we do the same thing as we did here see when the pattern repeats. So, the periodicity means you get a difference in phase of 2 pi between 2 different crust or 2 different trough and so on. So, 2 pi f times t minus X over C plus lambda, lambda is the wave length minus phi naught minus 2 pi f t known as X over C minus phi naught equal to 2 pi. So, you can see this term cancels with this, this cancels here, this cancels here and what you would get? Is 2 pi f lambda or should have lambda over C.

X is X plus lambda over C equal to 2 pi or you would get f lambda equal to C or lambda equal to C over f so that, is if you derive the speed of sound divided by number of waves per second you would get the wave length. I think this is also a intuitive result and we have derived this somewhat rigorously here. In other words within this time interval T the peak has travelled the distance lambda at the speed c, c is the speed of sound.

(Refer Slide Time: 12:54)

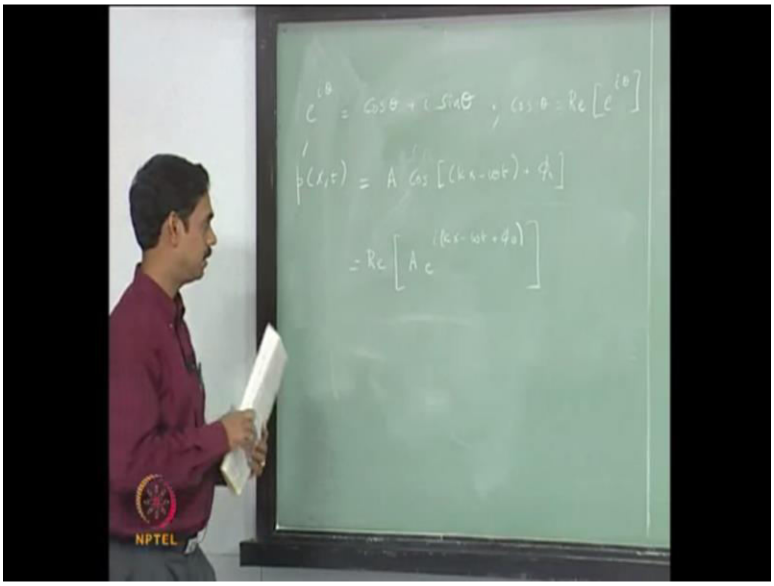


Now, I just want to do one more definition which is a wave number. K which is defined as 2 pi over lambda and which is equal to 2 pi f over C this also define omega over C. So, k is the wave number which is defined as 2 pi over lambda which is omega over C omega is refer to as the angular frequency. So, we can now write the harmonic wave expression for a harmonic wave as follows, this is what we have to begin with. We can equivalently write this in terms of omega and also in terms of wave number we can write because omega over C is K.

This is written for the right running wave we can or similarly, get expression for the left running wave as well. How many waves are there per meter or or centimeter? Whatever so this is the number of waves per unit distance it is very physical concept. In fact this concept came from electromagnetic theory and sound where they were actually in spectrum counting how many waves over there in per unit distance but, same thing cannot close here.

Now, the next think you want to do is you want to use the complex notation because it turns out that your algebra is very convenient in complex notation. Its more convenient than using trigonometric functions and so, let us try to do the derivation try to work out acoustic in the complex notation. Its equal and you can either view complex numbers or you can use Cos and Sin and I will talk about the equivalence.

(Refer Slide Time: 15:35)

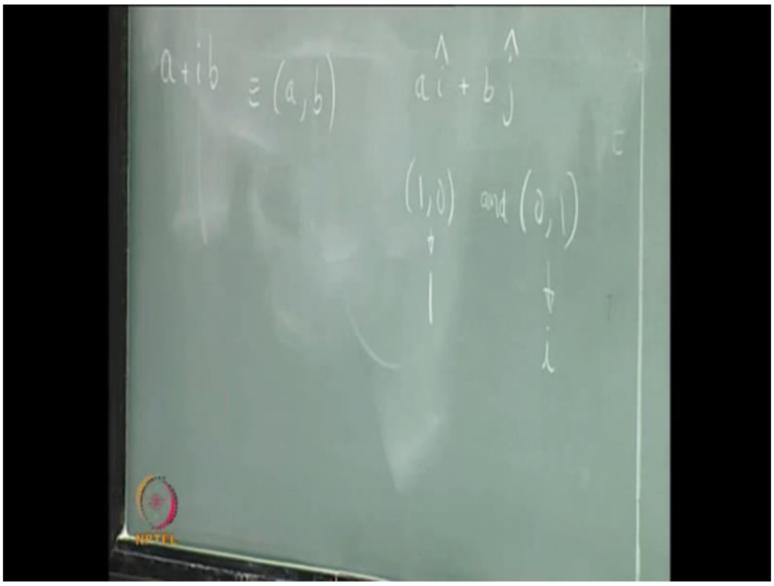


So, we know the formula e power i theta equal to Cos theta plus i Sin theta. So, this is the basic formula how to convert bit how to go between complex numbers and Cosine and Sin’s. And lets all we would need of the course nothing more than that. So, what we know now is that when you say you have again i say mentioned earlier sometimes we drop the primes but, there is really primes they are fluctuating pressure but, if I drop it that is all that is because that is the standard convince to drop this but, I as far I remember I will try to keep the primes so, if you see this formula e power i theta is Cos theta plus i Sin theta.

So, Cos theta is really the real part of e power of i theta. If you want a function with Cos theta e and you insist and writing exponential e power i theta. Then what you can do? Is you can write Cos theta as real part of e power i theta. So, if you want let me write that r e refers to real part of ((Refer Time: 16:56)).

So, if I have a expression for a harmonic wave as follows I can now write it as real part of A e power i times k x minus omega t plus phi naught. So, when you do this e power i times this factor you will have Cos and Sin and we want only the Cos. So. we take the real part. Now, p prime is the real pressure there is ((Refer Time: 17:35)). So, complex notation is a only convenient way to do the algebra there is nothing complex about the whole idea it is just of we are using complex numbers. Now, we knew use the term complex number we say that the Cos theta is there plus i times Sin theta we say i is imaginary number.

(Refer Slide Time: 17:56)



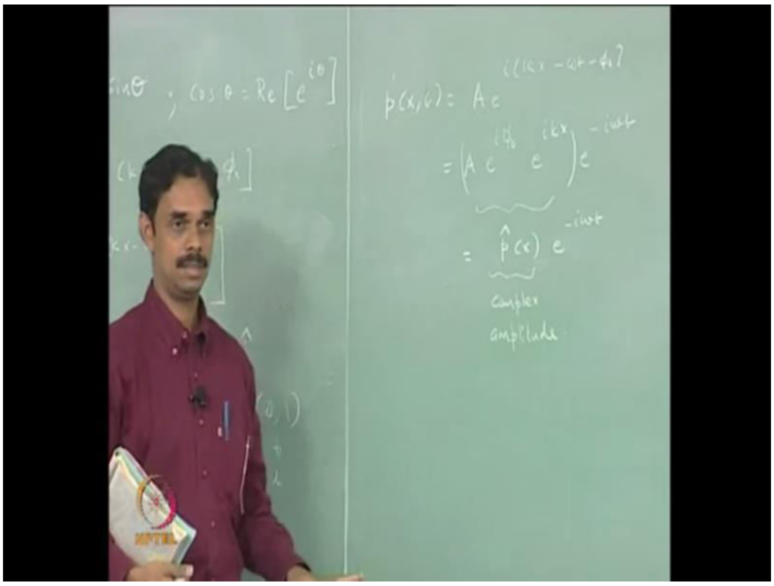
For example, we say that if you have a number a plus i b we call a the real part and we say b is the imaginary part i b is the imaginary part but, i want the emphases that there is nothing imaginary about the imaginary numbers it is just a bad nomenclature. So, you can think of a plus i b that we can alternatively think of it as 2 quantity is describing something. it is like if you are the vector a times i plus b times j.

So, suppose we have a vector a times i plus b times j this i and j are the basis function they are the unit vector along i is the unit vector along the x axis, j is the unit vector along y axis. Now, we write the vector along this basis function using this basis function. So, a and b are the components which are the projections along this basis functions. So, instead of using i and j the basis functions that we use are 1 coma 0 and 0 coma 1, 1 coma 0 in conventional convent ally we refer as 1 and 0 comma i we refer to as i.

So, it is just complex numbers is a quality which helps you to did in some point which need too thing to be specified. So, complex numbers enable to do that. So, although we have a real part and imaginary part. The imaginary part is not only imaginary thing it is just a convenient way of notation and you see that here in complex. When use this we actually use imaginary numbers to denote phase as I will show you. Now, in a few minutes time.

So suppose we where to expand this out.

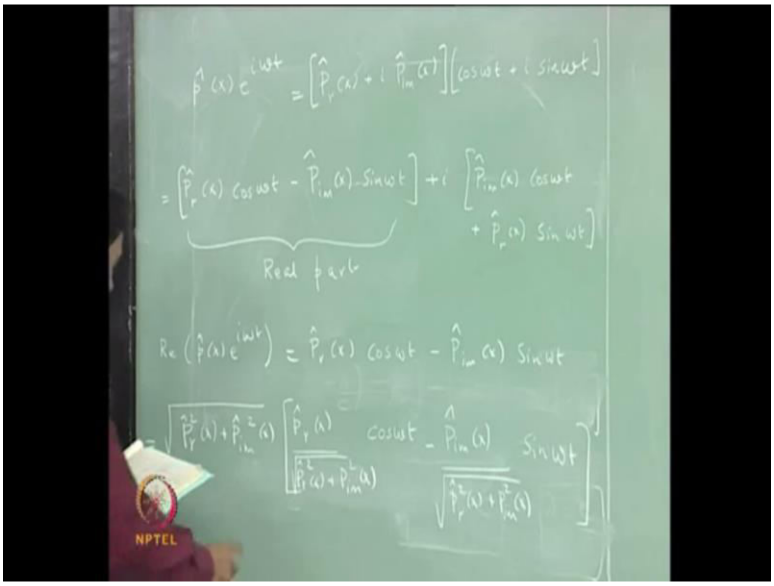
(Refer Slide Time: 19:41)



So, we can say that p of prime of x comma t equal to A e power i times K x minus omega t minus phi naught. This can be recast as A e power i phi naught times e power i k x times e power minus i omega t. Now, what we can do? Is we can club these thing together this is the function of this. So, we can call it p hat of x. This is a function of space times e power minus i omega t. This amplitude is called the complex amplitude p hat of x is the complex amplitude. I must emphasis 1 thing here we should not take the complex amplitude and take its real part that is meaningless.

We take the complex amplitude multiplied by e power i omega t and then take the real part of the whole thing that will give the real instantaneous pressure is that clear. Let me repeat again we take the complex amplitude multiplied by e power i omega t, e power i omega t or here its e power minus omega t taking is both this will be Cos omega t i Sin omega t. So, you do whole multiplication and then take the real part and that is what gives instantaneous pressure. Next I will show the equivalence between these 2 notations.

(Refer Slide Time: 21:42)

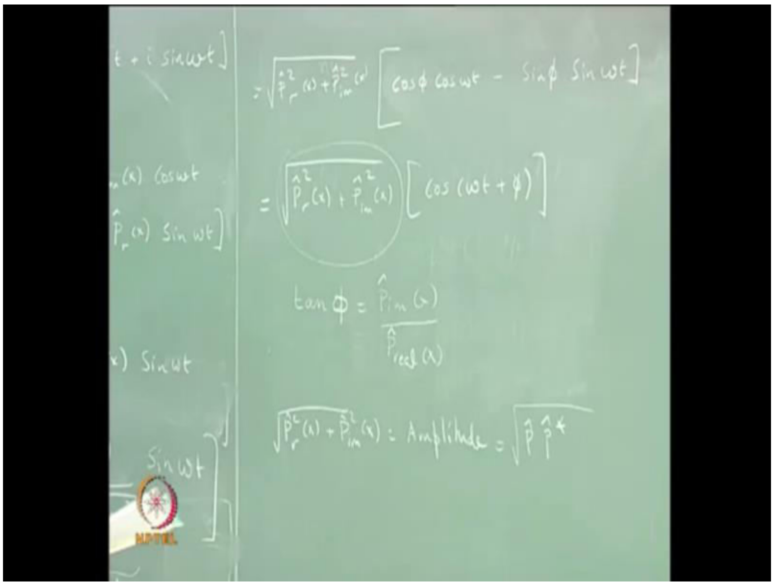


So, let us say we have the quantity p hat of x times e power i omega t, we can do the same thing with e power i omega t. If you stick with on you should stick with the same thing throughout with your calculation that is what is important. If you can show where you use e power i omega t or e power minus i omega t you will get the same results. So, just start to be consistent and so, if you expand this out you will get p real of x plus i times. So, the hat usually denotes complex amplitude.

So, let us multiply this thing out. So, what you get? Is p l of x times Cos omega t minus p imaginary of x times Sin omega t plus i times p i needs to hats sorry x times plus peril of x times Sin omega t. So, this is the real part and what we are interested in as I mentioned the instantaneous pressure is take the in this pressure this whole thing p hat x e power i omega t.

So, this is the real part and that is what we are interested in. So, this it is this quantity I will let me just write it again. Now, I will do algebraic manipulation. What I will do? Is take the square of p r square plus p imaginary square its square root and we will multiply and divide by that. So, what we get is ((no audio 24:13 to 24:45)). So, already it is multiply the numerator and denominator by the same quantity. So, it is really the same factor this will cancel minus. Now, we can say that you know very simple algebra.

(Refer Slide Time: 25:16)

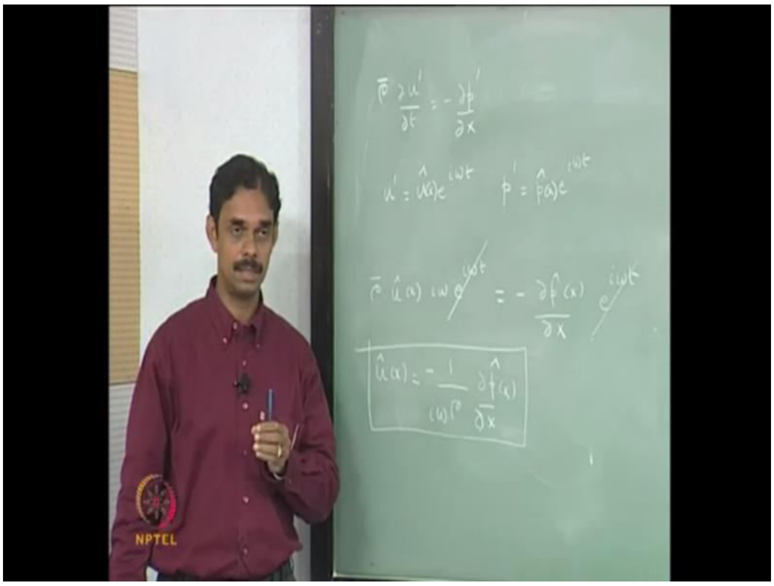


Now, we can say this is expression is same as p here squared of x plus p imaginary squared of x times. We can write this term as let us say Cos phi and this term as let us say Sin phi and you know that if you square this plus square this you will get 1. So, this its p r square plus p imaginary square divided by this quantity square. So, you will get 1 add this square this term and square this term add them.

So, this we can write as Cos phi Cos omega t minus Sin phi Sin omega t. This could be recast as root of p real square plus p imaginary square times Cos omega t plus phi where we form therefore, phi is tan phi equal to p imaginary of x divided by p real of x needs to hat. So, you can see that the imaginary part actually, is the another way of the presenting the phase imaginary part is another way of representing the phase. So, nothing imaginary about it is just it is just a reference. And this quantity is this is actually is the amplitude of the signal. So, this is actually the amplitude which could be written as square root of p hat star.

So, we need 2 more things to be done we have to talk about, How to express acoustic velocity in terms of acoustic pressure? We also need to talk about acoustic displacement amplitude in time domain we derived a relation for acoustic velocity in terms of acoustic pressure and actually there are 2 relation. 1 for the forward running wave and 1 for the backward running left running and right running wave. Let us do that for the case of harmonic domain 0 frequency.

(Refer Slide Time: 28:10)



So, if we remember our momentum equation was rho bar ((Refer Time: 28:16)) equal to minus rho p prime by rho x. So, let us say you hat equal to when I write this it is implied that the actually taking the real part. So, even when you say text book they would not often write u prime is real part of you had e power of i omega t that is implied enough to states similarly, we can write p hat equal to now we remember that this complex amplitude is actually a it can be a function of space so that, is why put the function of dependence. So, if you now substitute the equations here I will get rho bar u when you differentiate u of x it just states, it is not a function of time times differentiate e power i omega t you will get i omega e power i omega t.

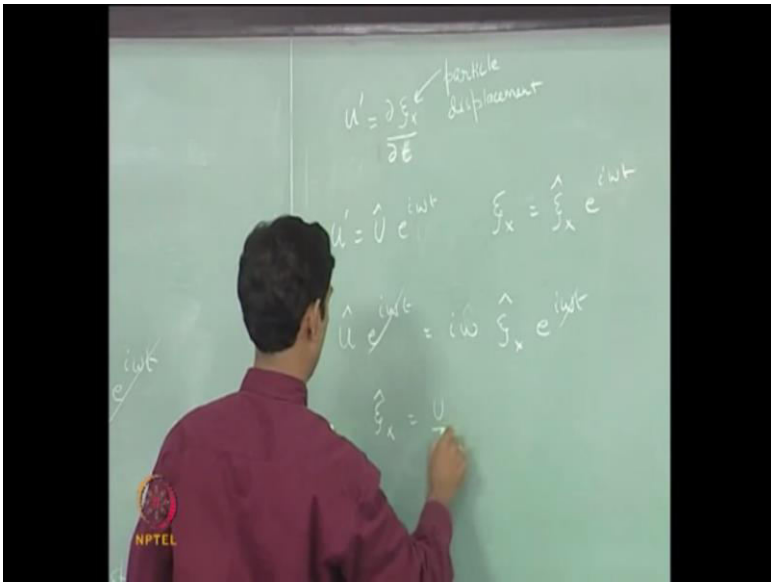
So, this would be minus rho p hat x over rho x or e power i omega t does not have any x. So, it justice now e could cancel this. So, you can say that you hat of x equal to minus 1 over i omega rho rho p hat by rho x. So, this is the relationship between acoustic velocity and acoustic pressure. In the harmonic domain I must emphases that when you say u hat of x that is really the particle velocity that is if you think of a fluid partials it is the velocity of the fluid particles.

That is different from the speed of the way. Now, just to give a analogy to make this things clear. Suppose you are standing in a line let us say you are going to a movie and you are standing in a line and somebody pushes the last fellow in the line and then he pushes the guy in front and he pushes the person in front and so on. So, eventually the push was started at the end of the line travels forward and it will be going till the guy the front of the just before the ticket counter.

Now, the wave itself actually move across the line although each person only move little bit. So, the push will go at certain speed all through the line that would be equivalent to our probation of the wave. The pattern itself moving but, each person moves little bit when the push happens that would be like our particle velocity so that, would be a very cued analogy but, I hope it helps you to understand the difference between particle velocity and phase velocity.

So the see which is the speed of sound that is the rated which the wave propagates that is different from how much the fluid particle moves this is very important and when we say you had a fax this is u is called acoustic velocity. And this is different from C, C is the speed of sound. So, please remember to distinguish between these 2. The last thing I want to speak about is the relationship between particle replacement and particle velocity.

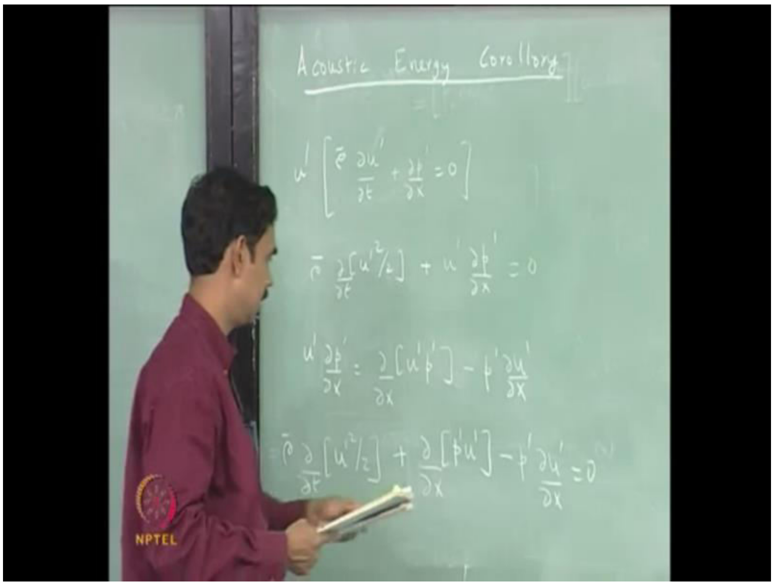
(Refer Slide Time: 32:10)



So, its quite simple displacement and velocity can be written by sorry differentiate the displacement and you get the velocity. So, this is the so, we say u prime equal to u hat e power i omega t and we say xi of x equal to xi of x hat times e power i omega t. So, you substitute it here you get u hat e power i omega t equal to i omega. So, times e power i omega t.

So, this cancels so, you get the simple relationship between the acoustic velocity and the particle velocity and the acoustic velocity. So, we did not say anything about the energy in the wave. So, the next issue we are addressing is to look at how much energy is there in the wave and try to writ the expression for it and try to see expression to look at evolution energy.

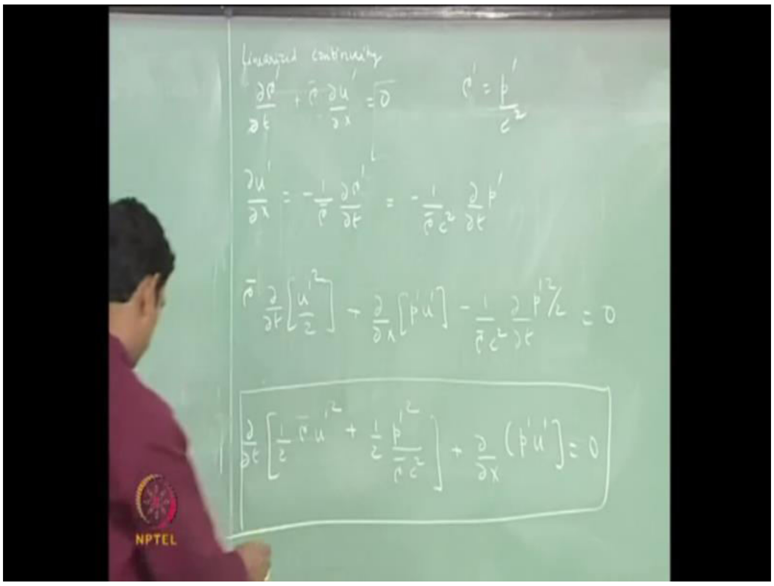
(Refer Slide Time: 33:43)



So, this would be called Acoustic Energy Corollary. Corollary because, we are not deriving anything from first principles. We are taking the momentum equation and the continuity equation and from those equations we are getting another equation. So, we are not deriving equation from scratch the existing equations the continuity momentum equations are there and we deriving energy equation which is compatible with that so that, is why it is called corollary it is from something which is already existing.

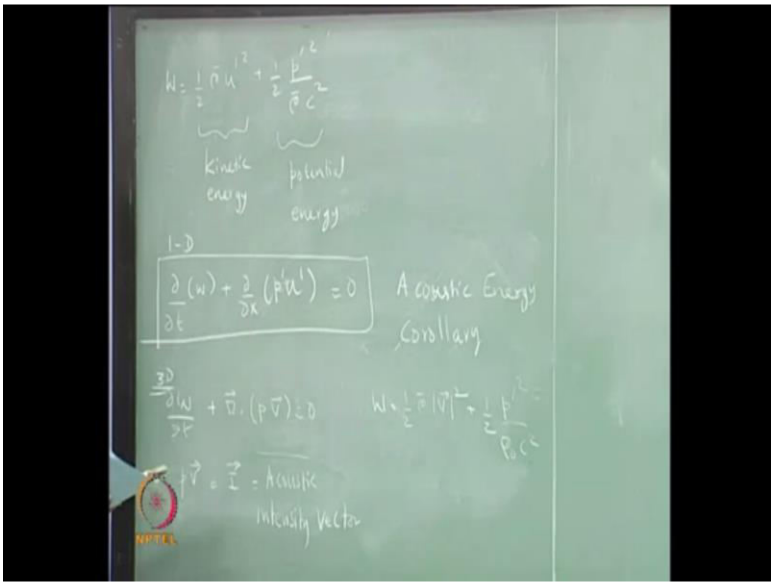
Recall that the momentum equation or the Euler's equation form rho bar dou u prime by dou t plus dou p prime by dow x equal to 0. What I will do? Is to multiply this equation by u prime and now I have rho bar what I will do? Now, is to write u prime dou p prime by dou x can be written as dou by dou x of u prime p prime minus p prime. So, why I am doing it you will see it in a minute it leads to some important physical research of this be patient. Now, let us substitute this over here you will get rho bar dou by dou t of u prime square divided by 2 plus dou by dou x of p prime u prime this time here minus p prime dou u prime by dou x equal to 0. Now, what we need? Is a expression for dou u prime by dou x. So, we can get that from the linearized continuity equation.

(Refer Slide Time: 36:09)



So, now we have a explained for dou u prime by dou x which is here. We also note that remember last class we said rho prime equal to p prime over c square. So, you substitute that in here we will get minus 1 over rho bar dou by dou t of p prime. So, what we going to do? Is to substitute this equation here and you will get rho bar u prime square over by 2 plus dou by dou x of p prime u prime minus 1 over rho bar c square. We have a p prime multiplying d t prime by d t. So, you get dou by dou t of p prime square over 2 equal to 0. So, we can drop this term and this term together and write very nicely. So, we can say. So, this would be the Acoustic Energy Corollary.

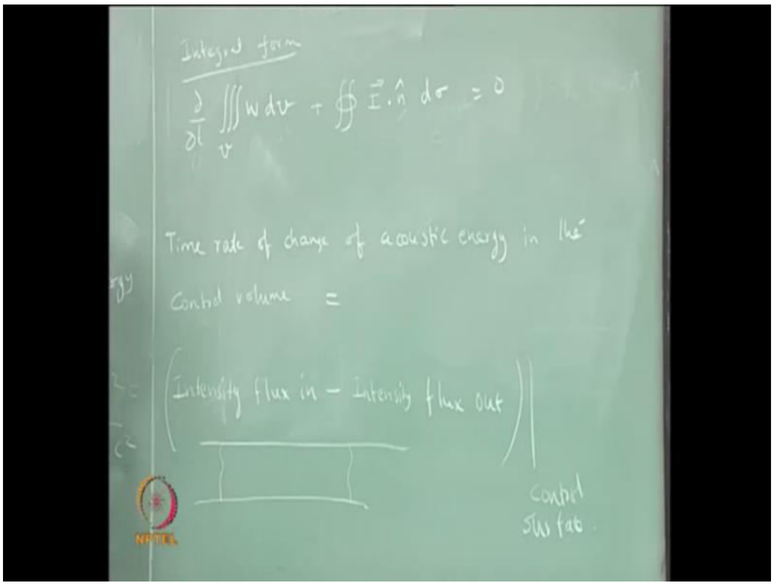
(Refer Slide Time: 38:23)



So, let us call that will be w of the term half rho bar u prime square plus half p prime square over rho bar c square. This is the term which was inside the derivative. So, we can what is this w? You could imagine w is nothing but, kinetic energy and this would be potential energy. So, we can say that dou by dou t of w plus dou by dou x of p prime u prime equal to 0, this is the so called Acoustic Energy Corollary.

So, this above equations is for the 1 dimensional case if you where having a general 3 dimensional case you would still have a equation which is similar this is for 1 dimension case for 3 D. What you would have? is a instead of dou by dou x, we will have del dot p v equal to 0 where w will now, be defined as half rho bar e dot v or this negative dot velocity square plus half p prime square over 2 p naught c square. And this term inside the radian that is p v vector quantity this is called I this is the acoustic intensity vector. This is in a differential form what we can do is we can integrate this over a control volume.

(Refer Slide Time: 40:55)



And then what we can do is let us say we are integrating this equation over a control volume this is the control volume plus now, you can you use Gauss theorem to convert this sorry this diversion term into a surface integral. So, this is the integral form this has very important physical meaning. What this means? Is time rate of change of acoustic energy in the control volume equal to the flux of intensity which is coming in minus the flux of intensity that is going out. I will write this in words ((no audio 41:58 to 42:39)). So, this intensity coming in of flux ((Refer Time: 42:43)) its flux coming in minus intensity out across the control surface.

So, what does this physically mean? So, if you have some amount of acoustic energy hat is coming in from the coming from the boundaries. Let us say we have a nut here and here from boundaries. So, if lot of energy is more energy is coming into the control volume from the boundary's and if more is coming in than what is going out then naturally the energy in the control volume will increase. Now, if more energy is going out compared to what is coming in then, you would have the acoustic energy in the control volume decrease. Now, if they are saying whatever is coming in whatever is going out acoustic energy will stay same or if nothing is coming out and nothing is coming in very ridge walls and cavity then the acoustic energy will stay same.

So, these are very beautiful result it is called Acoustic Energy Corollary. Later on in the course when we speak about the onset of thermo acoustic instabilities and how sound drives sound is driven by fluctuations and heat release rate we will actually be deriving extensions of e equation with source term right. Now, there is no source term here there if you can have things coming from you can have things coming from boundary in a row but, nothing produced inside in reality you can imagine things sound can be produced.

Sound is always produced somewhere the other h. So, we are concerned about democratic instability. So, if there is a flame here and if there is some mechanism with which sound is produced then we have to account for it. So, we have to modify this equation to account for source term there could also be volumetric terms which take away the energy for example, here could be if you think about solid rocket kind of situation n solid rocket motor.

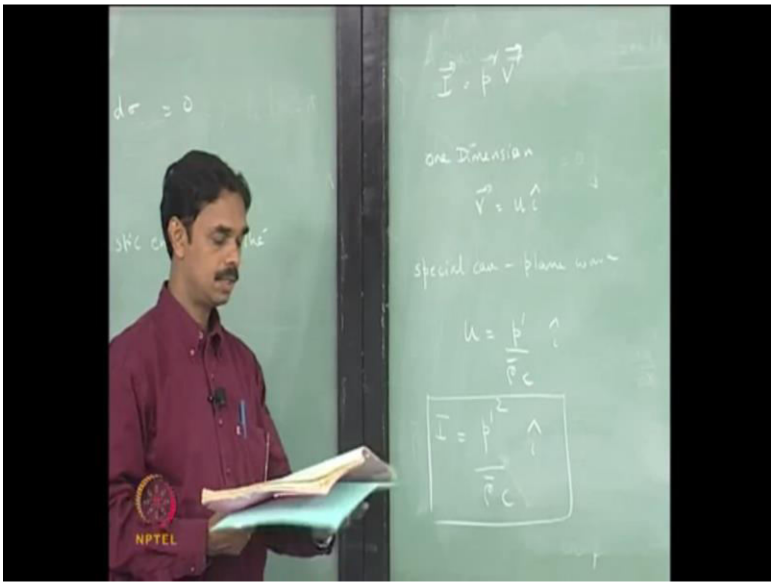
How does instability very big problem of democratic instability solid rocket motors? So, what they do is solid rocket motors use aluminous propellant and now, aluminum severs 2, 4 four purpose the alumina gives the percent of alumina, alumina is very advantage from a specific impulse point of view aluminum burnt from alumina and this actually has a very high heat release weight. And we get very high temperature rise because of this therefore, you get extra thrust you get higher specific impulse.

But, also source of another important purpose from point of view of combust instability there these particles alumina particles in, they are molten form. In which is there in the gas form to face flow mixture and there actually dam the oscillation in the particles. They start moving with the oscillations d they take away the energy from the oscillations. So, that would be like a sink so that be like a volumetric sink. So, you can have inside a volume you can have both volumetric sources and volumetric sink at the moment. We have not does with any of them but, when we the evil in the second of the class derived expressions which you deal with volumetric sound sources and volumetric sound sink.

Now, does 1 more thing that I want to say here actually that is really throwing a spanner in the works. We spoke about considering acoustic energy but, really in our situation that is for example, in engines or any such flow situations we actually have 3 different types of fluctuations acoustic is just 1 of them we have vortices fluctuations. And we have entropy fluctuations and I must also say that we have written this whole wave equation and this flow analysis everything in the absence of mean flow. We are looking at a quiescent medium.

So, in the presence of moving medium the corollary gets much more complex and it and further if you have entropy disturbances or entropy fluctuations. And if you vortices fluctuations and there is a quite a bit of debate and what is the acoustic energy itself? So, many people are working on it its even it is a topic of current interest. So, you do not have a very simple research like this you love much more complex yourself. We will speak about it again in the second round of the class you have to dominantly daily with 1 dimension situation. So, what happens to the acoustic intensity in a 1 dimension situation?

(Refer Slide Time: 47:08)



So, in 1 dimensional situation if we have only the x component. So, we have only u. So, we just have v equal to u times i. So, and if you are looking at a let us consider a special case of plane way we have u equal to p over rho bar c. So, then we can say I equal to p prime square over. So, this the expression for acoustic intensity for 1 dimensional case its p prime square over rho bar c. So, it can be reduced purely from pressure measurements. So, I will stop today's lecture with this, what we did we looked at harmonic waves. We got expression for harmonic waves then we converted look found equivalent expressions in terms of exponential functions. So, we looked at Cos and Sin as well as e power i x and then we also derived expression for acoustic velocity in terms of acoustic pressure.

We also talked about acoustic displacement particle displacement then we also looked at the why we need a imaginary quantity or complex number? What is the relevance of the imaginary quantity? We saw that it is simply another way of denoting the phase. And lastly we derived the equation for the evolution of acoustic energy. It is a Corollary it was derived from continuity and momentum equations. Then we also introduced the concept to acoustic intensity. I will stop with this. Have a good day