Acoustic Instabilities in Aerospace Propulsion

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Lecture - 03
Wave Equation and its Solution in Time Domain

Good day welcome to the third lecture on acoustic instabilities in aerospace propulsion. In the last class we talked about very basic of acoustics. What is sound? How sound is characterized in terms of frequency, amplitude and so on. And then, we spoke about the definition of decibel and then we talked about various kind of sounds that we hear in our day to day life. And what are the typical decibel levels and so on? Then we said that we want to characterize sound the propagation sound mathematically. So, we are starting from the basic equations of fluid mechanics which is conservation of mass, momentum and energy and so on and the state equation.

So, we derived the conservation of mass which is the continuity equation and the momentum equation from basic first principles. We also spoke about the state equation and so on. Today we are going to talk about how we derived the wave equation from these equations? So, before I proceed to this let us start with a story. So, there was just a professor who was working very late and then he was trying to go home and he was let us say drunk and it was like some 3 a m in the morning and he suddenly discovered that he has lost his married his or her marriage ring. And then the professor decided to search for the marriage wedding ring. And the professor was searching on the street and wherever he could we think he a wherever he thought it will be there. And then comes a student graduation student was working very hard and he is seeing he saw the professor.

And he asked the professor, professor What are you doing? So, late in the night. Professor I lost my wedding ring and I am searching for it and the student said ok. Let me try to help you I will join you and I will try to help you. The student also wanted to impress the professor and so on. So, both professor, student are starting to search for the professors wedding ring. The professor was really want it because if he or she goes home the spouse will be very angry.

The professor can be man or woman does not matter and then pretty soon the student discovered that the professor searching for the wedding ring only under the lamppost.

So, the student found this very amusing and he went and asked the professor, professor is there any guarantee that you would have lost the wedding ring or what guarantee is there that you would have lost the wedding ring under a lamppost? Its highly unlikely the professor said look I made a assumption I assume that the ring will be there under the lamppost. Then the student ask I mean it is a very bad assumption the professor says this is the only place where I can look. So, I made the assumption that its under the lamppost. So, I look where I can.

So, this is kind of true with our mathematical analysis also that is we make assumptions so that, we can simplify things and we can deal with things. So, often the assumptions are not because they are the way it is. We make assumptions so that, we can do things with the equations and then we look for situations where the assumptions are applicable and then we say that the solutions were derived or the equations we have derived are applicable under these set of conditions and so on

So, we derive partial differential equation and there were couple partial differential equation and there were non linear and so on. And I said that it was very hard to get very general solution for these things. So, the idea is that we need to get some solutions because we cannot just sit there without any hope or helplessly and say that these are complicated equations we can do we cannot do anything about it. So, we do the same thing that the professor did. So, we can look under the lamppost where he we can look.

So, we make assumptions so that, we can solve the equations that we have derived under whatever conditions it is possible. So, what we are going to do? I will let me summarize what we are going to do? Is to assume that we have a very small amplitude disturbances and then we derive we will linearize this non linear equation look for a set of linear equations or suppose simplifies this non linear equations in to a simplified linerized version. And then we try to make more assumptions about the quantities and then try to see if we can derive some kind of simple equation, simple linear equation which can be solved with paper and pencil and that has and its on utility no matter how simplifying the assumptions that we have.

So, in order to refresh your memory I will write down the equations.

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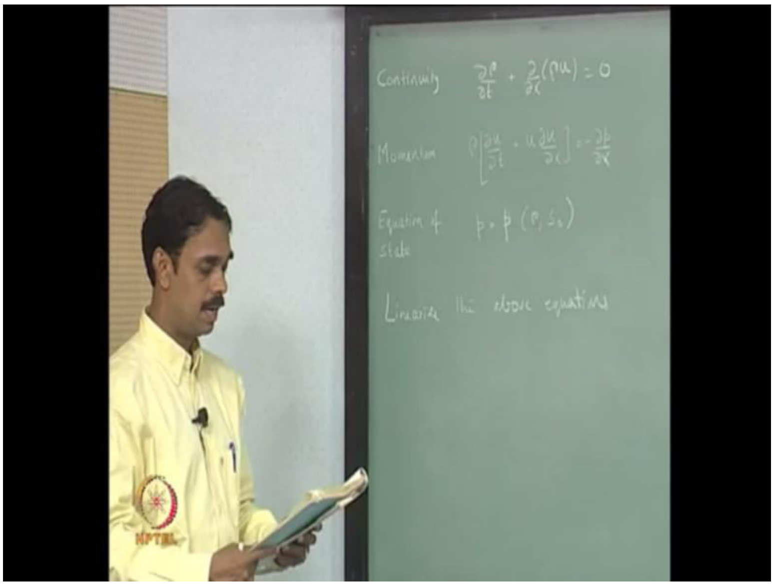

This is the conservation mass or the continuity equation. And the momentum equation is rho times and I want to say that I am doing everything in one dimension but the three dimension is not too difficult. And recall that s naught is constant that means you are saying that we are talking about constant entropy process. And now what we need to do? Is we want to linearize this equations. So, you can ((Refer Time: 06.29)) is the non linear term here. This is going to give lot of trouble in terms of solving the equations.

And so what we want to do? Is to say that the perturbations and pressure velocity density and temperature are fairly small compare to the mean values. So, we take the equations perturb the equations around its mean that way we derived the so called acoustic equations. And then we say that these perturbations are really small. So, the advantage of this is that you can actually drop out terms which are product of 2 perturbations quantities that way we get a linear equation and which we can solve.

So, this whole process called linear acoustic.

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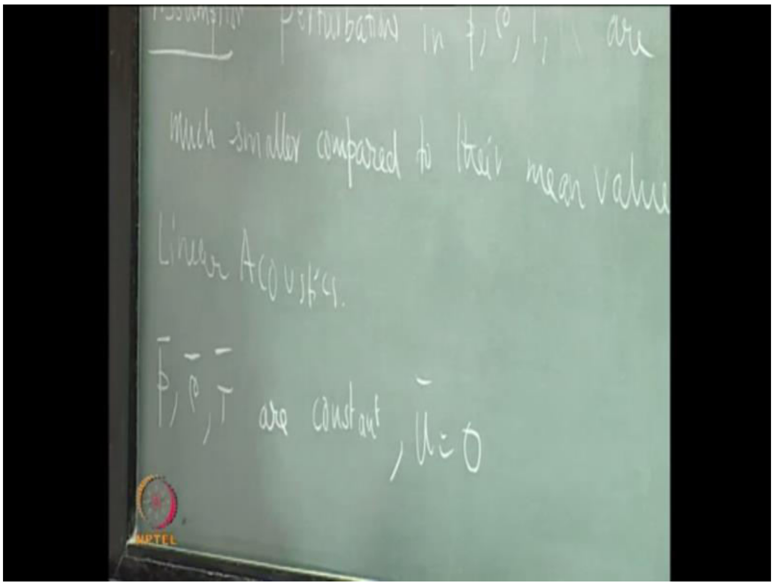

So, the assumption here is that perturbations in pressure,density,temperature,velocity are of much smaller compared to their mean values. So, this would be what we referred to as linear acoustics? And we also assume a homogenous fluid and we say that the mean quantities, the mean pressure, mean density and mean temperature independent of the position of the medium. And we will also say that we are a having a quiescent medium. That means there is no mean velocity associated with the medium. The mean quantities I will denote the mean by a over bar, p bar, rho bar, t bar are a constant and u bar equal to 0. So, we have constant thermodynamic mean properties and the mean flow velocity is 0 that means it is a quasion medium.

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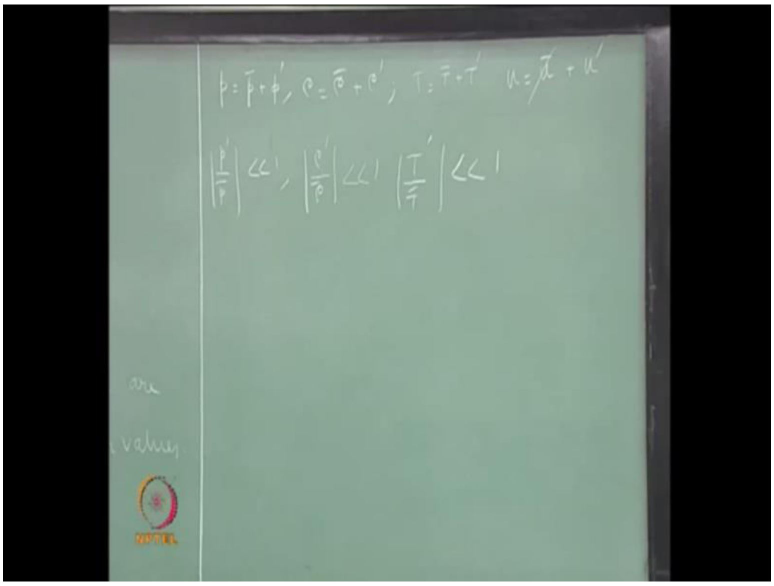

So, we now write all the variables so let us say pressure equal to. We write the variables as the sum of a mean quantity and the fluctuating quantity. Prime density equal to mean density plus fluctuating density temperature equal to bar plus t prime u equal to u bar plus u prime. So, here we are saying that we have a quasion medium so we drop this term and in general you could have the pressure and mean pressure and density temperature varying but, in this particular case h we are keeping them as constant. We also say that p prime over p bar a is much less than 1 rho prime over rho bar its negative it is much less than 1 t prime over t bar is much less than 1 and so on.

So, now that means if p prime p bar is much lower than 1 and rho prime by rho bar is much lower than 1 then this quantities are very small. So, we can actually drop terms that second order quantity that means terms that involves products of this quantities. So, let us now perturb up this equation and write this in terms of this mean and mean quantity and the perturbation quantity and then try to proceed the linearise the equation. So, let us start with continuity equation.

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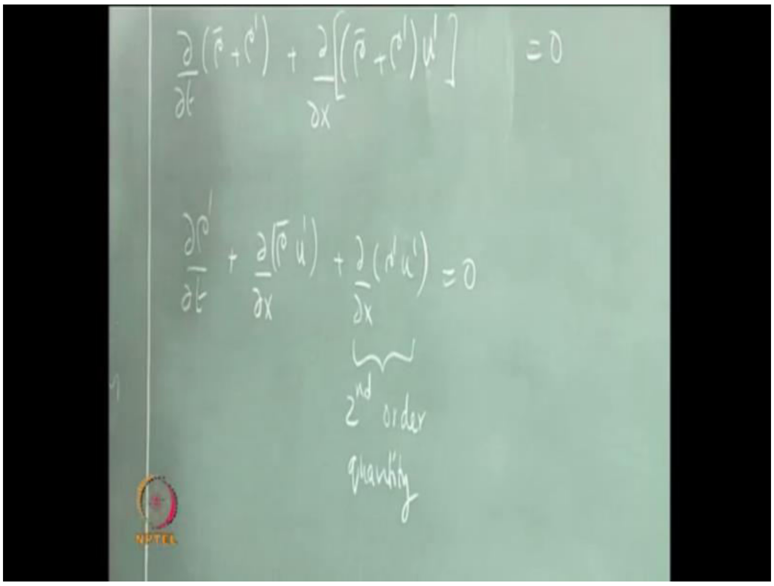

So, when you linearise this you will get dou by dou tau rho bar plus rho prime plus dou by dou x so rho bar plus rho prime into u bar plus u prime equal to 0. Now, I said that there is no u bar so we can simply erase this term and just say u prime equal to 0. So,. let us expand this equation and look at each term. So, rho bar is a time average value of mean time r value of density so that, is a average quantity in terms of time. So, by definition its independent of time. So, the first term will drop out see you keep only dou rho prime by dou t plus. Let us expand this term so we will have dou by dou x rho bar u prime plus dou by dou x so rho prime u prime equal to 0. Now, this is a second order quantity. Of course it is a non linear quantity and the trick here is that we do not deal with nonlinearity we are trying to linearise.

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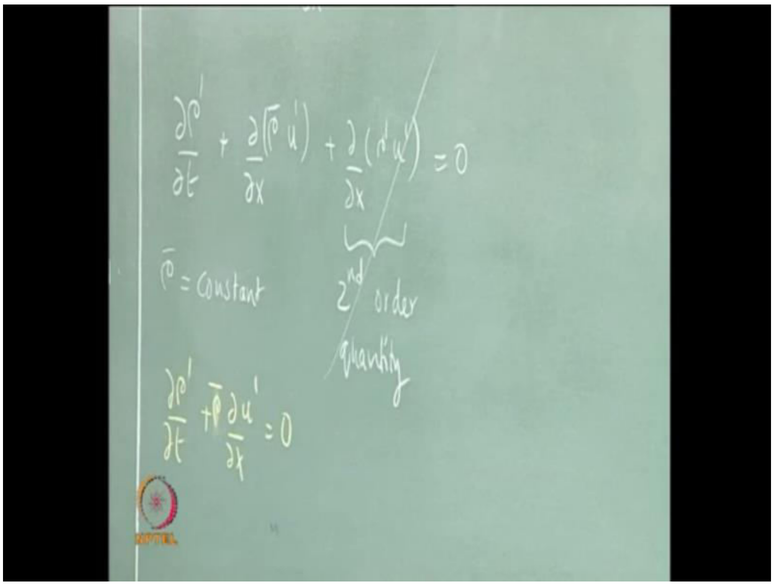

So, we will drop this we also assume that rho bar is a constant. It is independent of space we can take it outside the derivative. So, in the final equations become so let me say rho bar equal to constant. So, this would be the perturbation equation for the continuity equation. So, its like the acoustic equation for continuity. So, we need to do the same procedure next with the momentum equations.

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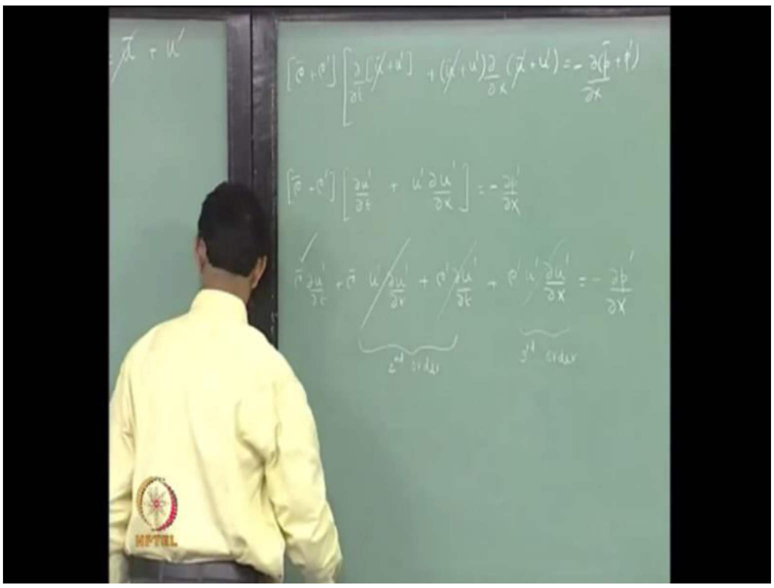

So, let us do that momentum equation you will have rho bar plus rho prime multiplied by dou by dou t of u bar plus u prime plus u bar plus u prime into dou by dou x. So, u bar plus u prime equal to minus dou by dou x. So, p bar plus p prime. So, let us simplify matters we said that, there is no u bar term. So, we can drop wherever there is u bar first. And so let us say rho bar plus rho prime into dou u prime by dou t plus u prime dou u prime by dou x equal to minus. We said that the mean pressure is independent of space so we can drop the term and dou p bar by dou x can be droped.

So, you will have just dou x. Now, lets expand this out. So, you will have rho bar dou u prime by dou t plus rho bar u prime dou u prime by dou x plus rho prime dou u prime by dou t plus rho prime u prime dou by dou x equal to minus dou p prime by dou x dou x. So, if you remember the primes stands for the fluctuating quantity or the acoustic quantity and once again the bar stands for the mean quantity. So, now we look so this is a first order quantity we can keep that. So, we keep this is a second order quantity will drop this also a second order quantity we drop that the third order quantity. So, let us write this is second order this is third order. So, we keep only the leading order.

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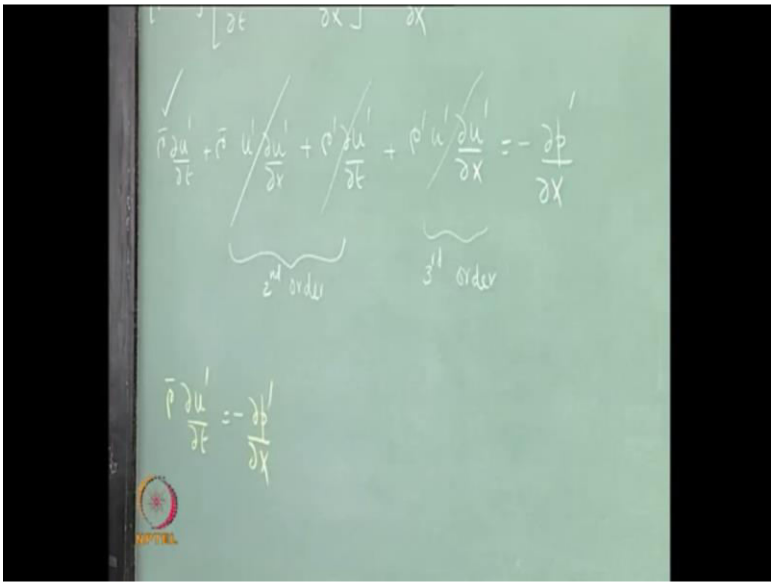

So, our equation now becomes rho bar dou u prime by dou t equal to minus dou p prime by dou x. So, this is the acoustic momentum equation. Next we want to linearise the equation of state.

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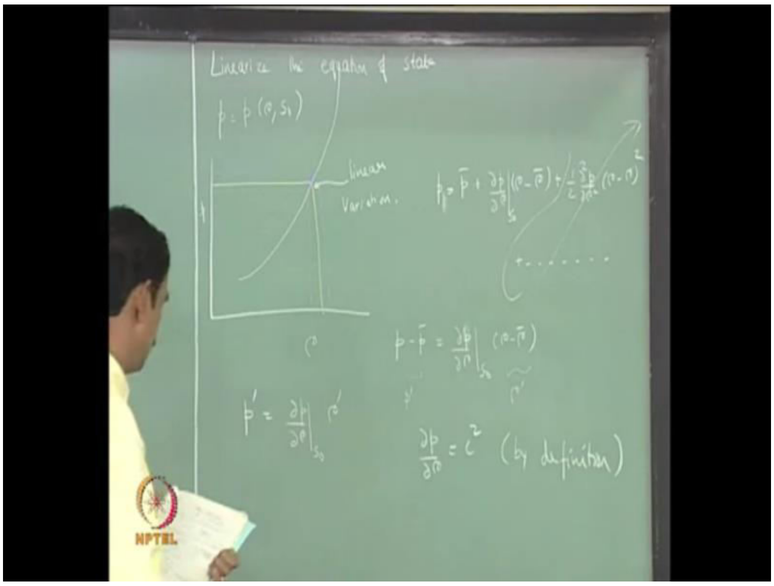

For that we said that p equal to p of rho comma s naught and remember that we said that we are having a isotropic process here. So, let us take a graphical approach. Now let us take a look at a graph of p versus rho and lets say its varying in some fashion. Let us mark our mean quantity of p bar and rho bar. Let us say this is our mean state and we perturb things around it. So, here this is the region we are looking and this region.

So, we although this function could be a general curve. In this small region the pressure is fluctuating about a mean value. And this fluctuating around mean value and this is a really small region and there we can say that this variation is linear. So, we have a linear variation in the vicinity of this base flow or the mean quantities we are having a linear variation. So, we can try it a linear relationship or write a equation of a straight line mathematically. We can use a tailor series. For example so pressure can be written as can we expanded by tailor series about its base value plus dou p by dou rho at constant rho p times rho minus rho bar plus half into dou square p by dou rho square into rho ms rho bar whole square plus higher order terms.

So, lets throughout the higher order terms and write a keep only the linear term. So, we can say we drop them. Now, we can rewrite this equation as p minus p bar equal to dou rho by sorry dou p by dou rho as naught times rho minus rho bar. So, this would be the linearised equation of state. And if you look carefully we know that this is our p prime and this is our rho prime. So, basically we are having p prime equal to dou p by dou rho at constant entropy times rho prime. Now, this is a constant value and we can denote it by some constant. So, we say that dou p by dou rho lets call its c square this by definition.

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So, we can say that p prime equal to c squared times rho prime. So, at this point it is not clear that c is its not clear that c is the speed of sound. At this point we will have to wait a little bit before we can see what actually see physically means? Whether it is speed of sound and so far.

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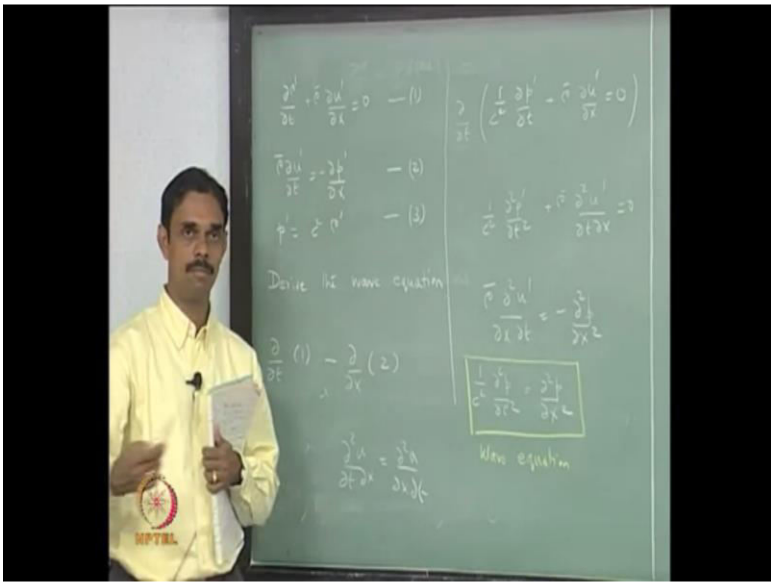

So, let me summarize the equations that we derived. So, let us call this equation 1 call this equation 2 call this equation 3. Now I must say that something really annoying in the convention used in acoustic that is they free when it explode u keep the primes to some point and then suddenly the author drops the prime. So, generally the quantity without prime. So, for example rho prime would actually be written as rho. I mean will be written as rho prime sometimes but p prime is staying as p prime. But if you the mean pressure will be denoted by p bar or something but after some time and I mean this dropping of primes is sometimes quite annoying but I will try to be consistent and keep this notations correctly yeah thank you so. And in the convention sometimes rho bar is written as rho and its very mixed up and so on. But its better to be consistent and keep the following notation about primes and bars and just keep it.

So, now what we do? Is we can so we next we proceed to derive the wave equation. So, what we can do? Is to take dou by dou t of equation 1 and subtract dou by dou x of equation 2 and then you will get. And we also use p prime equal to c squared rho prime. So, if you do that here you will get. So, you are going to get 1 over c squared dou p prime by dou t plus rho bar dou p prime by dou x equal to 0. And here so if I take a time derivative of this I will get 1 over c squared d squared p prime by d squared plus rho bar dou squared u by dou t dou x equal to 0.

If I take this space derivative of this equation 2. What you will get? Is rho bar dou squared u prime by dou x dou t equal to minus dou square t over dou x square. Now, you can subtract this equation from this subtract the bottom equation from the top equation. And then you can get 1 over c squared d squared p over d t square equal to dou square p by dou x square. So, this is the wave equation or this so called classical wave equation. And we having at look that the physical meaning of c although you might have I think all of you know that c stands for speed of sound but, we will have to see very carefully that what c is? So, let me so I must mention that I made a assumption here that dou squared u by dou t dou x equal to dou square dou by dou x dou t. That is why i could subtract the term of or cancel the term over. And of course that is ok when the velocity is a continues function of x and time then you solve the derivatives So, that is we have derived the wave equation and will be make a few remarks about the wave equation.

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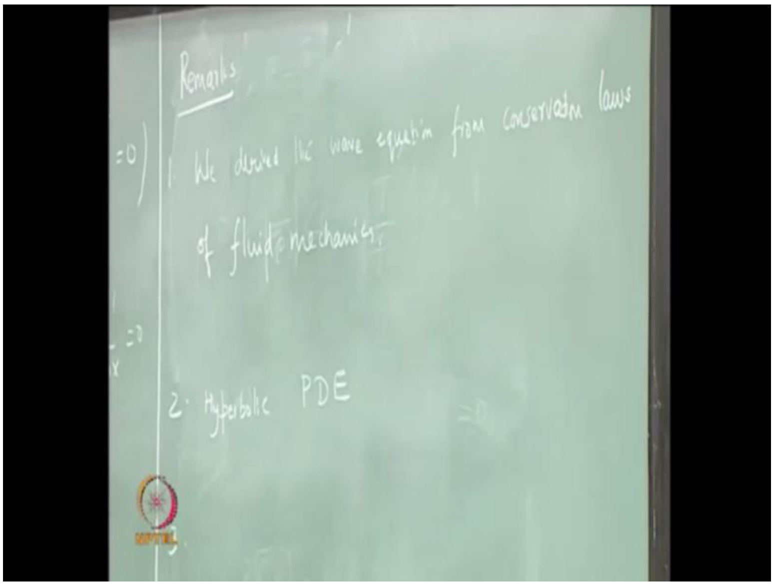

We derived the wave equation from a fluid mechanic conservation laws is just from fundamentals. The second thing I want to say those of you are mathematically incline will notice that this is a hyperbolic partial differential equation PDE stands for partial differential equation. In fact you see that the characteristics are the d x by d t equal to plus or minus c. I am not going to go in to much of a detail. But, if you are mathematically inclined you can persive this lineand we did not use the linearised equation of energy.

But, actually if you work out the problem and if you work out the linearised equation of energy it will turn that the linearised equation for continuity equation is identical to the linearised equation. For energy equation in this case if there may be different and some other case that is when you have temperature gradient or when you have combusting and so on. These equation may be different but right now they are the same. So, we see this 1 dimensional wave equation. We can I want to make a few more probations if you do a 3 dimensional derivation of it.

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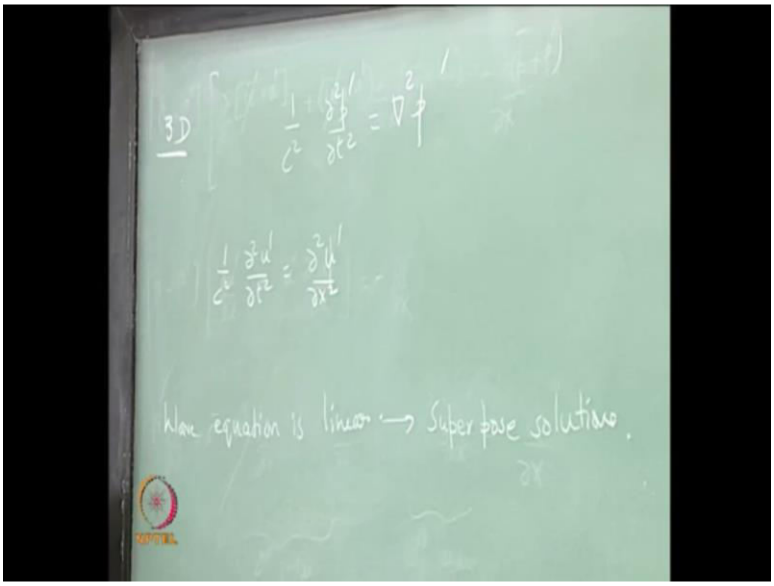

In 3 dimension you would have 1 over c square dou squared p by dou t squared equal to del square p. In fact I have started forgetting my prime. So, I think this will happens to you. Also so please remember to be consistent with the notation and you can also derive the wave equation in terms of acoustic velocity or velocity potential and so on. You can do this as a homework. So, in terms of acoustic velocity you will also get an identical wave equation. Now, the I want to speak about certain characteristic of this equation. This is a linear equation of course we linearized our original equation.

So, naturally we will end up with a linear equation and this is 1 of the most studied equation in maths. And in fact if you had a minus sign here or here if you had a minus sign then you know. What the equation this become? It is a laplace equation which has completely different characteristics. So, the advantage of linearity is that if you have linear equation will have linear solutions and we can superimpose the solution. So, if you have a 2 solutions you can add them or multiply by on constant multiply the other by another constant we can construct more and more solutions.

The wave equation is also homogenous. There are no other source terms if there are source terms then the wave equation will become in homogeneous. So, this wave equation does not speak about the production of sound it only talks about the propagation of sound. We did not consider any sound sources in this and sound source of course will come in second half of the course. A similar equation can be derived for the propagation of electro magnetic waves in free space. Of course it will look like a different. Similarly, you can have a wave equation for schodinger equation.Though these are very important class of equation that has been well studied by the mathematicians and the physicist.

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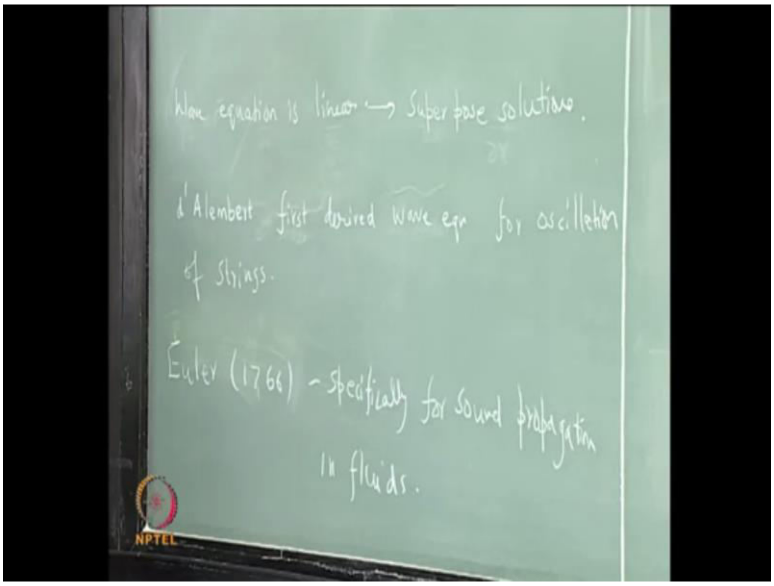

Historically, the 1-D wave equation was derived first by d' Alembert for oscillations of strings. And Euler in 1766s derived this equation specifically for sound propagation in fluids. One last comment the wave equation is a second order partial differential equation and it will have infinite number of solutions. This specific solutions can be obtained of course you need to specify the particular boundary conditions and the initial conditions and so on.

So, we will look for solutions for the travelling wave solutions, standing wave solutions etcetera. And first we will look for deriving generalized solution. So, the solution procedure is quite simple we will factorize this wave equation. That is the wave equation and we assume that d square p prime by dou x dou t equal to dou square p by dou t dou x. So, we can actually factorize this equation as follows, its quite simple to work out.

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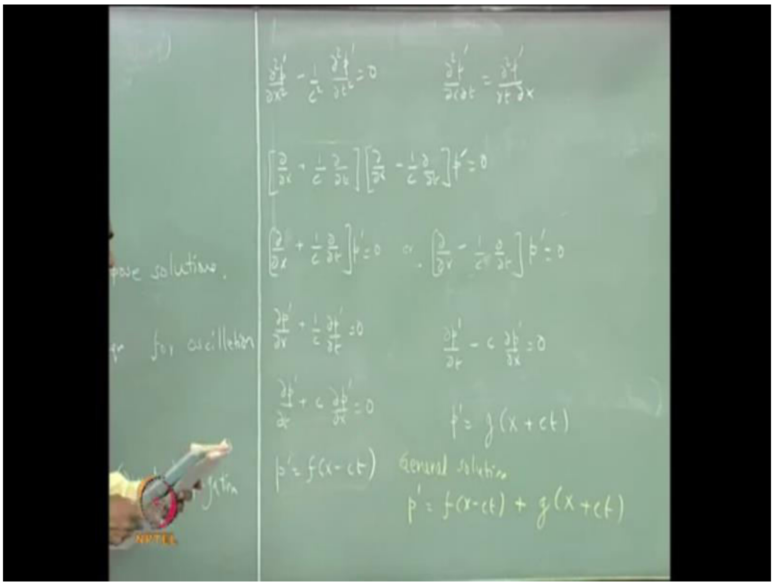

So, this is a same equation factorized it you expand this if you expand this equation out you will get the equation at the top. So, then we can have 2 possibilities 1 is dou p by you have the first operator acting on pressure equal to 0 or you can also have the possibility of the second operator acting on the pressure fluctuating pressure second operator acting on the fluctuating pressure makes it 0. So if you look carefully you can I mean you can rewrite this as dou p prime by dou x plus 1 over c dou p prime by dou t equal to 0 or we can write dou p prime by dou t plus c times dou p prime by dou x equal to 0.

So, you can easily see that the c is the characteristic speed d x by d t equal to c. So, the solution a generalized solution for this would be equal to f of x minus c t and similarly, if you look at you can write this equation as dou p prime by dou t minus c dou p prime by dou x equal to 0. So, the characteristic speed here is dou x by dou t equal to c. Yeah that should be c.

Thank you.

So, this would be p prime equal to g of x plus c t. So, f prefen a sound function g is another function. Now, you can write a general solution as p prime equal to f of x minus c t plus g of x plus c t. So, this would be the general solution. Now, its quite easy to convince yourself that this solution actually satisfies this first equation and this equation actually satisfies the top equation. I will just show that. Let us make a change of variable or a transformation that will make things easy.

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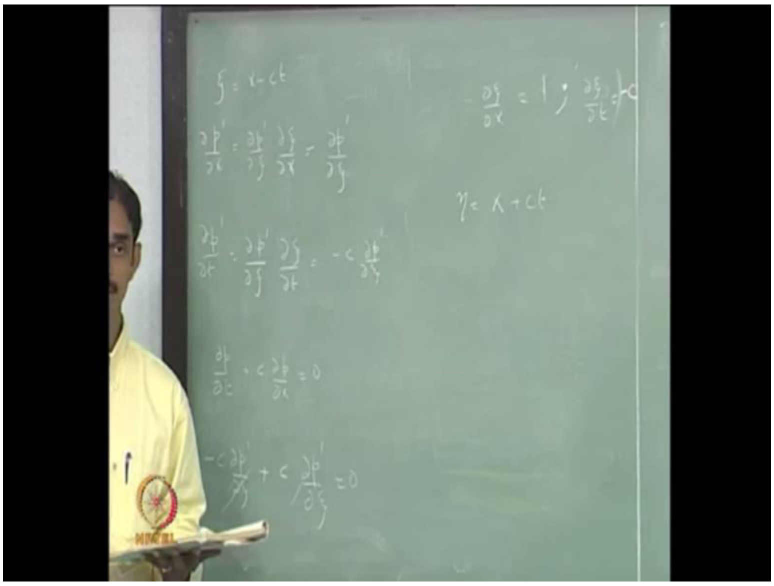

So, we say that xi equal to x minus c t. So, we say that dou p prime by dou x equal to dou p prime by dou xi into dou xi by dou x. Similarly, we can say that dou p prime by dou t equal to dou p prime by dou xi into dou xi by dou t. Now, if you see dou xi by dou x equal to 1 and dou xi by dou t equal to minus c.

So if you substitute this in here I will get. So, now if I substitute this in the wave equation. What we had? Was dou p by dou t plus c dou p by dou x equal to 0. So, if you make this substitution dou p by dou t is nothing but a minus c dou p prime by dou xi plus c times dou p prime by dou sei. You can see that this actually cancel this and you will get 0. So, therefore f of x minus c t is actually a solution to the wave equation.

Similarly, if you follow the same procedure except you use another variable. If you use eta equal to x plus c t you can a prove that g of x plus c t is also a solution to the wave equation. Therefore the general solution for the wave equation is p rime equal to f of x minus c t plus g of x plus c t. So, I must emphasize that f and g are at this stage simply orbiter function and we do not know what they are? They could be any functions but the specific form of the functions depends very much on the type of initial condition that you use and the type of boundary conditions that we use. So, now lets the big question we have to ask is what is c?

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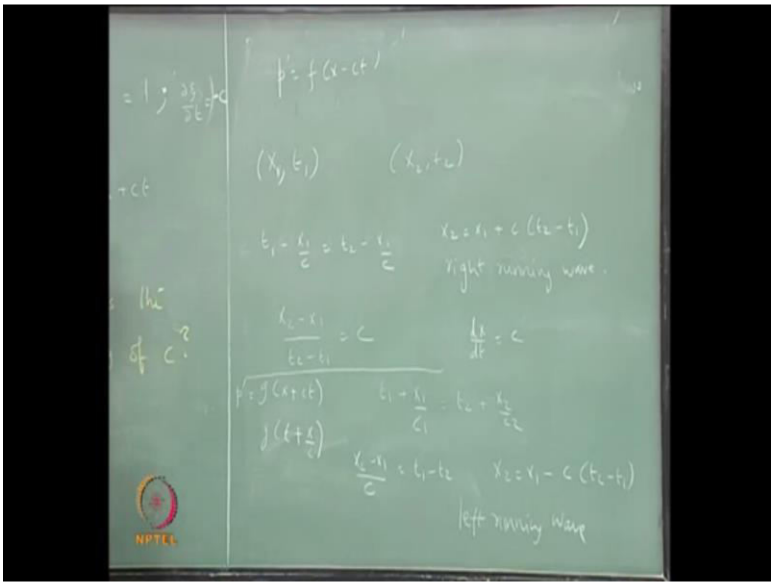

So, the next question is. What is the meaning of c? Towards this purpose lets just take a look at the first solution p prime equal to f of x minus c t. So, if you the meaning of this is that as long as you are on the line x minus c t equal to constant. You will see the same pressure so that, x minus c t some kind of preserved quantity. So, if you have a wave shape and it moves to some other place between similar points this quantity will be preserved.

So, lets say the you are at X 1 comma t 1 at sometime at 1 instant at another instant we are at X 2 comma t 2. Now, if you are on this solution then you have to make sure that x minus c t is constant that is the thing its preserved. So, now you will say that or t minus x by c some people write x minus c t sometimes you can say t minus x over c. Whatever t 1 minus X 1 over c equal to t 2 minus X 1 over c or we can rewrite this as xX 2 minus X 1 over t 2 minus t 1 equal to c.

So, you can really see that t is the time taken by the wave travel from X 1 to X 2 in this time interval t t 2 minus t 1. So, therefore c is indeed the speed of sound. Now, if you are looking at rite propagating wave. Now, f of x minus c t is right propagating wave because the velocity of the wave is its varying in to characteristic d x over d t equal to c. And so it is a right running wave. If you were to write the other solution we have g of x minus c t. If you look at this solution. What is preserved? Is or it can possibly written as g of t minus x over u x plus.

So, t 1 plus X 1 over c 1 equal to t 2 plus X 2 over c 2. So, What we can see? Is X 2 minus X 1 over c equal to t 1 minus t 2 or X 2 equal to X 1 minus c into t 2 minus t 1. That means actually the wave is here. You can see its wave is propagating to the left. So, this is left running wave. Whereas in the earlier case we see that the wave is propagating to the right because you can see that X 2 equal to X 1 plus c into t 2 minus t 1. So, this a so called right right running wave. So, now you can clearly see that why is c is the speed of sound. Now, you can clearly see why c is actually is the speed of sound anything else ok.

So, let us look at some more things about this solution p prime equal to f of x minus c t or g is equal to x plus c t. So, this means that along that x minus c t you have a certain shape of the wave or along x plus c t you have certain post shape of the wave. That means we are having a solution which says that the wave shape is constant. Wave shape does not change the amplitude doent change, only the wave just propagates.

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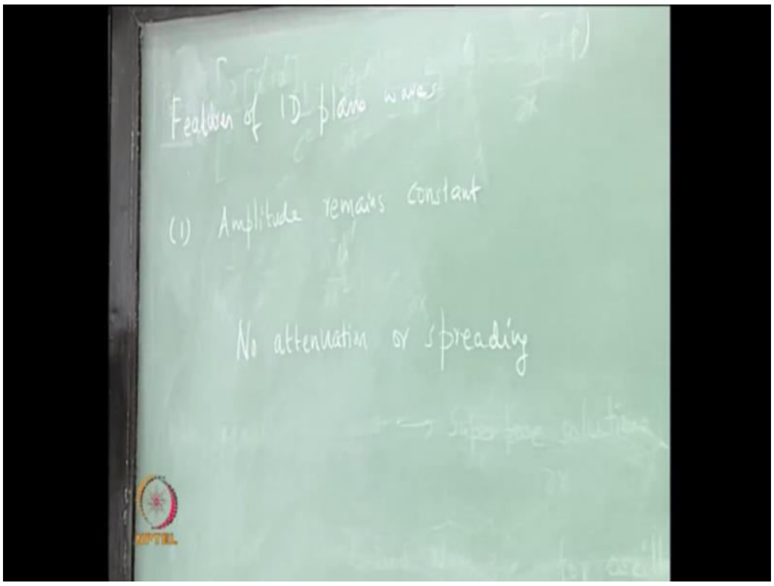

So, we are looking at 1D wave or 1D plane wave because we only have a coordinate x. So, it is a plane wave. So, lets look at the futures of the 1 D plane wave. So, with amplitude is all based is that f of g. So, there is no attenuation or speeding and the second feature is that the wave shape the wave shape does not change to any propagation. We will see later on in the course where will have the situations where the amplitude and the shape will change during propagation. But, at the moment from this classical case the wave shape does not change during propagation.

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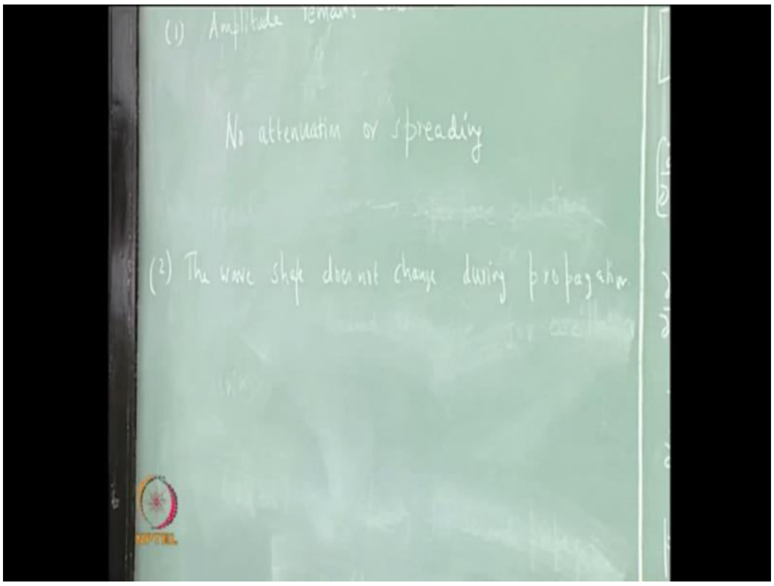

So, the last thing is I want to work out, is to work out the relationship between the acoustic pressure and the acoustic velocity. How do we go about doing this? So, we can actually what this out by using the acoustic momentum equation or the euler equation its linerized version.

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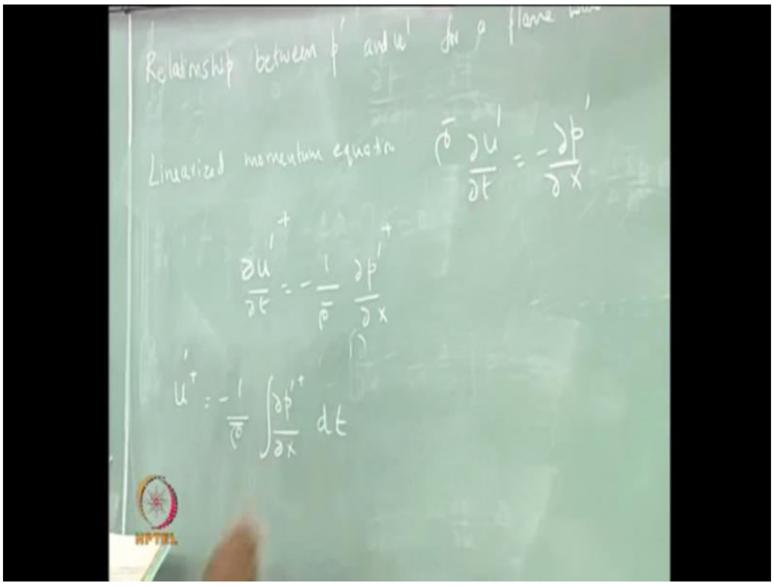

So, we are trying to derive the relationship between acoustic pressure and acoustic velocity for a plane wave. So, use the linareized Euler equation or the linerized momentum equation which is rho bar dou u prime by dou t equal to minus dou p prime by dou x. So, we can do this relations separately for a right propagating wave and then for a left running wave. So, let us denote the right propagating wave by plus and the left propagating wave by minus. So, we will write dou u prime plus over dou t equal to minus 1 over rho bar dou p prime plus rho dou x. So, all we need to do is to say that. So, you can integrate with right hand side with respect to time and you will be able to get the velocity.

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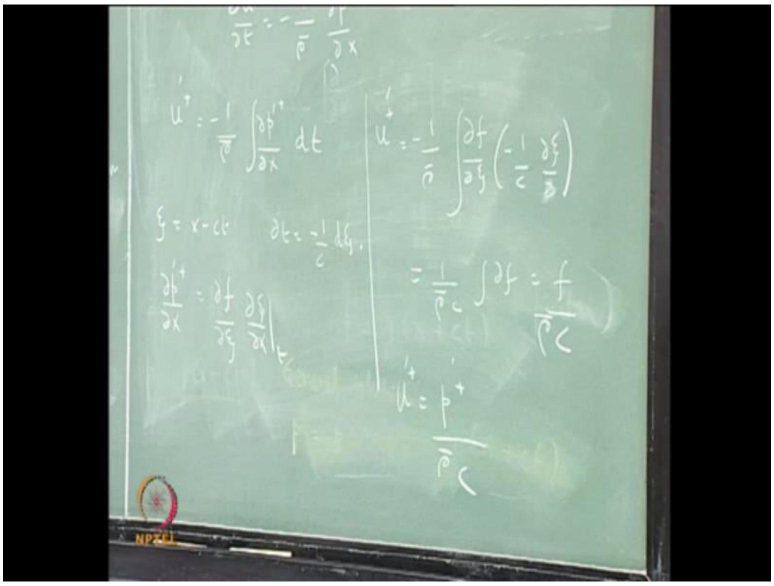

So, we can change the variables to xi equal to x minus c t. Then we can see that d t equal to minus 1 over c d sei. So, d p by d x equal to and now if you substitute this in here. What you can get? Is u plus equal to minus 1 over rho bar integral dou f by dou xi into minus 1 over c dou xi by. So, this would be nothing but 1 over rho bar c integral dou f which is f over rho bar c. If you are starting with initial conditions that are a question. So, basically we can say that u plus equal to p plus divided by rho bar c. Now, if you were to workout this relation would the left running wave and that is quite easy.

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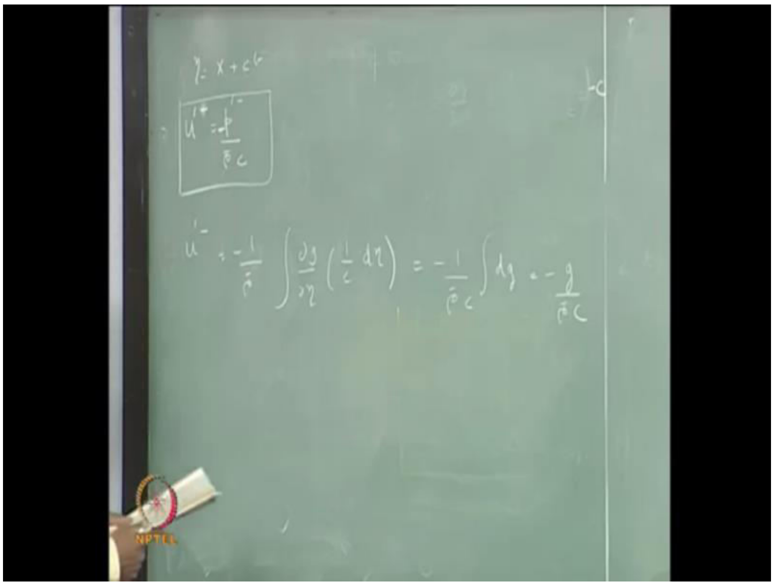

So, we say that eta equal to x plus c t and we can follow the same procedure and you can then derive the relation. Sorry, it is a you follow the same procedure as for integration you can and then you will get the results. So, it's a just to you would get this relation. So, I there is a minus sign in front of of p prime. So, in either case either for a left running wave or a right running wave the magnitude of the velocity is the magnitude of the pressure divided by rho bar c rho bar c is quantity called characteristic impendence. Now, the only thing is for a right running wave the velocity the front face velocity is u plus equal to p plus divided by rho bar c.

For a left running wave u minus equal to minus p minus divided by rho bar c. So, these are opposite sign this is although it looks strange that there is a different sign its quite intuitively obvious. So, let us look at the compression wave which is going to your right side. Now, if the wave is going to right side the gas particle will if the compression wave the gas particle will will move behind the wave to the right. So, then the sign should be plus because you are having the wave also move behind the the velocity move also to the positive x axis. So, there should be a plus.

Whereas the compression is going to the negative x axis then the particle will try to that compression wave to negative x axis. So, its naturally have a minus sign because its going into opposite direction. So, that is what that is the reason for this left running wave having a minus sign in front of it for the expression for the acoustic velocity. Whereas right running wave has a plus sign or a positive sign before the expression for acoustic velocity.

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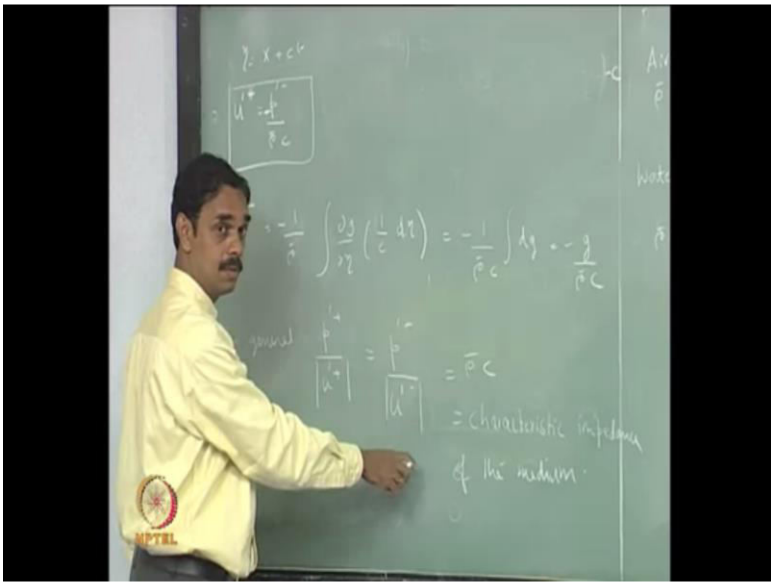

So, in general p plus over u plus equal to p prime minus divided by. So, if you take the region and the magnitudes will obviously rho bar c and rho bar c is called characteristic impendence of the medium. Different mediums will have different value for example, a would be of the value would be around 400, for water it would be some other value.

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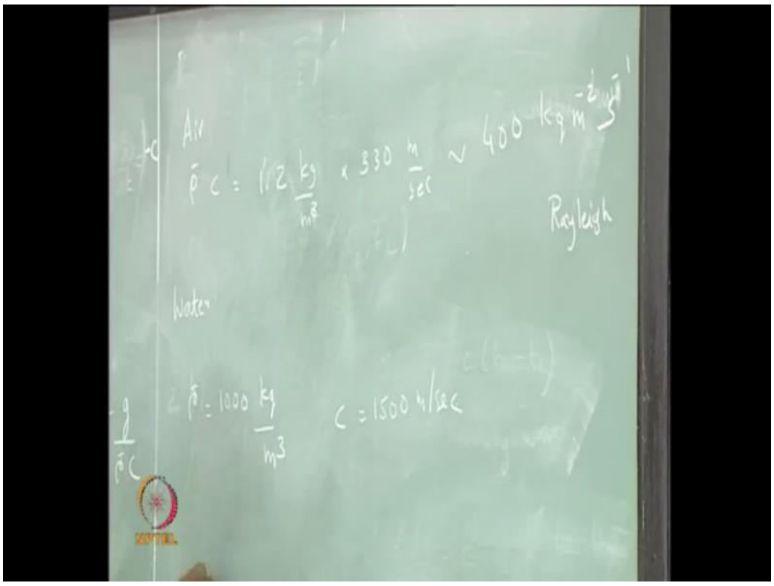

For example, let us work out rho bar c would be typically rho would be let us say 1.2 kg per meter square into speed of sound would be typically 330 meter per second. So, you will get approximately 400 kilogram meter or minus 2 second power minus 1. So, this unit kilogram will meter per minus kilogram per meter square second. This is called the this unit is referred to as rayleigh in order to honor the great scientist dort rayliegh who did a law of pioneer in working acoustics. Now, if you are having a ((Refer Time: 49:53)). Now, if you are having another medium medium water. So, you would have rho bar is 1000 kg per meter cube for water speed of sound is of the order of 1500 meter per second.

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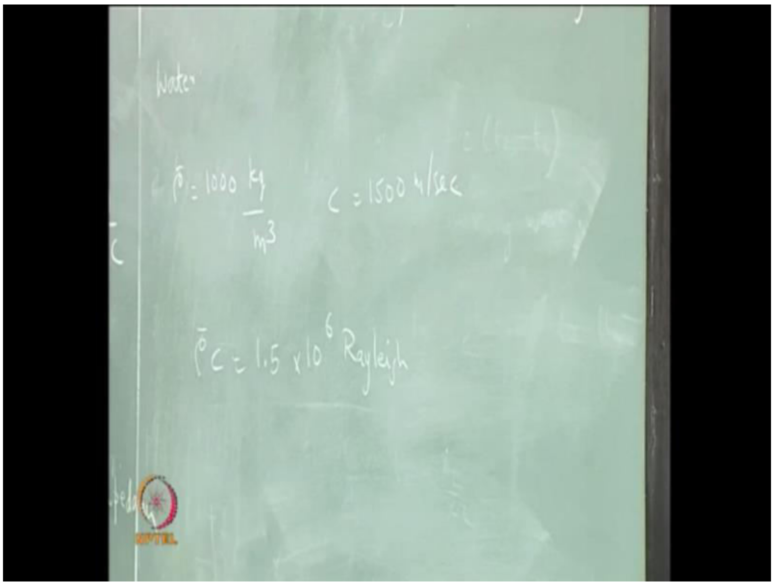

So, rho bar c would be of the order of 1.5 into 10 power 6 rayleigh. So, you can see that for different mediums the value of the rho bar c the characteristic impendence is quite different for different medium. So, we will stop here at this point. What we did? Is yes here I put a to summarize what we did? Is we took the equations of fluid mechanics which we derived in last class. Then we made assumptions of liner acoustic that means we we split the variables into a mean quantity and a perturbation quantity. Then we said that the perturbation are very small compare to the mean.

So, we actually neglect we kept the quantities which are first order in the perturbations but we neglected everything that the second order or higher. And then we massage these equations and derived the classical wave equation. Then we actually proceeded to find the solutions to wave equation which is as I showed earlier f of x minus c t in g of x plus c t. And then we also derived expression for acoustic velocity in terms of the acoustic pressure. So, we will stop here today.

Thank you. Have a good day.