Acoustic Instabilities in Aerospace Propulsion
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Lecture - 19
Modal Analysis of Thermo acoustic Instability - 1

Good morning everybody. We spoke about relatending, we derived expression for relatending, and we actually derived the jump conditions across the flame and jump conditions were that pressure was continuous and velocity was jumping. We said that flame acts like a monopole source and there is it is creates volume and which is what creates the acoustic field or drives the acoustic field. Are there any questions from last time? No, so we are now going to move to a model problem, solving a model problem.

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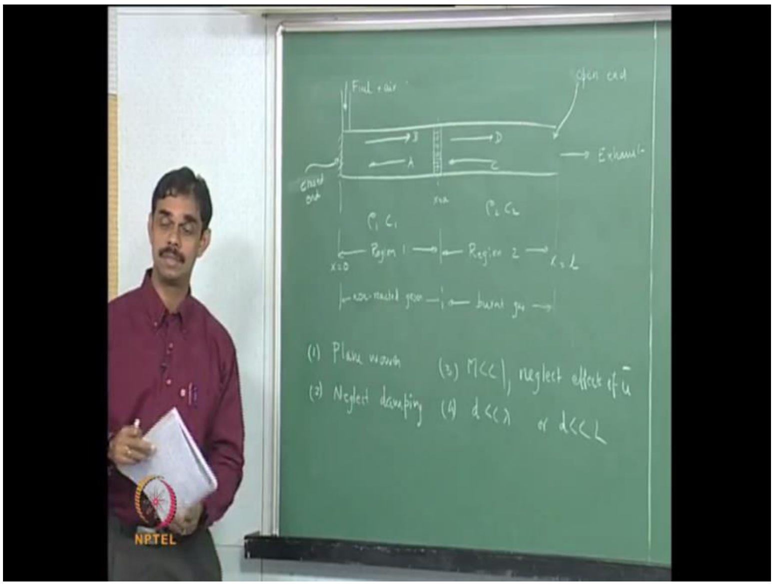

Whereby, we can actually proceed to use our information about the waves A B C D, the left and right running waves on both sides of the plus we include some kind of expression for the heat release rate as the function of the acoustic field and then we will try to whether we can calculate the growth rate and the frequencies. So, acts like the simplest problems and I have drawn a, so I have the geometry represented here. So, we consider a duct which is closed at one end and open at the other end. These are simple boundary conditions, that is the reason we are having it and we have a flame holder here.

It can be a wire mesh or grid something like that at x equal to A and the flame is sitting on flame holder.

You divide the region into 2 region, there is a region one to the left of the flame which is cold unreacted gases and there is a region to the right of the flame which we call region 2, which has burnt gases which are hot and there is no unsteady heat release to the left of x equal to A or to right of x equal to A. All the unsteady heat released is compacted here and we are assuming a compact heat releaser, so on either side we can use the solutions to the acoustic field, which we already derived, but the we will have to match the solutions for pressure and velocity across this unsteady heat is released.

So, that is the matter so we will we are of course, making some assumptions in addition to the earlier set of assumptions, so we are looking at plane waves, so we say that our frequency is quite low that there is no radial mode or tangential mode setup and its only the plane waves that are there. We are neglecting the dissipation of the acoustic field, so we are not really accounting for damping here and we also neglect the mean flow.

So, and the flame length is much smaller than the acoustic wavelength, so D is much less than lambda or we can also say that D is much less than l, both kind of mean the same thing because l is the order of lambda. So, given this assumption, we can write expression for the wave propagation on the left of the flame and the right of the flame and then we can do the procedure which I just described.

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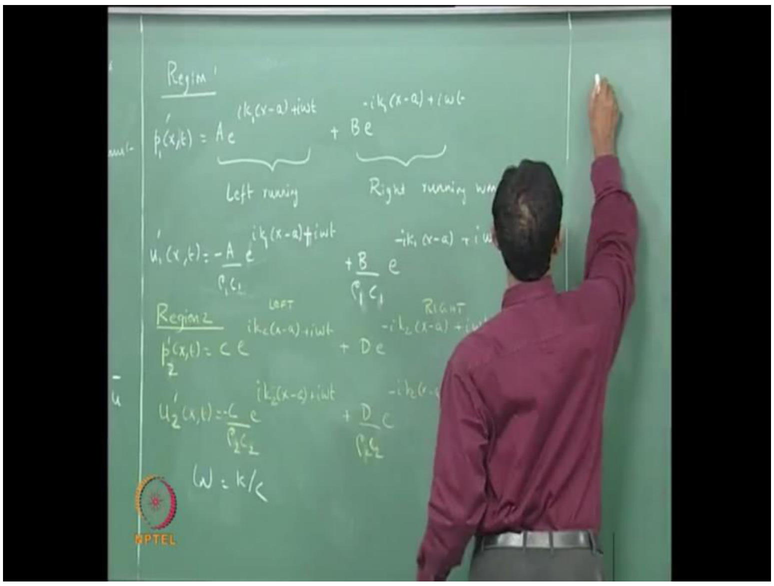

So, we will first examine region 1, so the pressure there you can also, this as A e power i k x, but I have put i k x minus just B e consistent with the notation of and also, it makes the expressions look final expression make simpler or prettier. So, on otherwise you do not I mean you can choose your x equal to 0. Anywhere, but here we are choosing x equal to 0 at the left end, if you choose it as the flame then you have some other expressions, you can choose whatever you want any reference is ok.

So, this is, I mean A and B gets adjusted accordingly. So, this is the left running wave and this would be the right running wave and lets call this p 1, so all the things in region one will denote by the sub script one, things in region 2 we will denote by sub script 2. So, for left running wave what did there relation between the pressure and velocity, you divide the pressure by minus rho c and you get the velocity for a right turning wave. You divide by plus rho c, so well I should say k 1 to be precise to represent region one.

Similarly, we can write expression for region 2, so you will say p prime of x comma t equal to c e power i k 2 x minus A plus i omega t plus D e power minus k k 2 into, so again this is left and this would be right. You can say right presume this is, so far and we can say that omega equal to k or c omega is angular frequency which is 2 pi tenth f and k is like the wave number, with this omega over c sorry, i is missing in your right hand .

Now, we need to apply the boundary condition, so we know the boundary conditions are, we know that the boundary conditions are, this is the closed end. So, here velocity will be 0, this is a open end, so pressure should be 0, so we can apply that. So, let us say x equal to 0, this velocity.

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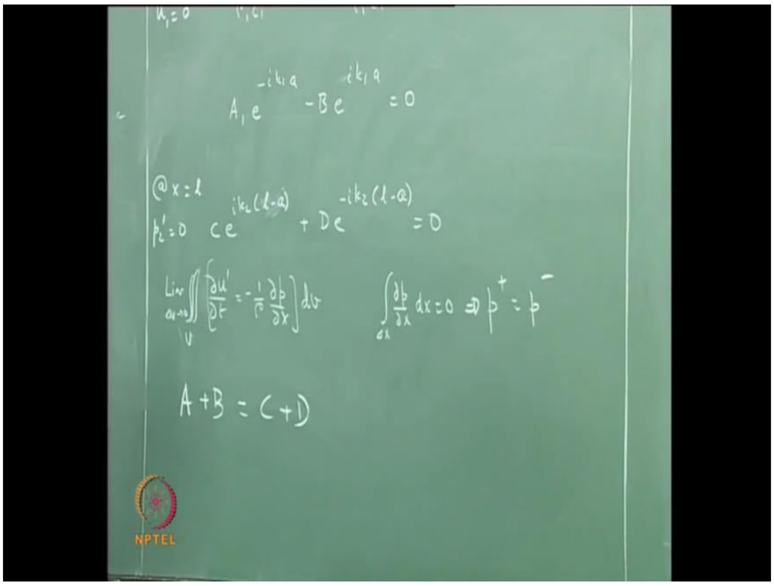

So, it is u on there, so you will get minus A 1 over rho 1 c 1 e power minus i k 1 a plus B over rho 1 C 1 e power sorry, it is ok or I can say, A 1 into and, so the next condition is at x equal graph here u 1 prime 0, so here p 2 prime equal to 0. So, we have to take this expression here and put x equal to l, so you will get C e power i k 2 and x minus sorry, l minus a plus D e power minus i k 2 into l minus a equal to 0.

Hope, this is no, so we have 2 boundary conditions and then we will have to enforce 2 jump conditions. So, just to look at it, so if you integrate this over the distance volume or distance whichever way limit delta v tends to 0 d v. Of course, you can rewrite this one dimensional integral and then you will get the first term is a finite quantity integrated over a in front decimal volume. So, that will go to 0 and this will come from the flux terms, so you will get p plus equal to p minus because integral d p is p, so p 2 minus p 1 or p plus minus p minus equal to 0. So, this will lead to a very nice condition, so at x equal to a. You will get this x minus a equals to 0 you will get A plus B equal to C plus D. Next, we have the jump condition for velocity, so and here we have to worry about the effect of heat release rate.

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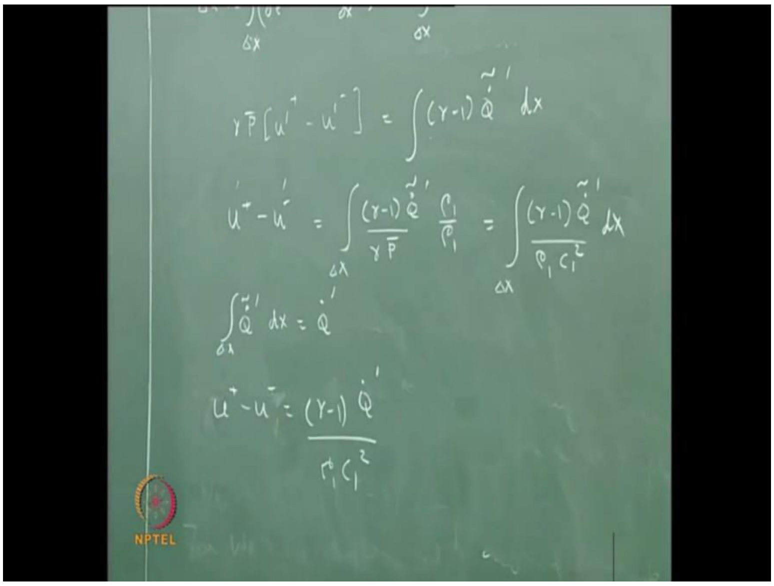

So, if you do that we will say, well I note this, i missed the primes here, so when you take the limit of delta x tends to 0, this first term will drop of because again it is a finite quantity integrated over a infinite decimal volume. So, this will give gamma p bar equal to integral r minus 1 Q dot prime d x, so you can write u plus minus u minus x multiply top and bottom by rho 1 and this should give, this is let me put tilde over here, so that I can reserve the Q dot for, so this integral q dot prime tilde dx i will call it Q dot over the heat released.

So, then I can say, u plus minus u minus equal to rho minus 1 Q dot prime over rho 1 c 1 square hope this is any questions? So, we can include damping. Then, you will have to have some kind of admittance condition here and admittance condition here and you will have to write a modified solution which has damping and so on. I just wanted to climb up claiming to be close to the reality or something we are just working out a model problem. So, in this case, the end result to lets fast forward and look into end result, end result will be that under some time delays the flying will actually drive under some delays flying will damp, but the region of will be more please note that, ok?

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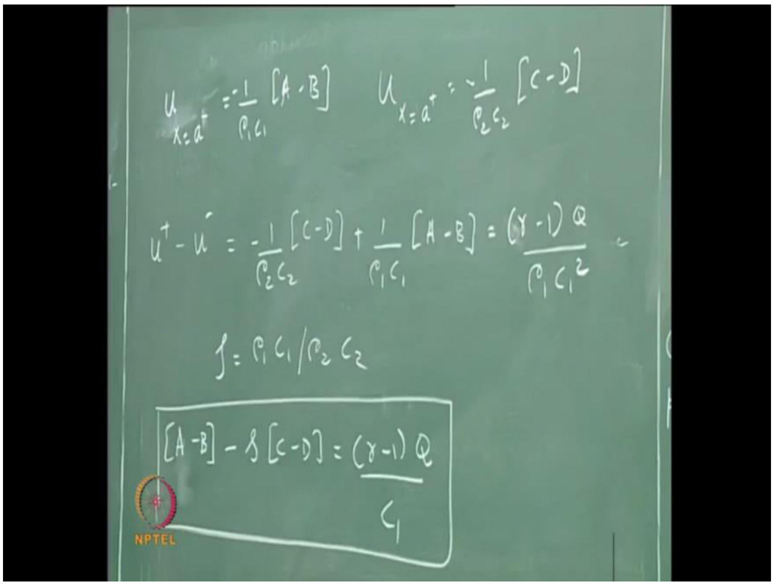

So, if you substitute x equal to a in the expression of velocity, I just erased it, you will get u at x equal to a minus, that is in this region we write the solution in terms of A and B. So, minus rho 1 c 1 into A minus B, just to the right to this plane, we will have minus 1 over rho 2 c 2 into C minus D, is that? So, We have to take the jump in velocity u plus minus u minus this will be, this is c plus, so minus minus is plus. I will say that Q dot prime equals to Q e power i omega t. We are in the harmonic domain and, so we just put that in. So, let us define a new symbol zeta which is equal to rho 1 c 1 over rho 2 c 2.

Then, I can say A minus B minus zeta into C minus D equal to r minus 1 Q over c 1, so we really have 4 equations. Now, we have this equation number 1, this is equation number 2, this is equation number 3 and this is equation number 4. How many unknowns are there? What are they? Now, we know Q dot or we have a relation for Q dot in terms of A B C D. That is, there we will come to know that we say, I mean, I already told that we will express Q heat list it in terms of velocity. So, that is not a unknown, I mean we have a coefficient there what are the unknown A B C D, is that all? Sorry, K1 k 2, K 1 k 2 and between k 1 k 1 there is a relationship k 1 and k 2, because temperature is there.

So, you can, it is enough to count all of them, so A B C D and k k 1 or k 2, so there are five unknowns and there are 4 equations. So what is the? Is it ok? I mean 5 unknowns and 4 equations. We are actually solving for omega right, so k 1 and k 2 can be written in terms of omega, so A B C D and omega is that to have 4 plus 1 5 unknowns and equations what do you catch here? sorry. we cannot solve for the no, we can solve for them. What kind of theory are we using? what kind of equations are we using?

That means for a omega we are using no. We do not know omega, we have to solve for omega. I think there is a very important message which we discussed when we looked at 1 problem when we are looking at some admittance one end and we converted that to growth rate. There we discussed this, I just wanted to go back and think what theory is it? What kind of theory is this? I mean what kind of equations are this all non-linear equations, linear equations, linear equations then linear theory.

It can be predict amplitude, can we lead to amplitude prediction in linear theory like if sin k x is a solution will 2 sin k x will it also be a solution by definition. It will be linear solution f of x is a solution 2 x is also a solution f of, what is the definition of linear theory. If you have f of x plus y equal to f of x plus f of y f of alpha x equal to alpha times f of y. So, if you have a solution you can multiply by any number and it will still be a solution. So, the best we can get is you can write between relations between A B C and D.

You can write, let us say C D and e or c B C and D in terms of A or A B and C in terms of D or A C and D in term, so f b or something like that you really can get all of them. You should not get all of them, if you get it is a problem right because we do not have any mechanism in which non-linearity is acting, but definition we have a linear frame work is it clear. So, we have to get 1 equation less not making any sense. Let us go back to the close open tube or something where we study can you take look at the note.

What was the solution? Can you please look at your notes and you have that closed open classical quarter wave tube or something we got cos k x something like that as a solution right and 2 a cos k x and we could not solve for a. There was A B and omega there, right? So, 3 variables and we had how many boundary conditions 2 boundary conditions and we actually got expression for omega we got c by 4 l 3 C by 4 etcetera, those where the roots of omega, those were Eigen values only for those omegas.

You will have the solution and then between A and B we got a solution, we got something like A equal to B or some other tube A equal to minus B, so it is clear that we cannot get all 3 if you get, if you get you can get amplitudes in a non-linear frame work, but not in a linear frame work, but what you can get is relation between the waves all in terms of 1 or the relative amplitude between them and the omega which should be the Eigen value and this are Eigen values is this concept clear. Now, say yes or no.

So, that I can proceed further you understood, so the catch here is to proceed. We still have a block that we do not know, heat release rate and we have to have a function for Q prime right. That is what you guys said right at the beginning that is absolutely perfect and determining the functional form of Q prime as a function of A B C D which is Q prime depends on the acoustic feel. Let us say fluctuating heat is depend on the fluctuating acoustic field, that is a reasonable assumption to make. I think it is probably the truth, so the acoustic field is expressed in term, so of A B C and D. So, we have to write Q prime in terms of A B C and D and there must be a.

There is a really complex relationship and it is really not easy to determine the functional form or anything other, so and from the more we may not be able to model the heat release rate as the function of acoustic waves alone, in reality nay combustor will have flame holding effects and there will be flow. For example, even if you have the flame stabilized on a burner. You would have the fact that flame stabilizing means the burner is holding flame that means the flow at the tip of the flame is playing a role, flow and heat transfer is playing a role in holding the flame there and an so on because there is a little recirculation region where the flame holds on to and so on.

In reality all that is very complicated to model and you really have a 2 scale power as I mentioned here, where over this scale the acoustic variables act and over this scale hydrodynamics acts and you have to really separate out the scales and write a 1 set of equations as we have done for the acoustics. Another set of equations for the hydro dynamics, but all that very complex and we cannot get a simple solution. The idea is to somehow make the problem tractable and get a tractable solution. So, we make, we make some simplification about the nature of Q prime and we write down the solution.

So, we use what is called Crocco's n term model or sensitive time lag model and we use that to model Q prime. I am going to erase all of this. is that? I hope you have written down the equation 1 2 3 4, ok?

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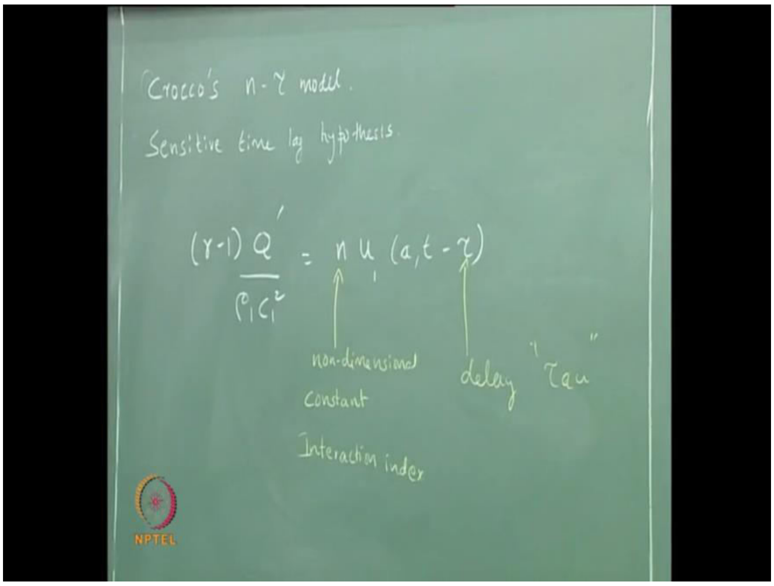

We are having one dimensional problem and in reality the hydro dynamics or the combustion zone will be quite 2 D, but we assume it 1 D. We say that the heat release is proportional to velocity at the flame hold region, but delayed by some time delay and there is a scaling factor. So, the scaling factor here is this n and the delay is tau, so that is the whole idea. So, we will take this term gamma minus 1 Q over c 1 and we model it with this hypothesis, so the why it is written is, so traditionally it is written as the velocity at the cold side.

So, we will write a and t minus tau t minus tau is a time, but it is not the, not actual time at the time, but it is a delayed time and n is a non dimensional, non dimensional constant. So, this is called interaction index and this is the delay tau, so tau is the time delay between velocity fluctuation and the heat delay, so velocity fluctuates now, but we are not using that velocity the heat release depend the velocity fluctuation at some time earlier, that is because this fluctuations will any process as a delay. For example, if you are injecting droplets there droplets, so it is some velocity acted and then that increase the evaporation rate and then it burn.

So, all these things take some time to happen, so that the time delay after which you are having the heat release occurring. If you are having a velocity fluctuation over a heated cylinder it takes certain time for the distribution fluctuations to be felt by the cylinder because they delay at the boundary layer and after that delay the heat will be released.

So, it is like if you are writing exam today, you should have studied yesterday or something like that. What you studies today will affect the exam tomorrow.

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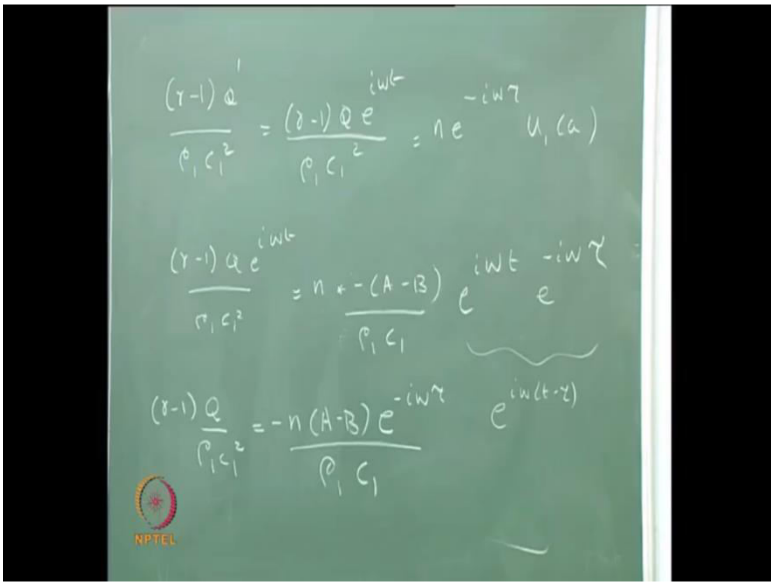

Let us say, so there is always a delay, so we also know that if you are in the harmonic domain we can write everything in e power i omega t kind of thing, but you can also do this kind of assumption in time domain also and people have done that. So, we can this is e power i omega t minus tau and this is recursion in front A minus B and minus n over rho 1 c 1 sitting in front. So, we can simplify this and say we can use this expression, we can use this expression for Q right here.

So, if we are write A minus B minus zeta into C minus D equal to minus l into m over b A minus B everything else goes. You get see there you can cancel the rho and c 1 and there is a c 1 left that will go with them, you will get an neat expression. So, all this factors here are put, so that you can get a simpler expression here that is all, so this would be the new equation 4, is it? So, let us that is let’s collect our equation and write them together and take a look at them.

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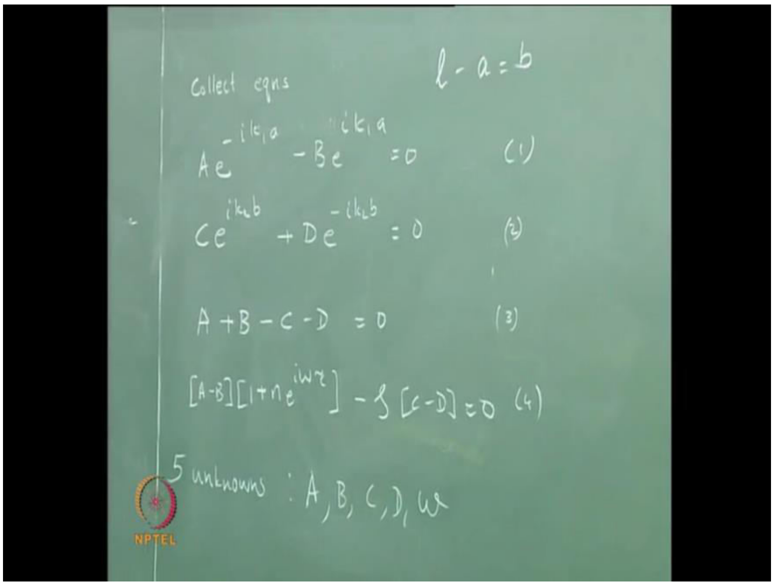

So, A e power minus i k 1 a minus B e power plus i k 1 a equal to 0. This was our equation 1, C e power i k 2 into l minus a, so we will call l minus as b it is for convenience. So, let us write this down here, let us say l minus a equal top b, so that I have less terms to write and then I have A plus B equal to C plus D. So, I will write this as A plus B minus C minus D equal to 0 and then the last equation is here A minus B, so I bring these terms over the left side and club it together with A minus B, so I will get A minus B. I will take it out and I will say, 1 plus n e power i omega tau minus. The second term zeta into C minus D equal to 0.

I have 4 equations and as I mentioned, there are 5 unknowns, we have they are A B C D and omega. Here, it is in terms of k 1 and k 2, but both can be written in terms of omega and we know c 1 and c 2, so k over c 1 c 2 is omega over c 2.

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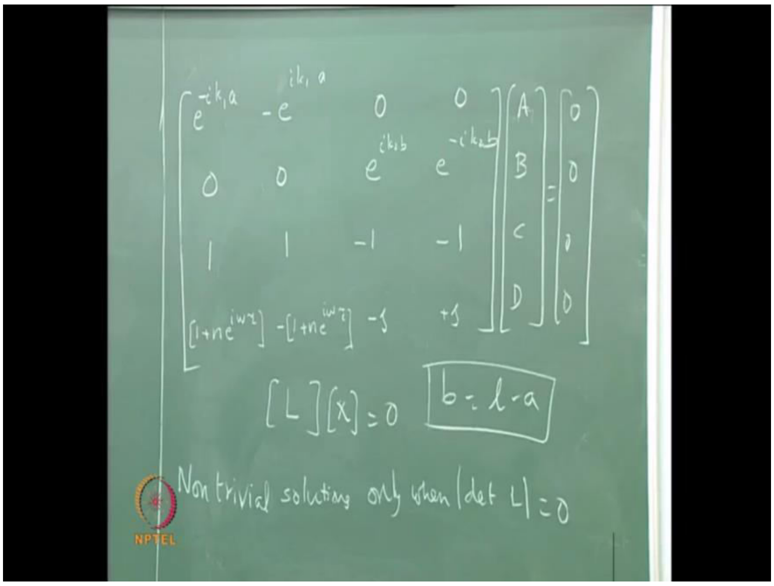

So, we can write this in a neat form e power i k 1 a minus e power. So, we get a nice expression in matrix form, but we can note that the entire right hand side is 0. Yes, is there a mistake here, right? That is a mistake here anything else we will check sorry, this thing should have the reverse because one is e power minus i k and the other one is l minus a comes here 0 minus a. So, l minus a is b 0 minus a is minus is that any other problem? There could be mistakes, so we have a matrix, let us say l times your variables, so our variables are contain this letter type or this vector type.

Now, if you will have non trivial solution only when the determinant of l is your the determinant of this big matrix is 0, so you can try to evaluate the determinant you can first expand it in terms of 3 by 3 matrices and then you can do the by 2 and try to get the determinant. I worked it out, you can try if you are very fast in algebra, I think, I will just write the first step and then I will leave it otherwise it is too boring in case you missed b is actually l minus a? We are writing that for convenient.

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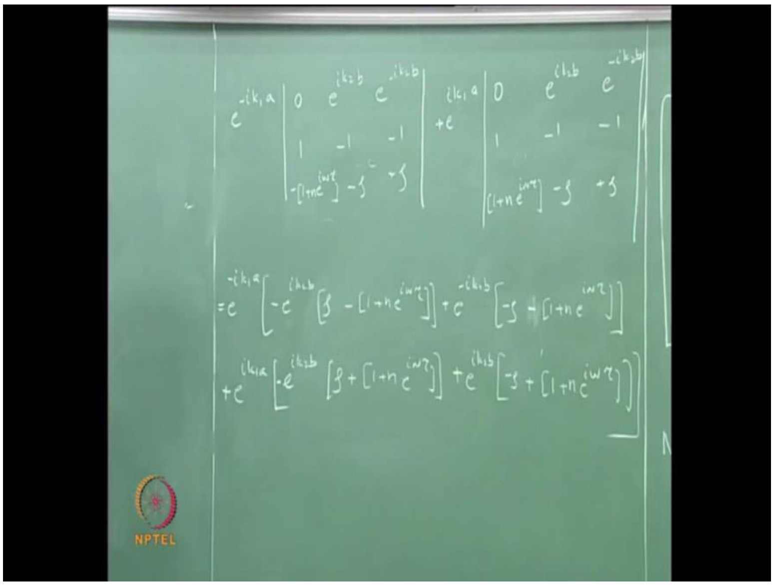

So, if we expand this, you will get e power minus i k 1 a times 0 e power i k 2 b e power minus i k 2 b 1 minus 1 minus1 and minus 1 plus n e power i omega tau plus, minus minus is plus, e power i k 1 a and 0 1 1 plus n power i omega tau e power i k 2 b minus 1 minus zeta e power minus i k 2 b minus 1 plus zeta. Now, you can write this is a e power minus i k 1 a times the first under that contribute times minus e power i k 2 b into minus minus minus.

So, it is check, if this is I guess, is this now you can pluck this terms, so you can does e power i k 2 b coming from here and e power minus i k 2 b coming from here and so you can use e power i k x plus e power minus i k x divided by 2 is equal to cos k x. Similarly, sin k x is e power i k x minus e power minus i k x divided by 2 i, so if you do that and if you simplify you will get.

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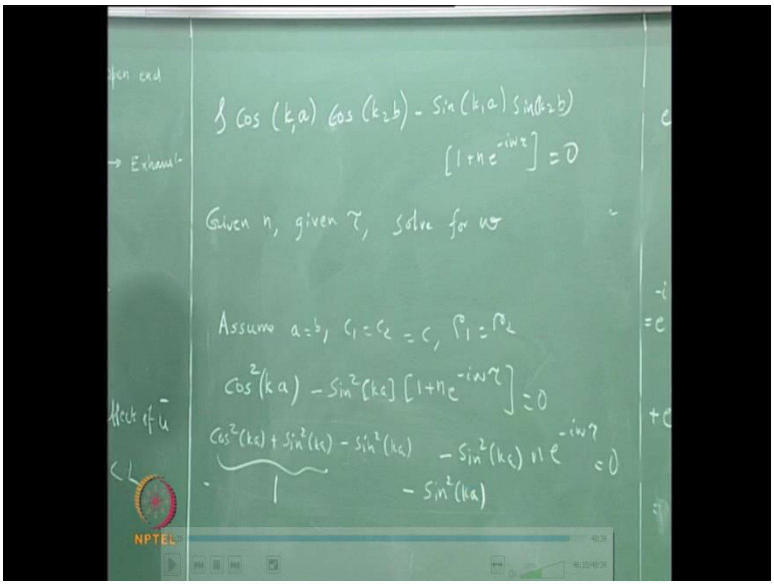

So, this is the answer I obtained, you can check this it should come straight forward. I got it in over 4 lines up to this. Then, I got this equation and we must remember that k 1 is omega over c and k 2 is omega over k 1 is omega over c 1 k 2 is omega over c 2. It is straight forward algebra just write it down, so for a given n and given tau should be able to solve for omega. So, this is the characteristic equation which you get, so you have to solve numerically, we also need the value of rho 1 c 1 over rho 2 c 2.

Then, you can solve numerically the computed program, but if you want to do things another degree which we can always decide because we can see things we can assume a equal to b c one equal to c 2 equal to c one rho one equal to rho 2. Then, lot of things simplify, but we can also get a solution for 2 over t 1 is equal to 4. If you think there is a temperature jam which is in a flame, there must be temperature jam and 4 is the reasonable value because you are having 300 degrees un reacted and 300 times for 200 in the, in the burns some, I think it is a reasonable number.

There also you can try to you can actually get a analytical solution. I will not work it out, but you can do it as home work. I will just do this case and we get analytical solution, but I want to emphasize that when you have t 2 over t 1 it is 4. We can literally get a solution may be I can ask this in the exam, so if you do this you will get what will you get. We can also assume that a equal to b that means that simplifies things further a equal to b really means that this plane is at the middle of the rocket, it just for simplifying this trigonometric formula. So, cos square k a minus sin square k a into 1 plus n minus i omega tau equal to 0. Now, we see if we can simplify this further, so we can add and subtract sin square k.

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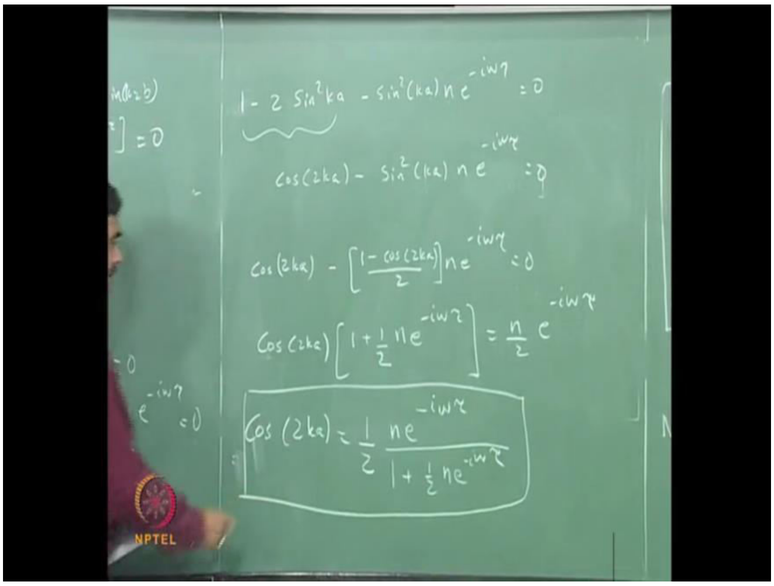

So, cos square k plus sin square k minus sin square k minus this could be 1 no. I missed one more, I missed this first term, so you will get 1 minus 2 sin square k minus sin square k n e power minus i omega tau equal to 0, it just take 2 more minutes. If you have a class right after this we can rush, so this would be what is this cos 2 k a minus sin square k a and minus omega tau equal to 0. So, we can write this in terms of cos 2 k a, so we can solve for this and say cos 2 k a into 1 plus half n i omega tau, so this is the expression form which we can get Eigen values. We will do that in next class.