Smart Structures

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Week - 02
Lecture No - 08

Analysis of composite laminate with piezoelectric patches (continued)

Welcome to the second lecture on Constitutive Modelling of Piezoelectric Materials.

So, in the last lecture we saw various approaches to model the constitutive relations and we started with electrostatics. So, today we will continue our discussion on electrostatics. So, we will start with Gauss law today it is a very important law. So, it says that if I have this electrical constant epsilon 0 and multiply that with del dot electric field that gives me the total charge and this total charge is summation of, I mean total charge within the volume that is the summation of the bound charge and free charge. So, this is called free charge.

$$\varepsilon_0 \vec{\nabla}.\vec{E} = \rho = \rho_b + \rho_f = -\vec{\nabla}.\vec{P} + \rho_f$$

$$\Rightarrow \vec{\nabla}\left(\varepsilon_0 \vec{E} + \vec{P}\right) = \rho_f$$

$$\int_S \vec{D}.d\vec{S} = Q_{free}$$

$$\vec{P} = \varepsilon_0 X_e \vec{E}$$

$$\vec{D} = \varepsilon_0 (1 + X_e)\vec{E} = \varepsilon_0 K \vec{E}$$

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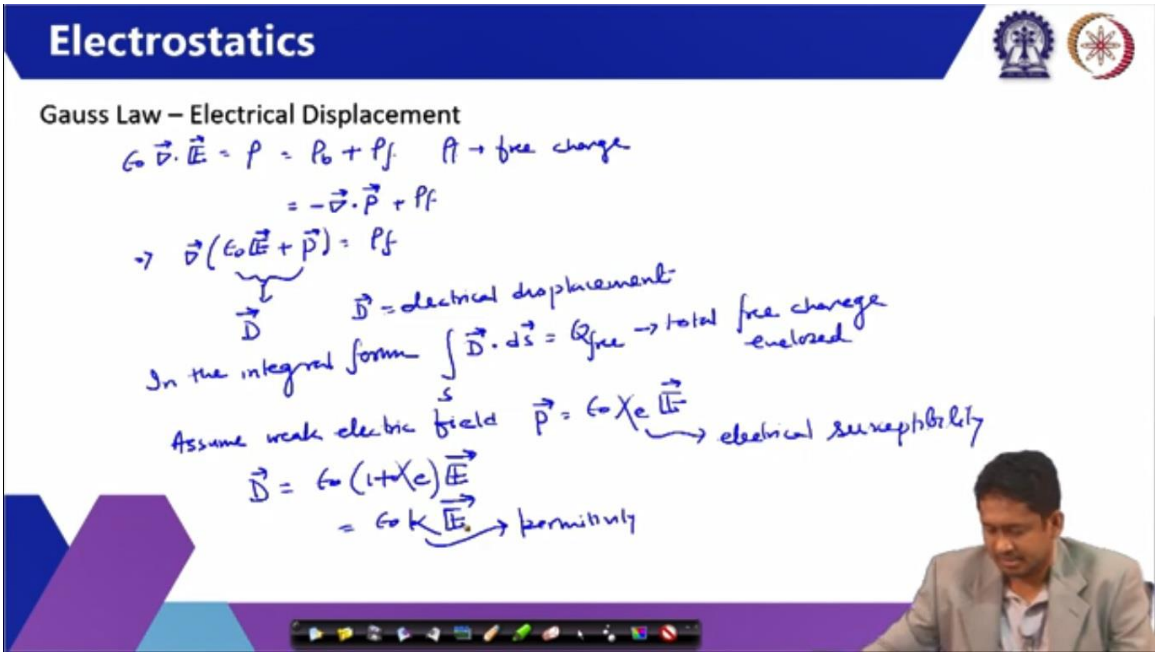

Now, our dielectric material does not have free charge. So, it is 0, but to be more generic we are writing this. Now, this as we know we saw it in the last lecture that rho b can be written as minus del dot p and plus rho f is here. Now, I can take it to this side and finally, we can write del dot epsilon 0 multiplied by electric field plus the polarization vector P and that is equal to rho f.

Now, this quantity gets a name and we call it electrical displacement. So, this is D is our electrical displacement and in the integral form this law can be stated as D dot ds is equal to total free charge. So, it is total free charge enclosed. Now, if we assume that electric field is weak and then polarization P can be written can be assumed to be proportional to the electric field. So, we can write this.

So, where this is called electrical susceptibility. So, this tells us how easily it can be how easy or how difficult is it to polarize this material. Now, we can combine these definitions and write the vector D as 1 plus susceptibility multiplied by the electric field and then we can give it a name and this we can called permittivity. So, finally, our electrical displacement D and electric field are proportional linearly proportional. Next, we will talk about force on a dipole.

So, let us imagine that. So, we will try to understand the force in a 2D setup and then we can generalize it to 3D. So, let us imagine that we have a charge q minus and we have another charge q plus. So, let us call it point 1, let us call it point 2. So, in the x y plane now the point1 is at a distance x 1 along the x axis from the origin and point 2 is at a distance x 2 along the x axis from the origin and there the distance between these two points is a.

So, let us define a vector a vector which joins point 1 to point 2. Now, this we can call a cos theta. Now, let us imagine that the electric field is oriented purely along the y direction. So, the value of the electric field at first point is E 1 y the value of the electric field as a second point is E 2 y. So, we can write the electric field to be electric field vector to be E as E y multiplied by j.

$$\vec{E} = E_y \hat{j}, \qquad F_{1y} = E_{1y}, \qquad F_{2y} = E_{2y}$$

$$F_y = F_{2y} - F_{1y} = qE_{2y} - qE_{1y} = q(X_2 - X_1)\frac{E_{2y} - E_{1y}}{X_2 - X_1}$$

$$= qa\cos\theta \frac{dE_y}{dX} = \vec{p}_x \frac{dE_y}{dX}, \vec{p} = q\vec{a}$$

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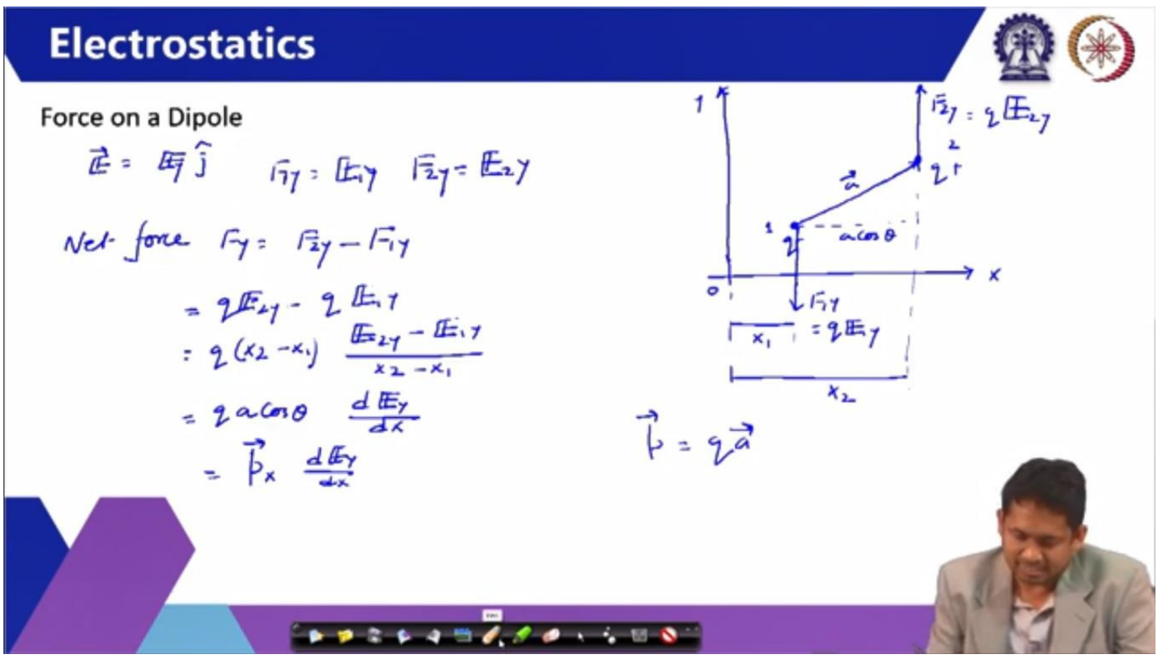

So, the force here is the force experienced by this negative charge is F 1 y which can be written as q multiplied by E 1 y and the force experienced by this is F 2 y which can be written as q multiplied by E 2 y. Now, so, F 1 y is equal to E 1 y and F 2 y is equal to E 2 y. Now, if if the field is uniform that means, E 1 and E 2 are same in that case the net force is 0 because these two cancels each other if they are not uniform then there is a net force. So, the net force is F y equal to F 2 y minus F 1 y which can be written as this and then we can rewrite this expression as x 2 minus x 1 divided by E 2 y minus E 1 y divided by x 2 minus x 1. Now, this x 2 minus x 1 is just this and which we can write as a cosine theta.

So, we can write this as q multiplied by a cosine theta and if we assume that x 2 minus x 1 is very small then this expression can be written as d E y by d x. And now this q a cosine theta this can be written as the x component of the dipole moment vector and this is the expression. So, we defined dipole moment vector type as q multiplied by a. So, its x component is q multiplied by the x component of a vector which is a cosine theta. So, it is p x and that is our next net force experienced by a dipole under the assumption that the electric field is oriented in the y direction.

Now, this entire expression can be generalized for a 3D case. So, without restricting the electric field to be acting only along y axis we can generalize it and the final 3D expression without any assumption becomes p x dE x dx plus p y. So, this is the x component of the force plus p x is J component plus again p x p z multiplied by the derivative of E z with respect to z multiplied by uni vector along z direction. And this can be written in a more generic form as I mean in a compact form as this. So, this force has 3 components along x, y, z and which we wrote in a compact form as this.

$$\vec{F} = \left(p_x \frac{dE_x}{dx} + p_y \frac{dE_x}{dy} + p_z \frac{dE_x}{dz}\right)\hat{\imath} + \left(p_x \frac{dE_y}{dx} + p_y \frac{dE_y}{dy} + p_z \frac{dE_y}{dz}\right)\hat{\jmath}$$
$$+ \left(p_x \frac{dE_z}{dx} + p_y \frac{dE_z}{dy} + p_z \frac{dE_z}{dz}\right)\hat{k}$$
$$= (\vec{p}.\vec{\nabla})\vec{E}$$
$$\vec{f_p} = (\vec{p}.\vec{\nabla})\vec{E} = B_{ex}\hat{\imath} + B_{ey}\hat{\jmath} + B_{ez}\hat{k}$$

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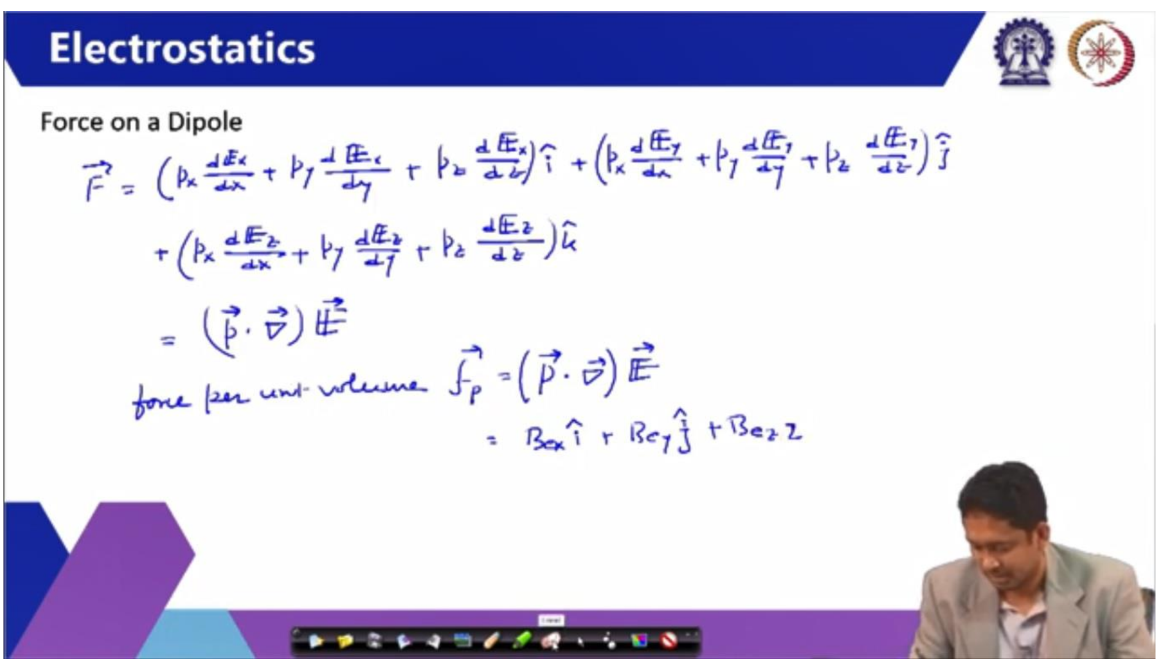

Now, let us imagine that there is a volume that contains these dipoles. So, force per unit volume in that dipole can be written as capital P here we will use capital P instead of small p because now we are talking about a volume containing dipole and we are concerned with unit volume. So, it is this now similarly this also can be written as a component along x component along y and component along z. So, B ex, Be y and B e z are the component of these forces along x y and z. So, now, we will go to moment on a dipole.

So, for this we are not going to derive the expression by using similar techniques we can find out the moment experienced by a dipole as. So, we will directly write with for moment per unit volume and that becomes m p vector is equal to P cross E and again it can have 3 components. So, we may call them as m e x into I m e y into j plus m e z into k. So, that this brings us to the end of electrostatics now we will go to the conservation laws which will help us getting the constitutive relations. So, first is mass conservation.

$$\vec{m}_p = \vec{P} \times \vec{E} = m_{ex}\hat{\imath} + m_{ey}\hat{\jmath} + m_{ez}\hat{k}$$

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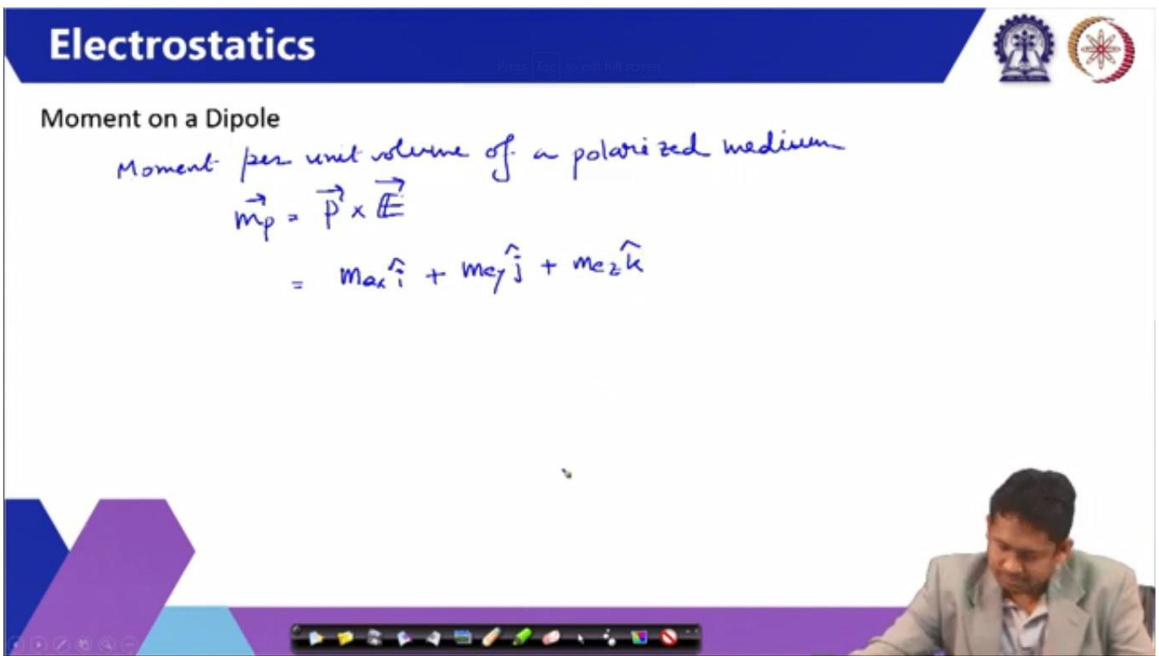

$$\rho\vec{\nabla}.\vec{V} = 0 \Rightarrow \vec{\nabla}.\vec{V} = 0, \vec{V} = \dot{u}\hat{\imath} + \dot{v}\hat{\jmath} + \dot{w}\hat{k}$$

$$\left(\frac{\partial \dot{u}}{\partial x} + \frac{\partial \dot{v}}{\partial y} + \frac{\partial \dot{w}}{\partial z}\right) = 0$$

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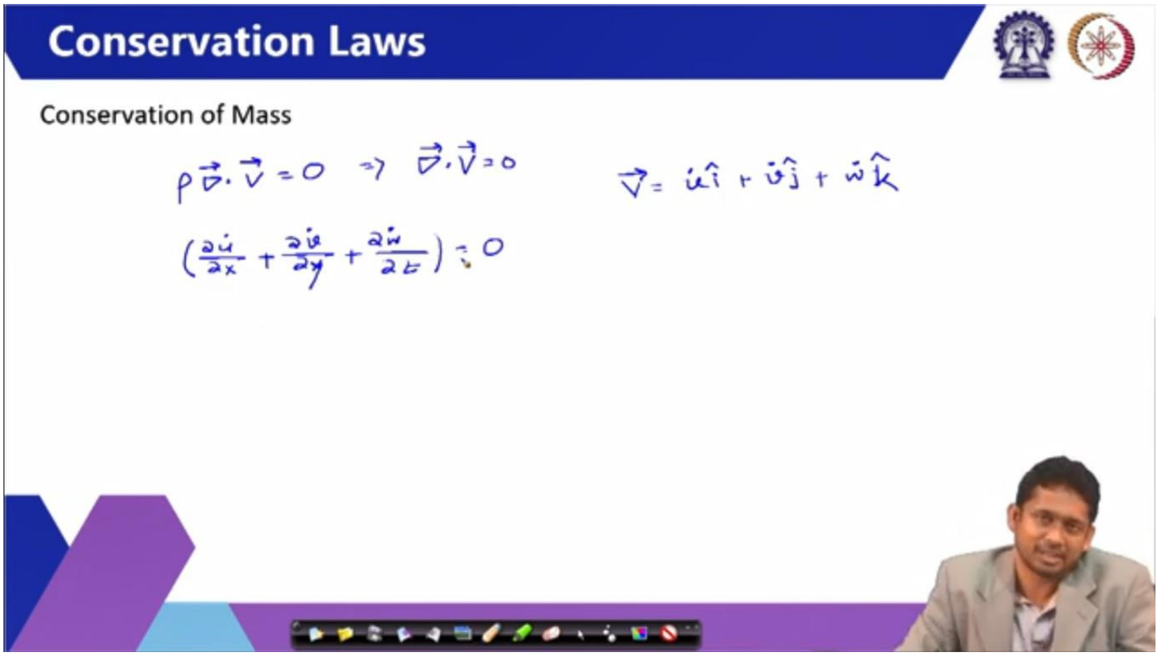

So, under the assumption that our density does not change the mass conservation law is simply this rho into del dot V 0 or we may write del dot V is 0. So, V is velocity vector which has 3 components u dot v dot and w dot. So, this x this can be rewritten in this way if I just evaluate the divergence this becomes this. So, del u dot by del x del u v dot del v dot by del y and del w dot by del z their summation is 0. So, u, v, w are displacement along x, y and z and their derivatives with times are the other velocities along x y and z ok.

Next is conservation of linear momentum. So, let us consider a 2D body like this. So, this is our x direction this is our y direction. Now this 2D body is of length delta x its length along x and delta y is a dimension along y. So, this is a negative x space where there is a normal stress sigma x this is the positive x space where there is a normal stress sigma x x plus some increment the increment is del sigma x x by del x into delta x and there is a shear stress and shear stress with some increment this is negative y phase where I have normal stress and positive y phase is normal stress with some increment.

$$\sum F_x = 0, \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\Delta x\right)\Delta y - \sigma_{xx}\Delta y + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y\right)\Delta x - \tau_{yx}\Delta x + B_{ex}\Delta x\Delta y = P\Delta x\Delta y\frac{\partial \dot{u}}{\partial t}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + B_{ex} = \rho\frac{\partial \dot{u}}{\partial t}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + B_{ey} = \rho\frac{\partial \dot{v}}{\partial t}$$

$$\sigma_{ji,j} + B_{ei} = \rho \frac{\partial V_i}{\partial t}$$

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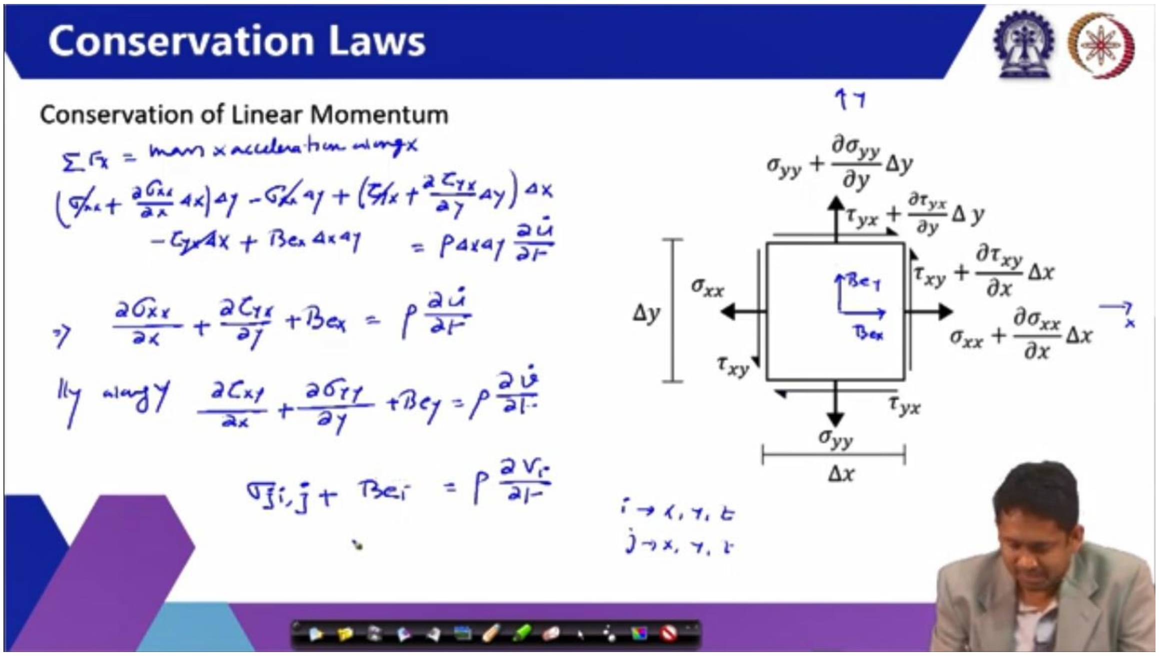

Similarly, I have some shear stress and shear stress with some increment and apart from that if it is a polarized medium under electric field there can be force B e x and B e y we derived it and saw that due to the interaction between electric field and polarization there are forces x component is B e x and y component is B e y. Now if we equilibrate it if we say that summation of forces along x is 0 then the equilibrium condition gives us that sigma x x plus del sigma x x by del x multiplied by delta x and this multiplied by delta y. So, this is the stress and this stress is multiplied by this length. So, that gives us a force along x direction similarly there would be contribution from this stress and that is sigma x x multiplied by del y plus we have tau y x plus del tau y x by del y multiplied by del y into delta x. So, this is a shear stress and it is oriented towards the x direction.

So, this stress multiplied by this dimension and minus we have tau yx multiplied by delta x plus we have to add this body force component along x direction. So, it is B e x del x del y now please understand that we are here dealing with only 2 dimensions. So, the dimension along the third direction which means the perpendicular to this plane is 1. So, this is our total body force and that is equal to 0. Now if you simplify this cancels and this and this cancels and we are left with this equation and that is equal to 0.

$$\tau_{yx} - \tau_{xy} = \rho_x E_y - \rho_y E_x$$

$$\tau_{yx} - \tau_{xy} = P_x E_y - P_y E_x, \tau_{zx} - \tau_{xz} = P_x E_z - P_z E_x$$

$$\tau_{zy} - \tau_{yz} = P_y E_z - P_z E_y$$

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

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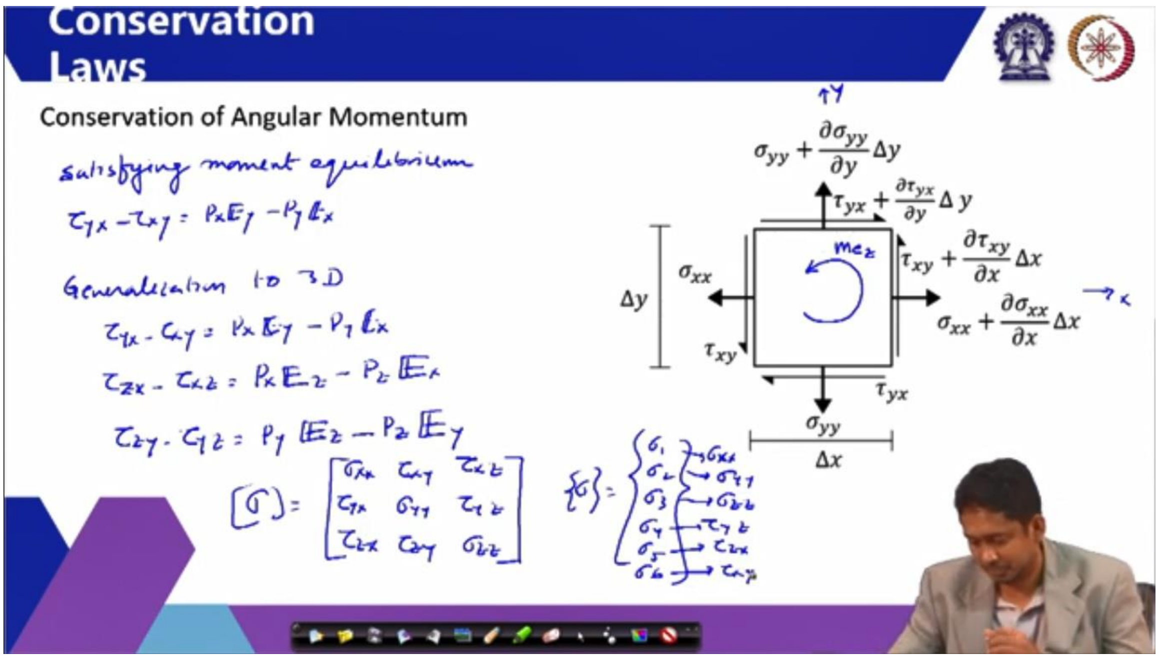

Now this was with the assumption that there is no acceleration. So, if this body is under some acceleration then this summation of these forces instead of being equal to 0 should be is equal to mass into acceleration. In that case we have to modify this as mass multiplied by acceleration along x and then right hand side we will have total mass total mass is rho into del x del y because the dimension along z is unity and that multiplied by del u dot by del t and similarly we will have the same thing here. So, rho del u dot by del t. So, that is our conservation of linear momentum along x direction.

Similarly, along y this equation can be written as tau x y by x a partial of tau x y with respect to x del sigma y y by del y plus d y is equal to rho del v dot by del t. Now this can be generalized to 3D. So, in 3 dimensions if we want to write it in a compact form using the indicial indicial notation we can write this as. So, this this is the equation for the ith direction. So, here it can be x ,y ,z similarly j also goes from x y z.

Now, here we considered only one body force and that is due to the polarization and electric field. Now if there are any other source of body force B i that can be from any other source in that case it has to be added to it. So, this was about conservation of linear momentum. Now we will go to the next one and look into the conservation of angular momentum. So, we have the same diagram and here so, this is our x and this is our y.

Now, here apart from this we will have a body moment which is m e z. So, this is x this is y and z direction are perpendicular to the plane. So, with that direction the moment is m e z. So, that is also going to be there. Now if I first satisfy the moment equilibrium.

So, after satisfying moment equilibrium we get this. So, the way we satisfy the force equilibrium we can satisfy moment equilibrium also which we are not going to do in details we will only write the outcome of that satisfaction. So, it becomes tau y x minus tau x y is equal to P x E y minus P y E x. So, remember the moment experienced by this polarized medium is and unit volume of the polarized medium is ah P cross E. So, if I take the cross product and take its k component, we get the expressions in terms of in in terms of P x E y and so on.

So, this is what we get. So, we get the expression in terms of P x E y P y E x and this is what we get by satisfying it. Now if I generalize it to 3D, we get 3 of such equations we get 3 such equations and those equations are the first equation itself that we wrote and then we have tau z x minus tau x z is P x. So, we are we have to put E as the way we have put E. So, this becomes P x multiplied by E z minus P z multiplied by E x and then we have tau z y minus tau y z and that is equal to P y multiplied by E z minus P z multiplied by E y. So, that is about the conservation of angular momentum.

However, in our case we we we would neglect the non-linear terms appearing because of this polarization and electric field in both the angular momentum equation and the linear momentum equation. So, from the linear momentum equation we will neglect the B E terms assuming that our electric field is uniform and also here also you will neglect these terms. So, if we neglect these terms our stress matrix becomes symmetric. So, when we when we neglect these terms, it becomes tau y x equal to tau x y tau z x equal to tau x z tau z y is equal to tau y z. In that case the stress matrix becomes sigma x x tau x y tau x z tau y x sigma y y tau y z tau z x tau z y sigma z z.

Now, if this and this are if this and this are equal this and this are equal, we have 6 independent terms. So, that way we can simplify it and write the stress as a vector. So, it becomes sigma 1, sigma 2, sigma 3, sigma 4, sigma 5 and sigma 6 where this is sigma x x this is sigma y y this is sigma z z this is tau y z this is tau z x and this is tau x y. Now, we will talk about the thermodynamics laws. So, our constitutive relation should satisfy the thermodynamics laws as well.

$$\frac{dU}{dt} = \dot{W} + \dot{Q}$$

$$\frac{dU}{dt} = \sigma_{ij}\dot{\varepsilon}_{ij} + E_i\dot{D}_i + \rho\gamma - q_{i,i}$$

$$q_{i,i} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}$$

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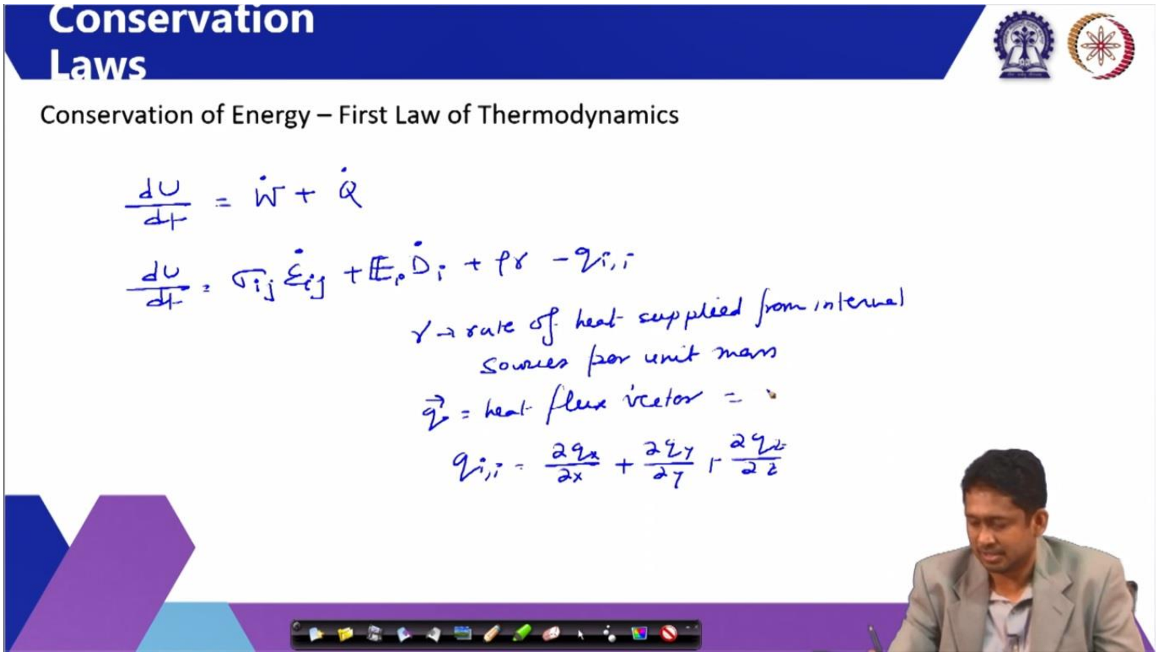

So, we will start with the first law. The first law in the rate form can be written as dU by dt is equal to W dot plus Q dot. So, dU by dt is the U is the internal energy and dU by dt is its time derivative and W dot is the rate of work done on the system and Q dot is the rate of heat supply to the system. So, for our piezoelectric medium we can write this as sigma ij multiplied by epsilon ij dot plus E as an electric field multiplied by D i dot plus rho gamma minus qi i. So, here are two more terms gamma is rate of heat supplied from internal sources maybe any internal source per unit mass. So, its per unit mass and q is our heat flux vector.

So, qi comma i just signifies del q by del x plus del q y by del y plus del q z by del z. So, it has three components qx, qy and qz and this quantity is this quantity is time derivative of internal energy per unit volume. So, here the work done comes from these mechanical quantities and the electrical quantities and this contributes to the heat. Now, the second law of thermodynamics written in the form of Clausius-Duhem inequality is Ts dot plus T multiplied by q by T comma i minus rho gamma which should be greater than equal to 0 and here S denotes entropy. So, with these two laws we will conclude the discussion here.

$$T\dot{S} + T\left(\frac{q_i}{T}\right)_{,i} - \rho\gamma \geq 0$$

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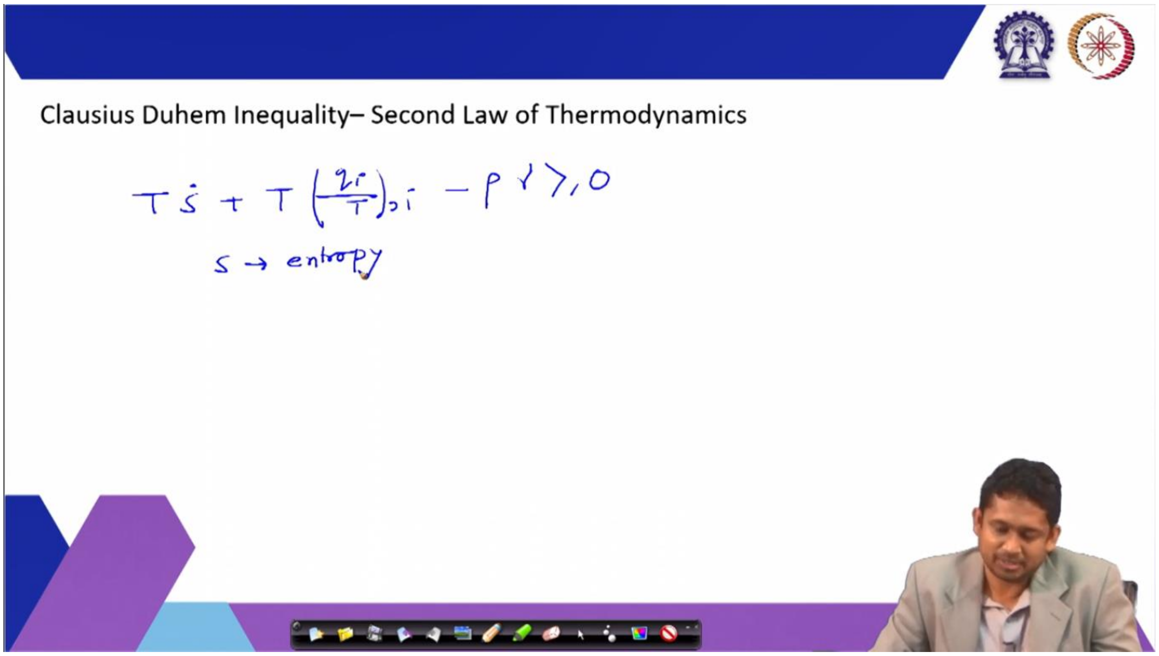

So, in the next lecture we would combine the two laws and gradually move towards the constitutive relation formulation with that let us conclude.

Thank you.