Smart Structures
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Week 12
Lecture No: 62
Analysis of a Beam with ER/MR Fluid Layer
Part 04

Today, we will talk about analyse of beam with ER and MR fluid layer.

Now, so far whatever we have discussed on the ER and MR fluids, they are different flow mode analysis. Those can be found in the book by Chopra and Sirohi that is our text book, and also the paper that we have referred to. For this discussion where we are analyzing beam with an ER and MR fluid, we would refer to this paper, that is referred here. In some cases, our way of formulation or notations can be little different from them just so that we are consistent with the kind of formulation that we have done so far. But the example and the development of the formulation is taken from this paper.

So, here we have a beam and this beam is a sandwich beam, sandwich in the sense that it has two layers at that two sides, top and bottom and the mid layer is different from the top and bottom layer. So, it has three layers. The top layer is an elastic layer, elastic means it is an elastic solid, the bottom is also an elastic solid, the mid layer, at the core of the mid layer, we have this red zone is ER or MR fluid. And this fluid is covered by a - I would say a box made of rubber. So, this rubber region we can see in the cross-sectional view in the x z plane that is at the periphery. Similarly, in the cross-sectional view in the x y plane, we also it is seen in the periphery. So, this entire rubber region which - so this is the wall of the rubber, the thickness of b_r. So, that rubber region contains the ER MR fluid inside it. Now, the dimension of the beam along the direction x is L, the thickness of the solid layers are - we can call h_1, h_2 and h_3. And then, we have here, the dimensions. So, if we look at the entire system in the x y plane, in a cross-sectional view, we can see that the width of the wall of the rubber is b_r, and the entire width is b.

So, when this beam vibrates, the rheological fluid, because of its damping, dissipates the vibration. So, that brings damping to it and that damping can be controlled by controlling the electrical, magnetic field in the rheological field. Now, in this analysis there are some assumptions - first of all the axial stiffness of the fluid layer is negligible. That means, this mid region, the core region, it does not have any axial stiffness. The only axial stiffness comes from this rubber region, but that is much less, I mean, the dimension of it is much less as compared to the dimension of the core fluid. So, this has negligible axial stiffness that also means that normal stress is 0. So, in the fluid region, normal stress is 0 and this thickness h_1, h_2 and h_3, they are much smaller as compared to L, and their sum h_1 plus h_2 plus h_3 is also quite small. So, we can have Euler Bernoulli assumption here.

So, we make Euler Bernoulli beam assumption - there is no slip between the elastic and fluid layer, and the elastic solid their damping is negligible and the transverse displacement is function of x only. So, we denote the transverse displacement as w. So, w is just a function of x, and that means, the quantity w does not vary along the depth or along the width. So, with these assumptions, we will do our analysis.

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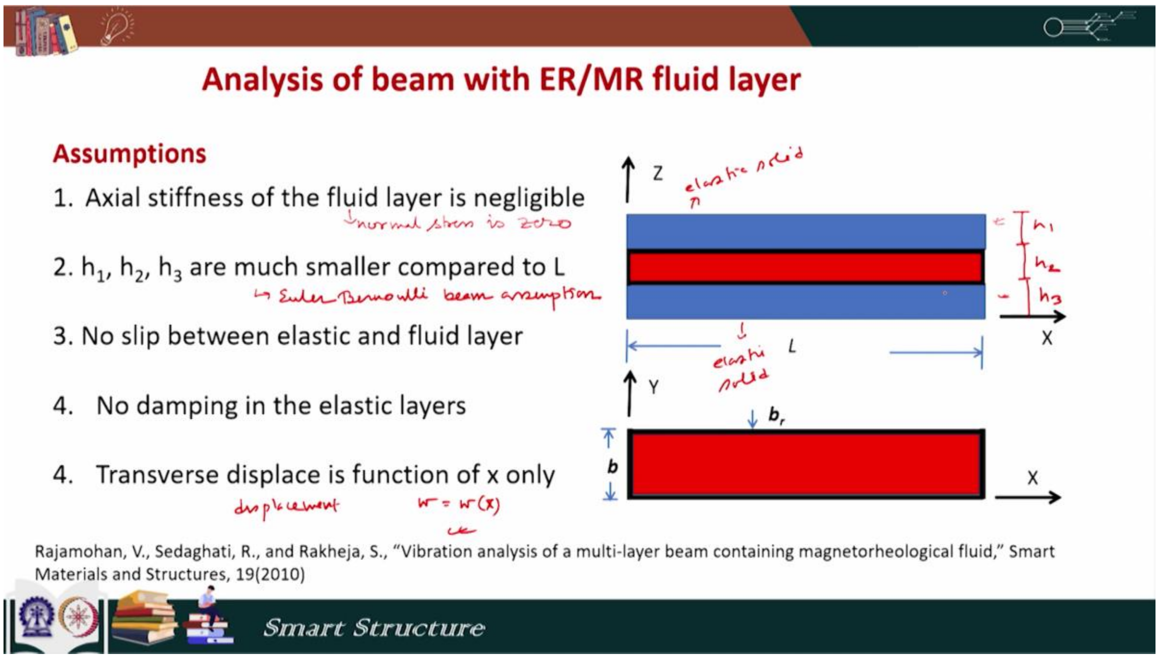

Now, from this kinematics, first we have to define the shear stress in the fluid layer. Now, the shear stress in the fluid layer is gamma, which we can say del w by del x plus del u by del z. So, that is the shear stress in the x z plane.

$$\gamma = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Now, del w by del x we know that remains constant along the cross section, del u by del z we have to find out in terms of some quantities. So, we find out del u by del z in terms of this w as well as the axial displacement, displacement along x direction at the middle of the top layer and the middle of the bottom layer. Now, let us denote u_1 as the displacement at the middle of the top layer.

So, at any x, u_1 is denoted as the displacement at the middle of the top layer and u_3 is the displacement at the middle of the bottom layer. So, u_1 and u_3 are displacements at the middle of top and bottom layers. If it is so, then the displacement at this junction here at the bottom of the bottom layer. So, the junction between the core and the top layer can be written as. So, we can say that - as we know this is as per our definition, this is h_3 and this is h_2, h_2 plus h_3. So, u at z equal to h_2 plus h_3 can be written as - u evaluated at z equal to h_2 plus h_3 is our u_1 plus h_1 by 2 multiplied by del w by del x.

$$u \text{ at } \quad z = h_2 + h_3$$

$$u|_{z=h_2+h_3} = u_1 + \frac{h_1}{2}\frac{\partial w}{\partial x}$$

Now, because this is a Euler Bernoulli beam, we know that the plane section remains plane before and after bending. So, if at any section, it has our 3 layers and if it bends, then after bending it takes a form like this, because we know that based on the Euler Bernoulli principle, plane section remains plane before and after bending. So, if the displacement at the middle of the top layer is u_1, then the displacement here is u_1 plus this additional amount and this additional amount is nothing, but del w by del x multiplied by this height and this height is our h_1 by 2. Now, because we are going towards negative z. So, we are adding it because we know that u as we go up along z direction is minus z into del w by del x, here we are going in the downward direction. So, this quantity is additive. So, we have u_1 plus h_1 by 2 into del w by del x.

If this same quantity was to be found out with respect to this, the mid plane of the entire beam, then it would be just minus of del w by del x multiplied by h_2 by 2, but here we are finding out with respect to the mid of the top layer. So, it is u_1 plus h_1 by 2 multiplied by del w by del x.

Similarly, u at z equal to h_3, if you want to find out that means, here and that if you want to find out based on u_3, then that becomes u at z equal to h_3 is equal to u_3 minus h_3 by 2 into del w by del x.

$$u \text{ at } \quad z = h_3$$

$$u|_{z=h_3} = u_3 - \frac{h_3}{2}\frac{\partial w}{\partial x}$$

So, we have found out u at the junction between layers 2 and 3 and layers 1 and 2. So, if I want to find out del u by del z now, then del u by del z is just the difference of u between here and here divided by the thickness of that layer h_2. So, del u by del x is u at z equal to h_2 plus h_3 minus u at z equal to h_3 divided by h_2. And if we do that, then the quantity that we get is - u_1 plus h_1 by 2 into del w by del x minus u_3 plus h_3 by 2 into del w by del x and then that quantity is divided by h_2. Now, we can put this. So, this is del u by del z, not del x.

$$\frac{\partial u}{\partial z} = \frac{u|_{z=h_2+h_3} - u|_{z=h_3}}{h_2} = \frac{u_1 + \frac{h_1}{2}\frac{\partial w}{\partial x} - u_3 + \frac{h_3}{2}\frac{\partial w}{\partial x}}{h_2}$$

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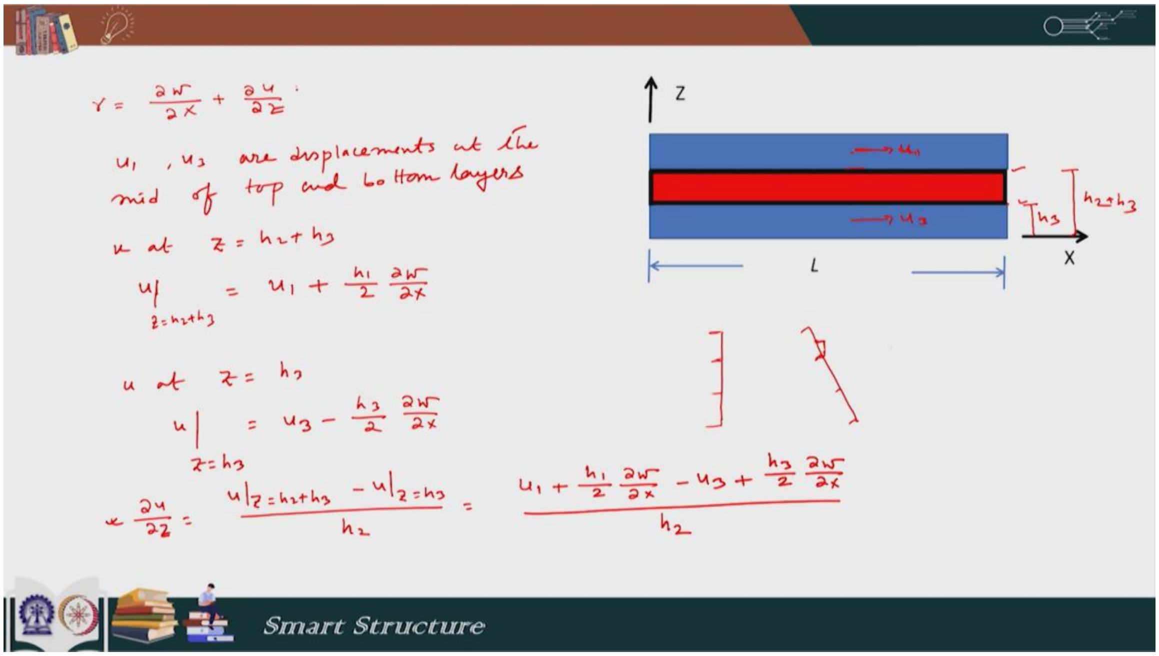

So, if we put that this del u by del z here, that gives me gamma. So, after putting that gamma becomes 2 into h by 2 plus h_1 plus h_3 divided by 2 h_2 into del w by del x plus u_1 minus u_3 by h_2. And then that quantity can be written as - D by h_2 into del w by del x plus u_1 minus u_3 by h_2, where D is defined as a quantity, which is h_1 plus h_3 by 2 plus h_2.

$$\gamma = \frac{2h_2 + h_1 + h_3}{2h_2}\frac{\partial w}{\partial x} + \frac{u_1 - u_3}{h_2} = \frac{D}{h_2}\frac{\partial w}{\partial x} + \frac{u_1 - u_3}{h_2}$$

$$\text{where,} \quad D = \frac{h_1 + h_3}{2} + h_2$$

Now, we have to find a relation between u_1 and u_3. So, that can be found out by balancing the normal forces along x direction. So, for that, we can write longitudinal forces along x in the elastic layers and those are F_1 equal to E_1 A_1 del u by del x, because we know del u by del x is strain, if I multiply that with the corresponding Young's modulus that gives me stress, and A_1 is the cross-sectional area of the first layer, the top elastic layer. And similarly, we have F_3 is equal to E_3 A_3 into del u_3 by del x. So, again A_3 is the cross-sectional area of the bottom layer.

$$F_1 = E_1 A_1 \frac{\partial u_1}{\partial x} \qquad\qquad F_3 = E_3 A_3 \frac{\partial u_3}{\partial x}$$

Now, we know that there is no externally applied normal force. So, F_1 plus F_3 should be 0, that is our equilibrium condition, and that gives me E_1 A_1 multiplied by del u by del x is equal to or minus of E_3 A_3 into del u_3 by del x. Now, we can integrate both sides and that gives us E_1 A_1 u_1 equal to minus of E_3 A_3 u_3. And then, we can write u_3 is equal to minus of e into u_1, where e_1 is defined as, small e_1 is defined as - capital E_1 A_1 divided by capital E_3 into A_3.

$$F_1 + F_3 = 0$$

$$\Rightarrow E_1 A_1 \frac{\partial u_1}{\partial x} = -E_3 A_3 \frac{\partial u_3}{\partial x}$$

$$\Rightarrow E_1 A_1 u_1 = -E_3 A_3 u_3$$

$$\Rightarrow u_3 = -e_1 u_1$$

$$\text{where,} \quad e_1 = \frac{E_1 A_1}{E_3 A_3}$$

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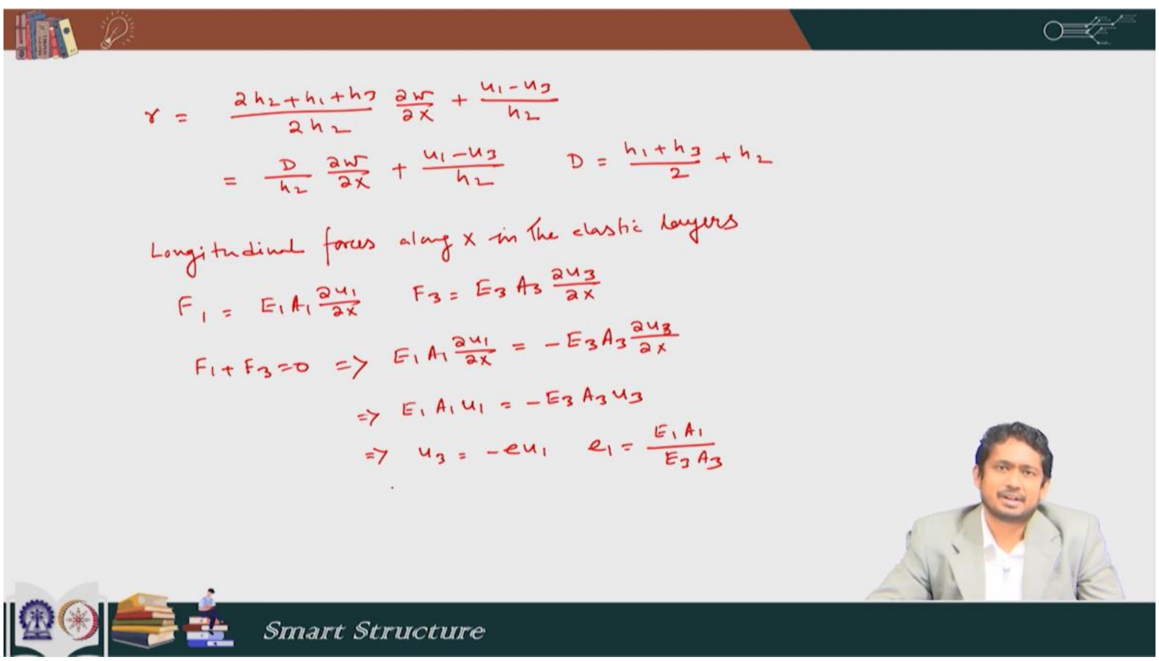

So, this relation helps me get rid of one of the unknowns in the problem. Now, in the problem I have two unknowns, one is w and one is either u_3 or u_1. So, now we will write in the formulation the unknown as u_1. So, we will consider u_1 as the unknown and we define u_1 as just u, and we will solve the problem.

Now, before doing that we have to write the shear stress also. Now, the shear modulus at the mid layer. So, mid layer has our rubber plus ER or MR fluid. Now rubber and ER MR fluid have different shear modulus. So, we define something called equivalent shear modulus of that layer and that we just do by the rule of mixture. So, for that we write G bar, the shear modulus at the mid layer is equal to G_r multiplied by b_r by b plus G star multiplied by 1 minus b_r by b.

$$\bar{G} = G_r \frac{b_r}{b} + G^* \left(1 - \frac{b_r}{b}\right)$$

So, G_r is shear modulus of the rubber that is used and G star is the shear modulus of the fluid. Now, here we can see that - this is the ratio of the width of the rubber multiplied by the entire width of the rubber plus the ER MR fluid. And this is the ratio of the width of the ER MR fluid layer divided by the width of the entire beam. So, the equivalent shear modulus G bar is just a weighted combination of the individual shear moduli, here the weight is the ratio of the width.

Now, for G star, which is the shear modulus of the ER and MR fluid, we use the constitutive relation that we discussed last week based on the paper by Chen and Yu. So, here we saw the viscoelastic property of the MR fluids. Here, the pre yield viscosity was viscoelasticity was incorporated in the model and that is what we are going to use here. So, G star is - again as we saw that G star has two parts - one is G prime plus i into G double prime. So, G prime is storage modulus and G double prime is loss modulus.

$$G^* = G' + iG"$$

So, this is the same model we are going to use for the pre yield condition, and for the post yield, it is just tau is equal to tau_y plus mu into gamma dot, which is the Bingham plastic model. So, this is Bingham plastic model.

$$\tau = \tau_y + \mu\dot{\gamma}$$

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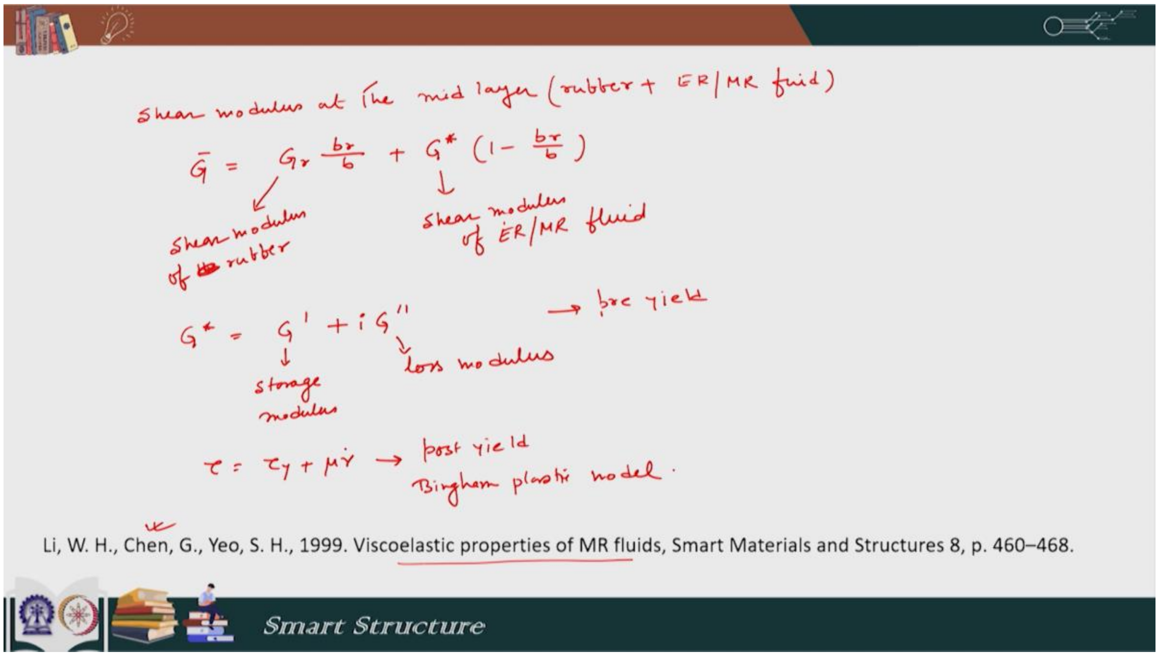

Now, with all these - we have defined the shear strain in terms of the displacement and we have defined the shear modulus. So, we can now write shear stress.

Now with all these, we can write all the different energies, and from there we will write the variational formulation. So, we have two elastic layers at the top and bottom. Now, the strain energy of the two layers can be written as this. So, half into the integral of E_1 A_1 multiplied by del u by del x whole square is the strain energy of the top elastic layer. So, we are writing u as just u_1, because we have got rid of u_3, because of the relation between u_1 and u_3. Now, this is E_3 A_3 small e square multiplied by this quantity integrated and multiplied by half, that gives me the strain energy of the bottom layer. Now, these strain energies are the strain energies due to the extension of those layers. Now, those layers bend also. So, because of the bending these are the strain energy.

Now, this I_1 and I_3 these are the moment of inertia with respect to their own centroid. So, now, I_1 and I_3 are moments of inertia of the top elastic layer and the bottom elastic layer, and these are defined with respect to their own centroid. Because we have seen that in these three layers, if I define the moment of inertia with respect to here, then I need to define the moment of inertia of this layer with respect to its own centroid and shift it using the parallel axis theorem. If I do that then, I am writing everything in terms of the centroid of the three layers combined. In that case, these energies do not come into picture and in that case after shifting by parallel axis theorem, whatever the moment of inertia expression I have, that gives us - because of that I get these two-energy combined in just this expression. But here, because I am writing these two layers separately, I am writing the strain energy due to the extension of its mid layer here and the bending of this layer itself where I is considered with respect to the centroid of this layer itself. So, that is why they are written separately.

So, if I can write them separately, if I take the centroid of the layer itself or I can write it in a combined way, if I take the centroid at the middle of the combination of these three layers. So, here we have written them separately. So, you have two variables w and u. So, these I's are moment of inertia with respect to their own layer.

Now, comes shear strain energy of the mid layer. We have defined the shear strain at the mid layer and we know the shear modulus which is G gamma. So, again we do half into G into A_2, and then we multiply the shear strain square and that gives us the shear strain energy of the mid layer. Kinetic energy is considering motion along z of the elastic and fluid layers, which is this. So, half into mass into velocity square in the transverse direction. So, here this is the contribution from layer 1, layer 2. This is the contribution from the rubber layer. So, this is rubber density and this is the density of the ER and MR fluid. This is the density of the bottom elastic layer. This is the density of the ER and MR fluid. So, this is ER MR fluid density. And, this is density of elastic layer 1. This is density of elastic layer 3.

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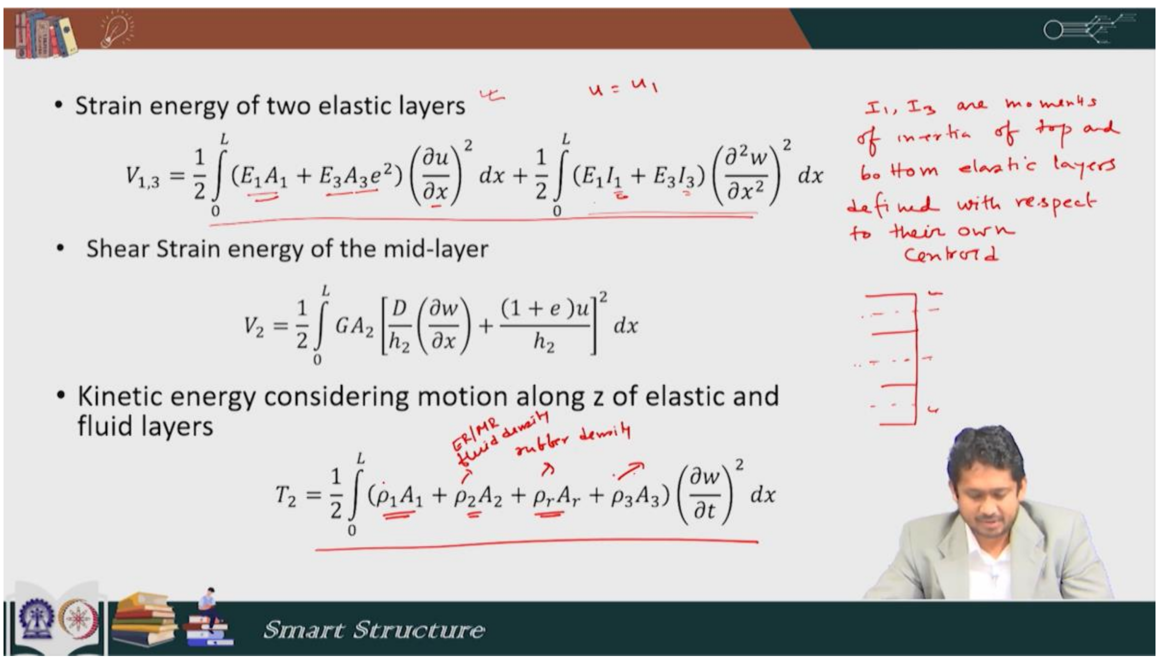

Now, we will write the other component of the kinetic energies which is T_2 here. So, these are the kinetic energy due to the extension of the layers, and then we have the kinetic energy considering rotation of the MR fluid layer. So, it is rotation. So, it is rotational motion. So, far we have neglected the kinetic energy due to rotation, but here it is incorporated. It can be neglected also, if the layer thickness is small. Now, here again I_2 is the moment of inertia of that fluid layer with respect to its own centroid. And here, rotation is defined as del w by del x minus del u by del z. So, we know del u by del z. So, we can write the rotation.

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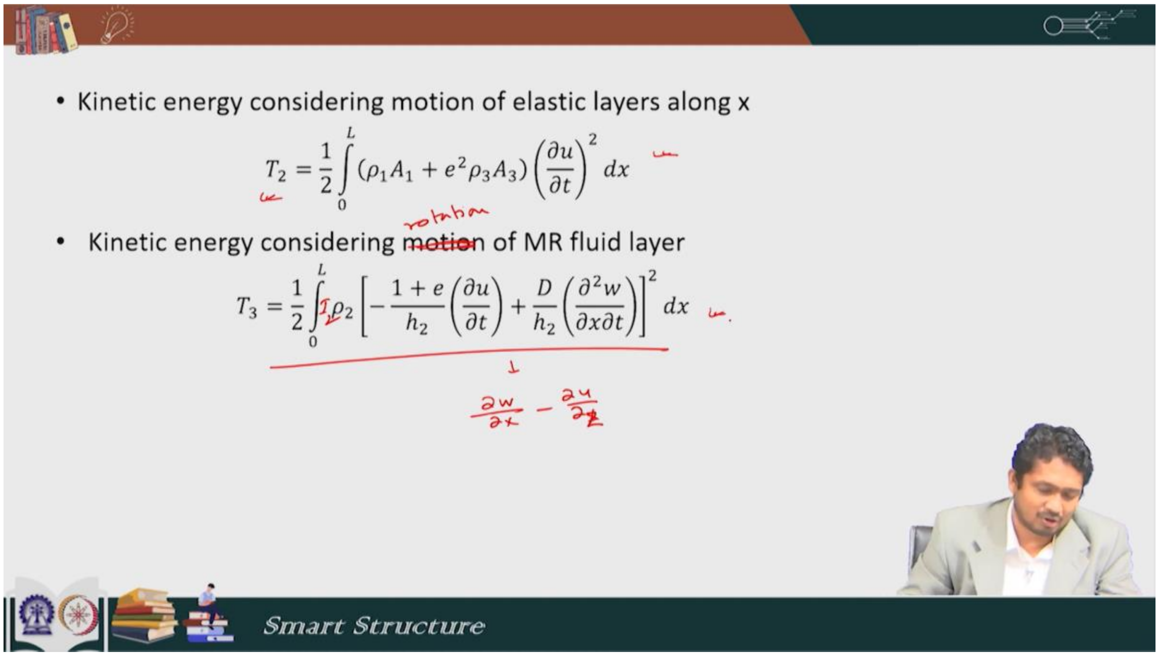

So, after having all the energies now we can make some approximation for our variables u and w, and put it here. And then, we can minimize T minus U that means, the total kinetic energy minus the total strain energy, or we can make a variational form, and then put it in the variational form. And then, form a set of ordinary differential equation that we want. So, this is the variational form, if I take variation of all the energies. So, we have T_1, T_2, T_3 and V_1, V_2, V_3. So, if I take T_1 plus T_2 plus T_3 minus V_1 minus V_2 minus V_3, and again we know the Hamilton's principle says that its integral over any two arbitrary time t_1 and t_2 is 0, and we know how to get into this kind of equations from this principle. Now, one thing to note in so far in all the formulations we have denoted our strain energies as U_1, U_2, U_3. Here, we have denoted them as V_1, V_2, V_3, but they are same.

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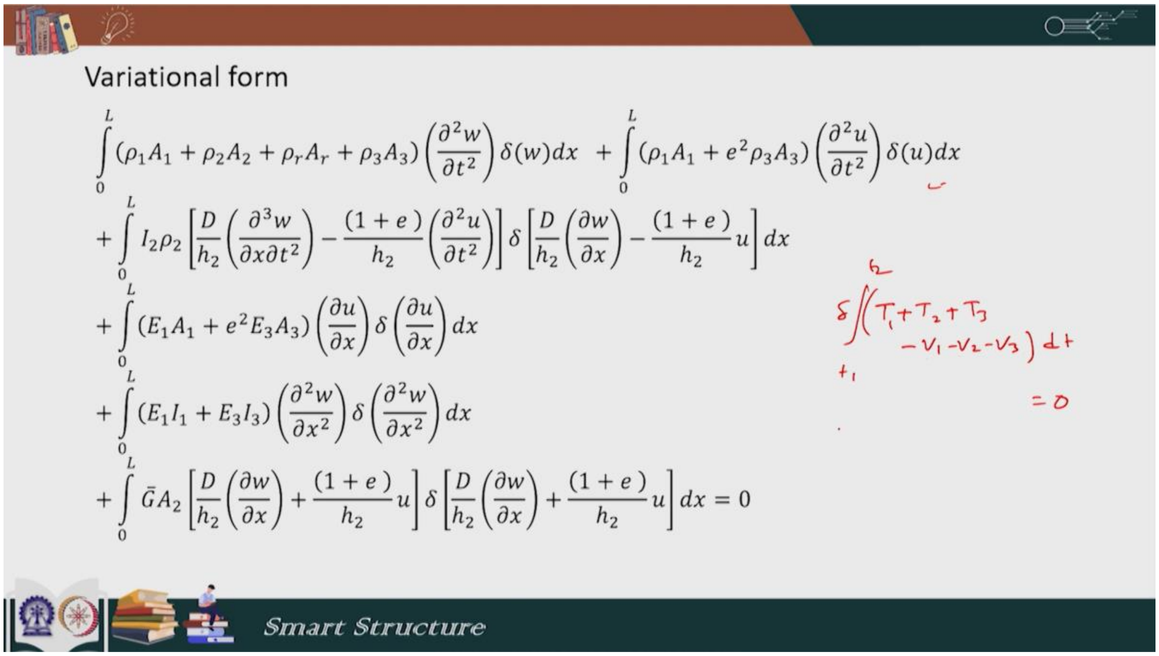

Now, after that the variational form that we get is this. And then, we put some approximation. U is approximated using a set of basis functions q_u phi_u and w is approximated using a set of basis functions phi_w q_w. And then, if we put it there, then we get equations of this form plus variation of q_{uNu}, variation of q_{w1}. We have done this kind of formulation several times in this course so far. So, we know that because these are independent variations. So, each of the terms in the bracket is 0 and that gives me a set of equations of this form. So, set of N_u plus N_w equations. So, in this equation the mass matrix M takes a form like this - M_{uu} M_{uw} M_{wu} M_{ww}. Similarly, the stiffness matrix takes a form like this - K_{uu} K_{uw} and K_{wu} K_{ww}. So, M_{uu} is this. M_{uw} is this.

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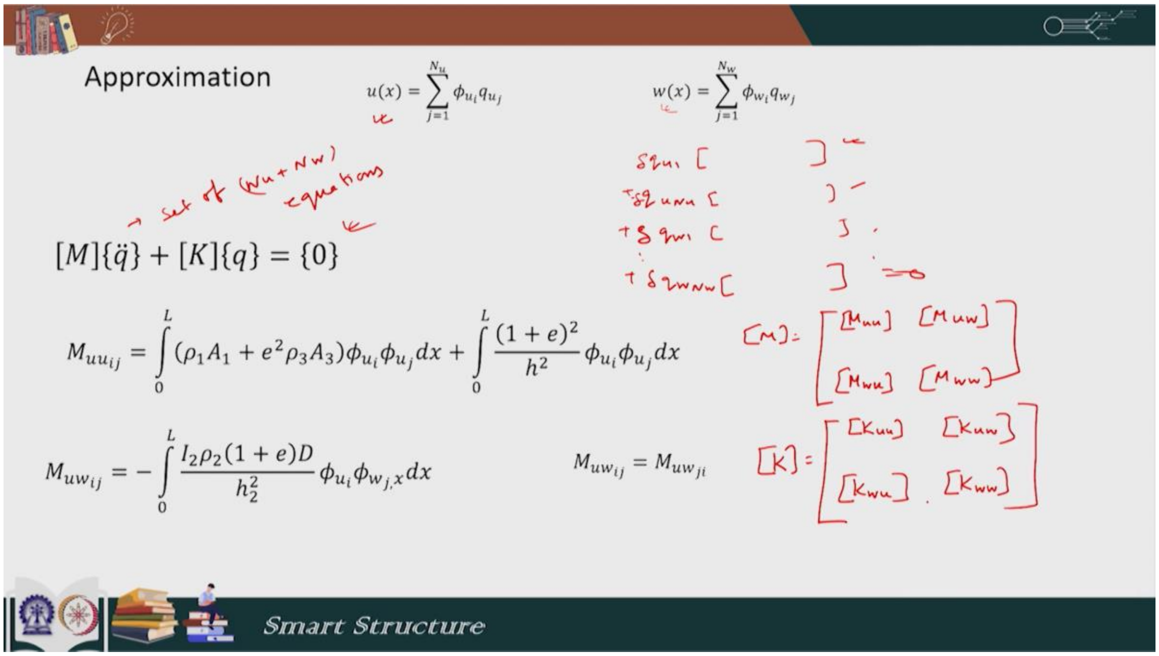

We can just look into the variational formulation and we can figure out the kind of forms this matrix elements take. So, this is M_{ww}. Similarly, these are the elements of the stiffness matrix.

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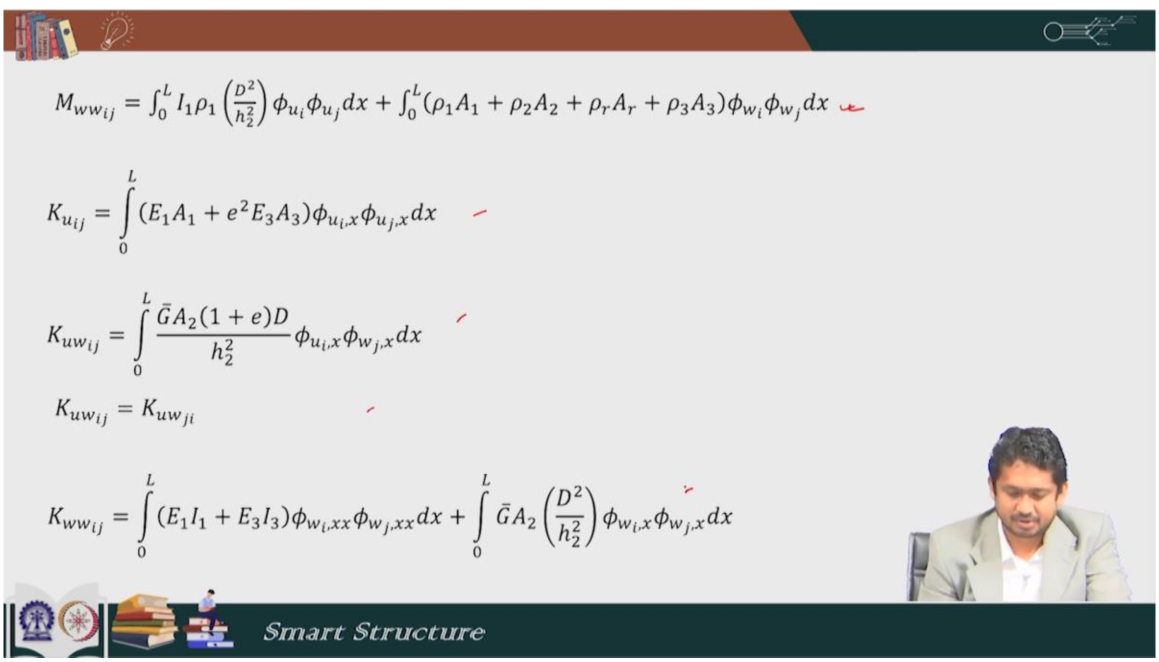

So, this is the final set of ordinary differential equations that we got. And these are free vibration equations, since there is no externally applied force. So, the right hand side is 0. Now, this gives rise to an Eigen value problem because if we consider q as q_0 multiplied by e to the power i omega t, and then put it here, that gives me minus of omega square M multiplied by q_0, multiplied by e to the power i omega t because if I differentiate this quantity twice with respect to t, then minus omega square comes out. And e to the power i omega t is common in both the terms. So, I do not need that. Plus, we have q_0 equal to 0. And then, we can write this entire quantity as - so, these are Eigen value problem.

$$\{q\} = \{q_0\}e^{i\omega t}$$

$$-\omega^2[M]\{q_0\} + [K]\{q_0\} = \{0\}$$

$$\big[[K] - \omega^2[M]\big]\{q_o\} = \{0\}$$

Here, one thing to note is that this K has contribution from the shear in the fluid layer and the shear modulus of the fluid layer was a complex function. So, by solving this Eigen value problem, from the Eigen values, we can estimate the damping present in the system. For different values of the electric field, we can find how the damping changes in the system and that kind of study can be done from this equation.

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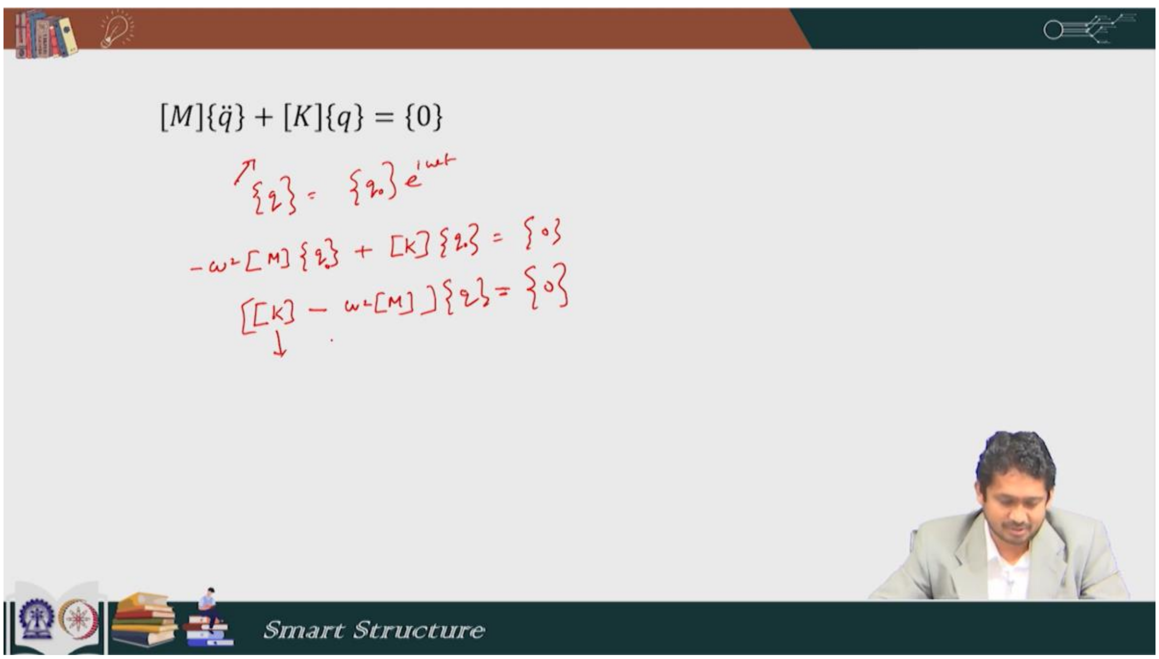

So, that brings us to the end of this lecture, and also it brings us to the end of this course.

I would like to thank all the participants for their interest. I hope you enjoyed this, and I also sincerely appreciate our teaching assistants Mr. Vaibhav Mishra, and Professor Sunny Akhtar, and I also thank the entire NPTEL team.

Thank you.