Smart Structures
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Week 01
Lecture No: 06
Mathematical Preliminaries

Welcome to the 6th lecture.

In the last lecture, we saw the basics of piezoelectricity. Now, we will gradually move on to the mathematical modeling of piezoelectric materials. Now, before going directly to the mathematical models of piezoelectric materials, we need to, it is better that we spend some time on the mathematical preliminaries that we are going to use there. So, we will be using some of the mathematical tools. So, let us spend some time on that, and then we will move on to the mathematical modeling of the piezoelectric materials.

So, we will be using indicial notation quite a few times while doing the mathematical modeling. Indicial notations are helpful in writing equations or expressions in small, or in compact form, helpful in writing equations or expressions in a compact form. So, in many cases, our expressions, if we write in fully expanded form, are going to be quite big. So, we will be using indicial notations for those cases.

Now, indicial notations, as the name says, it uses index, and these indices generally go from one to three unless otherwise stated. So, if you want our index to go from one to six or like that, we have to state it; otherwise, it is understood that it goes from one to three. So, there are two kinds of indices that we will deal with. One is a free index. Now, free index is something which remains the same, and it appears once in each term. So, suppose we have a vector F. Now, this vector F has three components: F_x, F_y, and F_z, which we can also write as F_1, F_2, and F_3.

$$\vec{F} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} \rightarrow \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

If we say that our x-axis is x_1, y is x_2, and z is x_3. So, x is x_1, y is x_2 and z is x_3.

$$x \rightarrow x_1$$
$$y \rightarrow x_2$$
$$z \rightarrow x_3$$

Now, we can say that the ith component, a component of F, is F_i, where i goes from one two three. So, I can be anywhere between one to three.

$$F_i \qquad \text{where}, i = 1,2,3$$

So, i is equal to one means F_1, which is F_x. Similarly, F_2 is F_y, and F_3 is F_z. So, here are some of the properties of the free index. Free index should remain the same. So, if I have F_i, then I cannot suddenly change into F_j, and F_i and F_j are different in general.

$$F_i \neq F_j$$

And in each term, free index appears once. So, let us suppose I write an equation like this: F_i is equal to m a_i.

$$F_i = ma_i$$

So, we have F_i here, we have i here, and i here, or we may write something like this: a_i b_j is equal to suppose C_{ik} d_{kj}. So, here i and j appearing once in this term, i and j appearing once in this term, or we can add some more terms into it, maybe plus e_{im} into f_{mj}. So, i and j are free indexes here.

$$a_i b_j = C_{ik} d_{kj} + e_{im} f_{mj}$$

Now, I cannot change suddenly the free index to something else. For example, if I write something like this: a_{ij} is equal to C_{ik} d_{kj} plus suppose e_{pm} f_{mq}.

$$a_i b_j = C_{ik} d_{kj} + e_{mp} f_{mq}$$

This is not allowed. I am suddenly changing the free index here. So, this is not possible.

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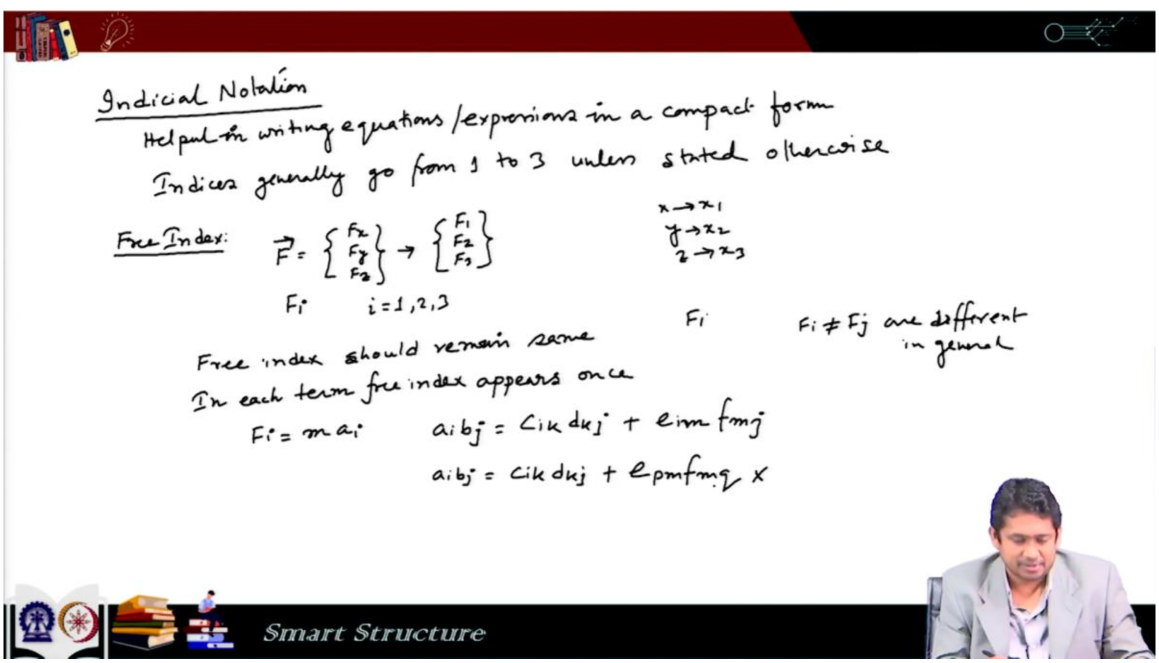

Now, there is another kind of index that is called Dummy index. Dummy index is local to an individual term and appears twice, and it is summed over. Here, name can change because it is local to each term, and it may not appear in all the terms. For example, if I have something like this a_i b_i, then i is repeated, it is appearing twice, and it is summed over. So, a_i b_i means a_1 b_1 plus a_2 b_2 plus a_3 b_3.

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

So, i is going from one to three, when i is equal to one, a_1 b_1 plus i is equal to two, a_2 b_2 plus i is equal to three, a_3 b_3. Similarly, I may have a_i b_j is equal to C_{ik} d_{kj}.

$$a_i b_j = C_{ik} d_{kj}$$

Here, k is the dummy index. So, i j, they are free indexes, and this k is a dummy index. Now, again I can expand it, and it becomes C_{i1} d_{1j} plus C_{i2} d_{2j} plus C_{i3} d_{3j}.

$$a_{ij} = C_{i1} d_{1j} + C_{i2} d_{2j} + C_{i3} d_{3j}$$

So, it is summed over, and it is appearing only once. It should not or may not be appearing in all the terms. So, we can see that in this term a_i b_j, it is not appearing whether, in this term C_{ik} d_{kj}, it is appearing twice. It is appearing twice. And its name can change. So, something like this C_{ik} d_{kj} can also be written as C_{im} d_{mj}.

$$C_{ik} d_{kj} = C_{im} d_{mj}$$

It does not matter here. Also, m will go from one to three, and finally, it will be expanded, and we will get this.

Now, if I write this a_{ij} is equal to C_{im} d_{mj} plus suppose F_{in} G_{nj}

$$a_{ij} = C_{im} d_{mj} + F_{in} G_{nj}$$

It is possible. This gets expanded, and there are three terms, and this gets expanded, and there are three terms. So, m, but I cannot change i and j here because free index does not change. And dummy index is local to one term. So, there what I use does not matter.

Now, we will talk about something called Kronecker Delta. Now, Kronecker Delta is denoted as del_{ij}, and it has a value of zero, when i is not equal to j, and it has a value of one when i is equal to j.

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

So, for example, del_{12} is zero because i and j are different, whereas del_{11} is one because here we have the two indices the same.

$$\delta_{12} = 0$$

$$\delta_{11} = 1$$

Now, if I have δ_{ii}, then i is a dummy index.

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

So, it should repeat. So, it becomes δ_{11}. Then i should become two, so it becomes δ_{22}. Then i should become three. It becomes δ_{33}. And each of them are one, one plus one plus one. So, that becomes three.

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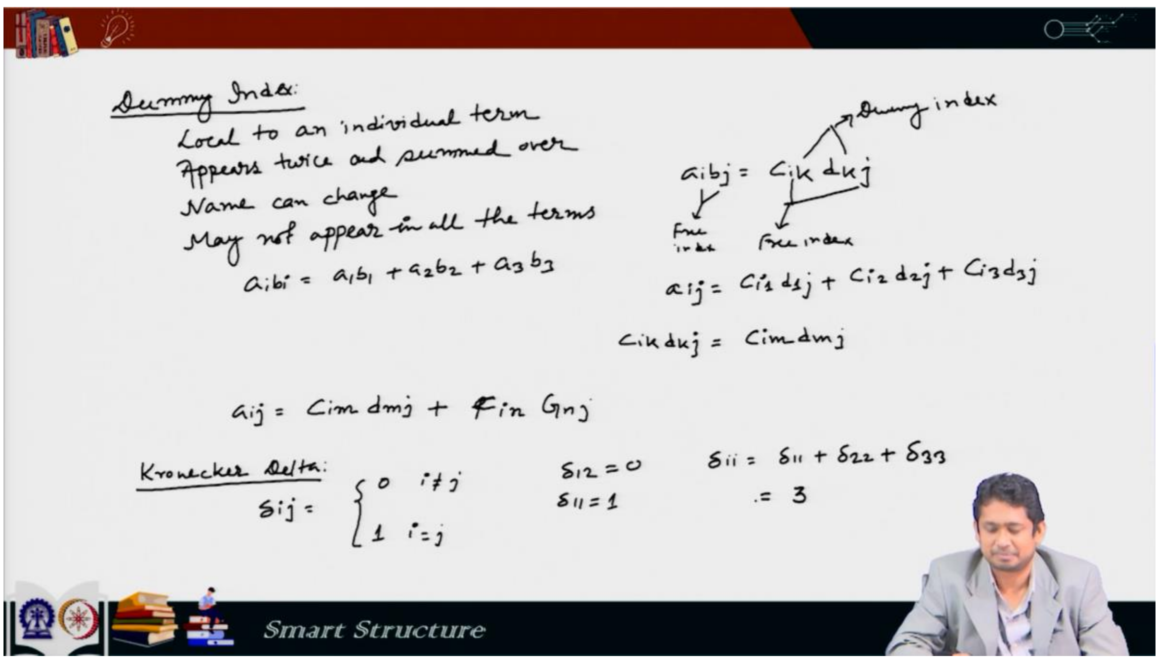

Now, we have something called Levi Cavita, ϵ_{ijk}. So, epsilon_ijk means it is one if ijk are even permutations, and it is minus one if ijk are odd permutations, and it is zero otherwise.

$$\epsilon_{ijk} = \begin{cases} 1 & i, j, k \text{ are even permutations} \\ -1 & i, j, k \text{ are odd permuations} \\ 0 & \text{otherwise} \end{cases}$$

Now, what is an even permutation, and what is an odd permutation? Even permutation means one two three, two three one, three one two, they are even permutations, which means ϵ_{123} is one, ϵ_{231} is one, and ϵ_{312} is one and one three two, three two one and two one three are odd permutations.

Now, here is an easy way to remember this: if I draw a circle like this. If I put one here, if I put two here, if I put three here, and then if I go clockwise, sorry, anticlockwise, then one two three, start from here, two three one we have it, start from here, three one two. So, that is an even permutation.

In the same circle, given that one two three are placed in a similar way, if I go in the other direction, clockwise then, one three two, I have here, three two one here, two one three here. This is an odd permutation, and for any other thing, it is zero.

Now, the relation between delta and epsilon. Epsilon_{ijk} multiplied by epsilon_{ilm} is equal to delta_{jl} delta_{km} minus delta_{jm} delta_{kl}.

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

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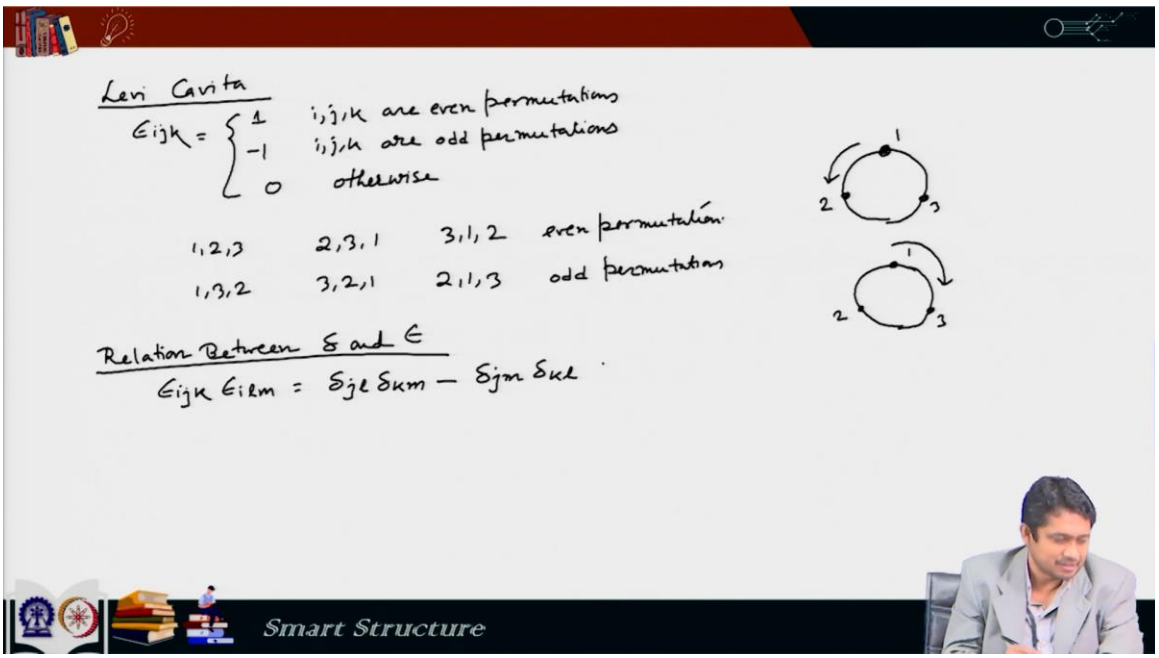

Then, there is something called contraction. The contraction says that C_{ij} multiplied by del_{ij} is equal to C_{ii}.

$$C_{ij}\delta_{ij} = C_{ii}$$

It can be proven easily. C_{ij} delta_{ij}, we have i as dummy index here and j as dummy index here. So, when i is equal to one, j is equal to two. Let us say, i is equal to one, and then j repeats. So, it is one one, one one plus i remains one, j becomes two. So, two plus i remains one, j becomes three. So, it is C_{13} delta_{13}. Then, we have C_i become two. So, it is C_{21} delta_{21}, then C_{22} delta_{22}, then C_{23} delta_{23}, then C_{31} delta_{31}, C_{32} delta_{32} plus C_{33} delta_{33}.

$$C_{ij}\delta_{ij} = C_{11}\delta_{11} + C_{12}\delta_{12} + C_{13}\delta_{13}$$

$$+C_{21}\delta_{21} + C_{22}\delta_{22} + C_{23}\delta_{23}$$

$$+C_{31}\delta_{31} + C_{32}\delta_{32} + C_{33}\delta_{33}$$

Now, as we know that, these all are zero, and these are one. So, this is one, and this is one. So, it becomes now this is zero, delta_{12} is zero, delta_{13} is zero, delta_{21} is zero, delta_{23} is zero, delta_{31} is zero, delta_{32} is zero, and this delta_{11}, delta_{22}, delta_{33} they are one.

$$C_{ij}\delta_{ij} = C_{11}\delta_{11} + C_{22}\delta_{22} + C_{33}\delta_{33} = C_{ii}$$

So, we have C_{11} plus C_{22} plus C_{33}, and we know that, this is C_{ii}. So, it is proved.

Now, vector operations can also be shown through indicial notations. For example, dot product, if we dot these two vectors a and b, then we can write a_1 b_1 plus a_2 b_2 plus a_3 b_3, which we can write as a_i b_i.

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = a_ib_i$$

Cross product. If we take a cross product of two vectors a and b, this is another vector a_2 b_3 minus a_3 b_2 i plus a_3 b_1 minus a_1 b_3 j plus a_1 b_2 minus a_2 b_1 k.

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\hat{\imath} + (a_3b_1 - a_1b_3)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}$$

So, the ith component of this, is this, which can be written using the Levy Capita as epsilon_{ijk} multiplied by a_j b_k.

$$\left(\vec{a} \times \vec{b}\right)_i = \epsilon_{ijk}a_jb_k$$

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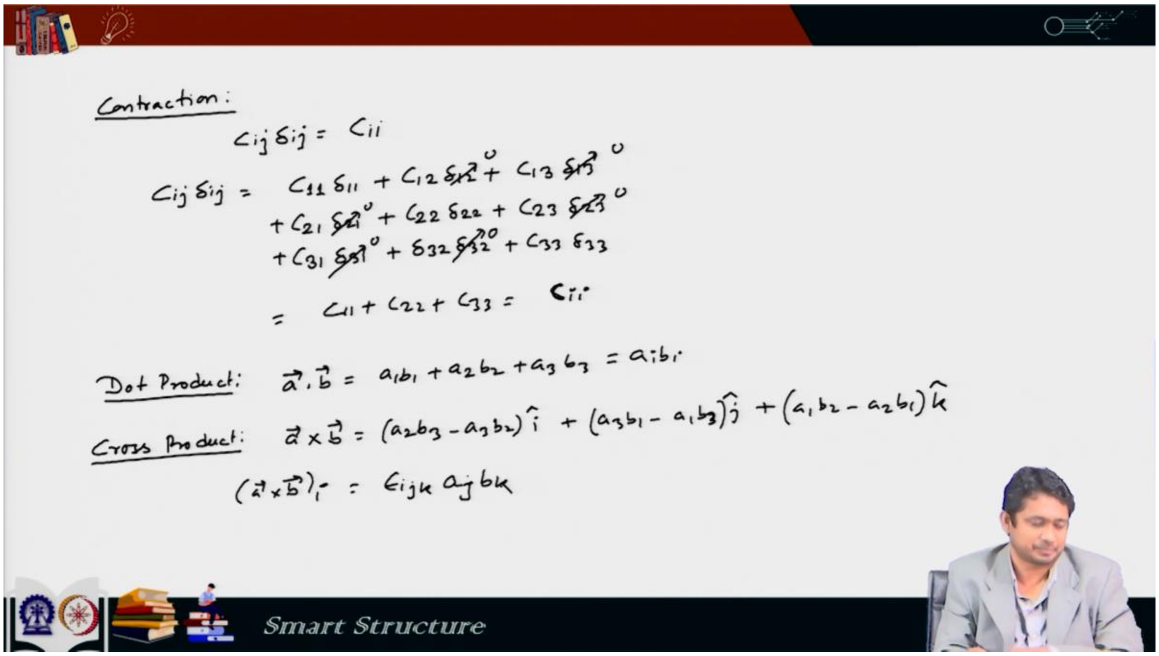

Now, let us look into matrix multiplication. Say, we have a matrix A, and we are multiplying it with b, and the result is a vector C.

$$\{C\} = [A]\{b\}$$

Now, this we can easily verify that C_i would be A_{ij} b_j as per the indicial notation.

$$C_i = A_{ij} b_j$$

And then, if I multiply two matrices, A and B, if the result is D. We can verify that D_{ij} is equal to A_{ik} and B_{kj}.

$$[A][B] = [D]$$

$$D_{ij} = A_{ik} B_{kj}$$

Now, let us look into something called Delta Operator, which we will be using a lot. Delta operator. It is a vector, and this signifies del del x operated on something i plus del del y operated on something into j plus del del z operated on something into k. So, i j k are unit vectors along x_1, x_2, and x_3 directions.

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)$$

So, we have x direction, y direction, z direction. We can call these as x_1, x_2, x_3, and we have i as the unit vector here, j as the unit vector here, and k as the unit vector here.

Now, if this delta operator works on a scalar phi, then it becomes delta phi by del x i plus delta phi by del y j plus delta phi by del z k.

$$\vec{\nabla}\phi = \left(\frac{\partial \phi}{\partial x}\hat{\imath} + \frac{\partial \phi}{\partial y}\hat{\jmath} + \frac{\partial \phi}{\partial z}\hat{k}\right)$$

So, if we say that the after delta operates on phi, the vector that we get is g. So, if g is a vector, then we can say that g_i is equal to phi comma i.

$$g_i = \phi_{,i}$$

So, the ith component of this vector g is phi comma i, which means phi differentiated with respect to x_i.

Now, let us say that g is equal to delta dot v, where v is a vector.

$$g = \vec{\nabla} \cdot \vec{\mathrm{v}}$$

So, it is a dot of two vectors: one is delta, and one is v. In this case, the result is, we can directly write the result here: del v by del x plus, so del v_1 by del x plus, or del x_1, plus del x_2 plus del v_3 by del x_3.

$$g = \vec{\nabla} \cdot \vec{\mathrm{v}} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

So, our v vector is v_1 i plus v_2 j plus v_3 k.

$$\vec{\mathrm{v}} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$$

Or we can write in a vector form as v_1 v_2 v_3.

$$\{\mathrm{v}\} = \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

And as we know x_1 x_2 x_3 are x y z. So, this can be written as v_i comma i.

$$g = \vec{\nabla} \cdot \vec{\mathrm{v}} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = v_{i,i}$$

So, now, i is a dummy index here, and it repeats three times. So, when i is equal to one, it means v_1 comma 1, which means derivative of v_1 with respect to x_1, plus i becomes two. So, derivative of v_2 with respect to x_2, plus i becomes 3. So, derivative of v_3 with respect to x_3.

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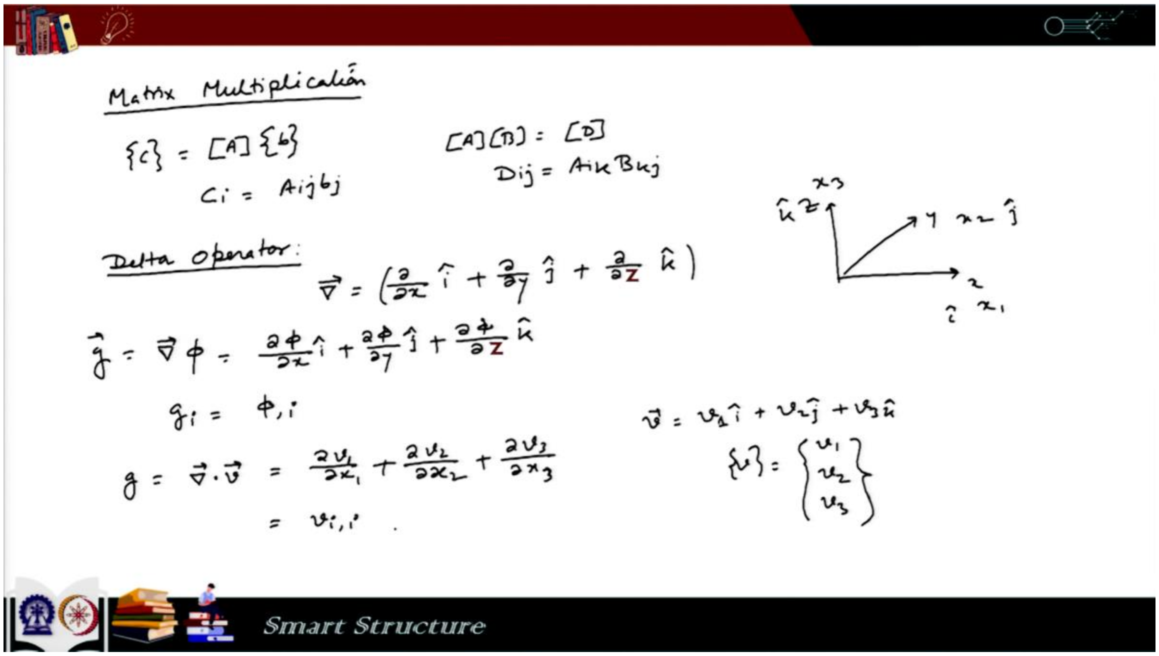

Now, let us assume that this delta, we take delta cross v. The result is a vector which we denote as g.

$$\vec{g} = \vec{\nabla} \times \vec{v} = \epsilon_{ijk} v_{k,j}$$

This we can write as epsilon_{ijk} multiplied by $v_{k,j}$ because we have seen that a cross b, its ith component is epsilon_{ijk} a_j b_k. So, in this case, my a is delta and b is v. So, we can write this.

Now, let us prove something. Maybe let us look into this: a cross b cross c. And this is a dot c b minus b dot c a.

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$$

To prove this, we will be using the Levy Cavita symbol. So, a cross b is multiplied by c. If I take the ith component of it. So, we can write. So, let us say this vector, which we get after doing dot. Let us give it a name d. So, it is d cross c, ith component of it that is what we want.

$$\left((\vec{a} \times \vec{b}) \times \vec{c}\right)_i = (\vec{d} \times \vec{c})_i = \epsilon_{ijk} d_j c_k$$

So, it is epsilon_{ijk} d_j c_k as per our definition, and then we have to look into d. So, now we have d is equal to a cross b, and we are interested in the jth component of d.

$$\vec{d} = \vec{a} \times \vec{b}$$

$$d_j = \epsilon_{jml} a_l b_m$$

So, that is going to be epsilon$_{jml}$ multiplied by a_l b_m. So, finally, we have epsilon$_{ijk}$ epsilon$_{jml}$ a_l b_m c_k.

$$\left(\left(\vec{a} \times \vec{b}\right) \times \vec{c}\right)_i = \left(\vec{d} \times \vec{c}\right)_i = \epsilon_{ijk} d_j c_k = \epsilon_{ijk}\epsilon_{jml} a_l b_m c_k$$

Then we already proved this can be written in terms of deltas. We did not prove it, but we know the relation. So, it is delta$_{kl}$ delta$_{im}$ minus delta$_{km}$ delta$_{il}$ then multiplied by a_l b_m c_k.

$$\left(\left(\vec{a} \times \vec{b}\right) \times \vec{c}\right)_i = \epsilon_{ijk}\epsilon_{jml} a_l b_m c_k = (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}) a_l b_m c_k$$

Now if we look into this expression delta$_{im}$ into b_m and delta$_{kl}$ into a_l, then with that we can write: a_k b_i c_k, and this can be written as: a_i b_k c_k.

$$\left(\left(\vec{a} \times \vec{b}\right) \times \vec{c}\right)_i = (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}) a_l b_m c_k = a_k b_i c_k - a_i b_k c_k$$

And then this expression a_k c_k is dot of a and c vector. So, a dot c and then it is multiplied with b_i, and here we have b dot c that is multiplied with a_i.

$$\left(\left(\vec{a} \times \vec{b}\right) \times \vec{c}\right)_i = a_k b_i c_k - a_i b_k c_k = (\vec{a} \cdot \vec{c}) b_i - \left(\vec{b} \cdot \vec{c}\right) a_i$$

So, we have done it for a component, ith component. So, finally, the vector can be written as a dot c multiplied by b minus b dot c multiplied by a.

$$\left(\vec{a} \times \vec{b}\right) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - \left(\vec{b} \cdot \vec{c}\right)\vec{a}$$

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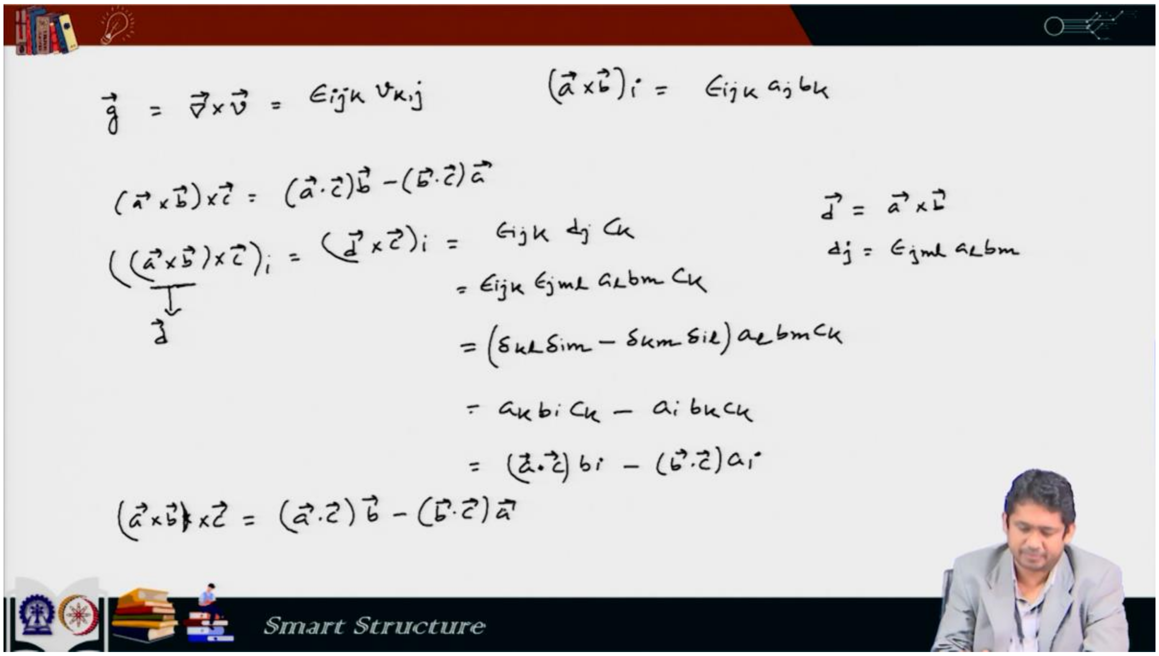

So, with this proof, we will end this lecture here. Now, we will move on to the mathematical modeling of piezoelectricity from the next week.

Thank you.