Smart Structures
Professor Mohammed Rabius Sunny
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur
Week 10
Lecture No: 53
Analysis of a Beam with Shape Memory Alloy Wire – Continued
Part 04

So, in the last lecture, we derived the final element formulation for the beam that has externally applied load, which is dampened out by shape memory alloy wires.

Now today, we will see, how the forces F_H bar F_V bar and M can be calculated. Based on the displacement that are experienced at node number 2N plus 1, sorry, node number N plus 1. So, to do that let us look into the diagram.

So, here this portion shows only the part of the beam which is from x equal to 0 to L_a. Now, due to the bending of the beam, this point moves here. Now, we have to find out the final length which is small l plus delta l. So, small l is the initial length of the shape memory alloy wire, and l plus delta l is the final length after the deformation. So, we have to find out this final length small l plus delta l.

So, the slope here is – this slope is del w by del x, because if we consider the Euler-Bernoulli assumption, then the plane section remains plane before and after bending. So, this slope is also del w by del x. Now, if this slope is del w by del x, then this offset is del w by del x, which is w prime multiplied by t_b by 2. So, in this diagram w prime is equal to w comma x equal to del w by del x. Then, if we know this quantity then finally, after applying trigonometry, it can be shown that our l plus delta l whole square is equal to h minus $d_{2N \text{ plus } 1}$ whole square, plus L_a minus t_b by 2, $d_{2N \text{ plus } 2}$ whole square.

$$(l + \Delta l)^2 = (h - d_{2N+1})^2 + \left(L_a - \frac{t_b}{2} d_{2N+1}\right)^2$$

Here, h is this vertical dimension, and as we know that $d_{2N \text{ plus } 1}$ as per our degree of freedom nomenclature is equal to w at x equal to L_a. And $d_{2N \text{ plus } 2}$ is equal to del w by del x at x equal to L_a. And also we know that l square, the undeformed length is h square plus L_a square.

$$l^2 = h^2 + L_a^2$$

Then subtracting, we can get 2 small l, del L equal to minus 2 h $d_{2N \text{ plus } 1}$ minus L_a, t_b $d_{2N \text{ plus } 2}$.

$$2l\Delta l = -2hd_{2N+1} - L_a t_b d_{2N+2}$$

And then, it becomes del l equal to minus sin theta $d_{2N \text{ plus } 1}$, minus cosine theta into t_b by 2, $d_{2N \text{ plus } 2}$.

$$\Delta l = -\sin\theta\, d_{2N+1} - \cos\theta \frac{t_b}{2} d_{2N+2}$$

So, strain in wire 1 increases by an amount epsilon_{1c} which is equal to del l by l, and that is equal to minus sin theta $d_{2N \text{ plus } 1}$ divided by small l, minus cosine theta t_b by 2l, multiplied by $d_{2N \text{ plus } 2}$.

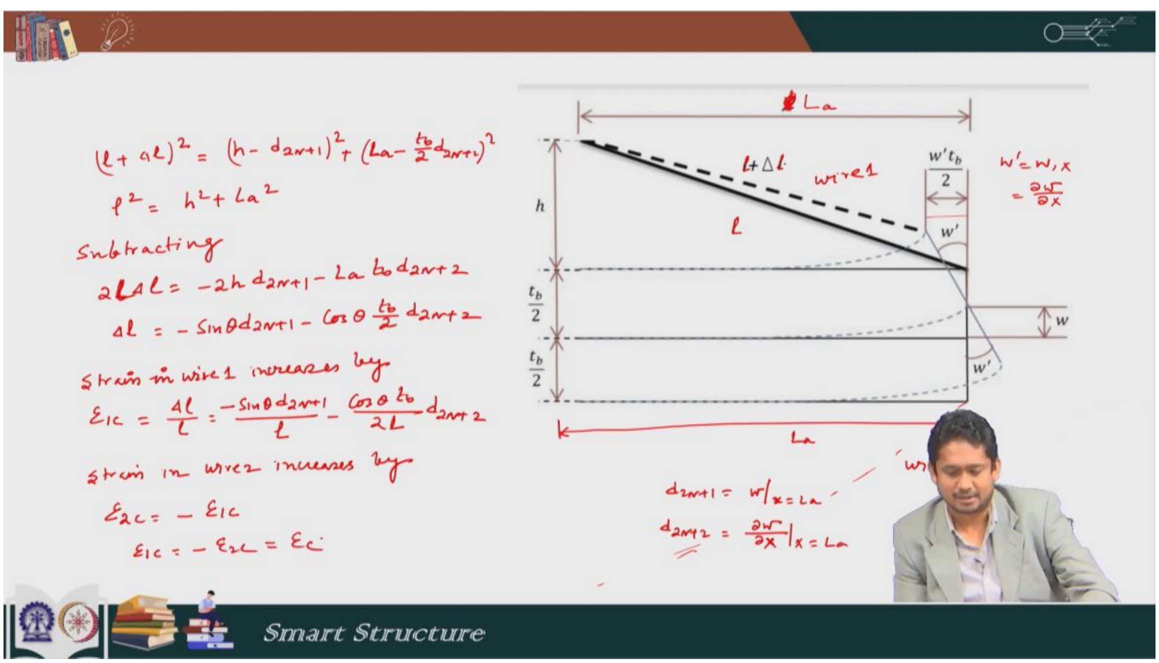
$$\varepsilon_{1c} = \frac{\Delta l}{l} = -\frac{\sin\theta\, d_{2N+1}}{l} - \frac{\cos\theta\, t_b}{2l} d_{2N+2}$$

Now, we have wire 2 below it. So, the upper wire, we denote it as wire 1. And then below also we have a wire, and the wire below is wire 2. So, the wire that is positioned here, we call it wire 2. The upper one is wire 1, and the wire below is wire 2. So, because they are symmetrically placed the amount of strain would be same here, but opposite because if this is experiencing a negative quantity as a strain this would experience a positive quantity as a strain it will try to increase in length. So, strain in wire 2 increases by epsilon_{2c} which is equal to minus of epsilon_{1c}.

$$\varepsilon_{2c} = -\varepsilon_{1c}$$

Now let us define a quantity epsilon_{1c} is equal to epsilon_{2c} is equal to epsilon_{c}.

(Refer Slide Time: 6:40)



Now, we can say that: strain in wire 1 is epsilon$_1$ equal to epsilon$_p$ plus epsilon$_{1c}$, where epsilon$_p$ is the amount of pre strain in the wire. So, when the beam is undeformed the amount of prestrain in the wire is epsilon$_p$. So, epsilon$_p$ is prestrain in the wires and they should be same because they are symmetrically placed, when the beam is undeformed. And strain in wire 2 is epsilon$_2$ equal to epsilon$_p$ minus epsilon$_c$. We have already defined a quantity epsilon$_c$. So, let us not use the suffix 1 or 2, say epsilon$_p$ plus epsilon$_c$ and epsilon$_p$ minus epsilon$_c$.

$$\varepsilon_1 = \varepsilon_p + \varepsilon_c$$

$$\varepsilon_2 = \varepsilon_p - \varepsilon_c$$

Now, if I know the strains, we can find out the stress by using the constitutive relation of the piezoelectric materials that we discussed. So, stress in wire 1 can be found from the equation, sigma$_1$ minus sigma$_{10}$ is equal to E xi$_1$ multiplied by epsilon$_1$ minus epsilon$_{1,0}$ plus omega xi$_1$ multiplied by xi$_1$ minus xi$_{1,0}$. And there could also be a term capital theta multiplied by the temperature difference. However, in this entire process the temperature is remaining fixed. So, that quantity does not have any contribution and anyway even if there is some temperature difference, we have seen that the contribution due to that capital theta term is generally negligible. So, we do not have that term.

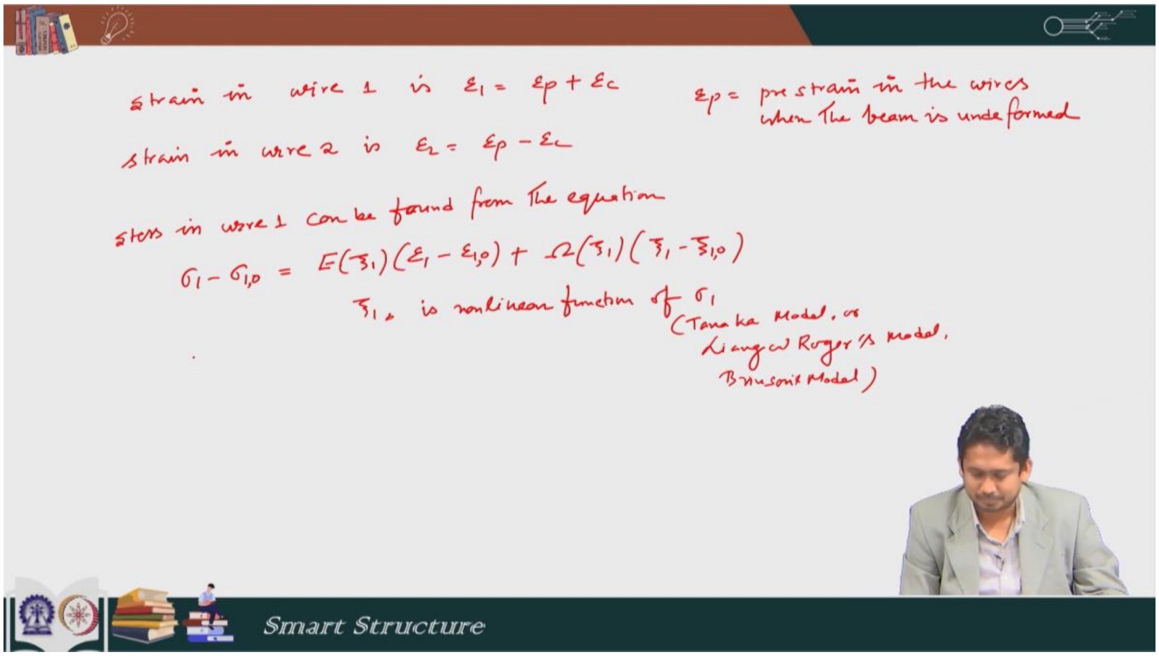
$$\sigma_1 - \sigma_{1,0} = E(\xi_1)\left(\varepsilon_1 - \varepsilon_{1,0}\right) + \Omega(\xi_1)(\xi_1 - \xi_{1,0})$$

So, here xi$_1$ is the martensite volume fraction in wire 1 and epsilon$_{1,0}$ is the initial. So, it is not epsilon$_{1,0}$. It is epsilon$_{1,0}$ maybe we can put a comma here. 0 means the initial configuration. Epsilon$_1$ means the stress in wire 1. 0 means initially how much it was. So, there is a change in the stress in wire 1. Similarly, here also it is, sorry, it is sigma 1 minus sigma$_{1,0}$ that is the change in the stress in.

So, here we have this term sigma 1 minus sigma 1 comma 0. So, it is not sigma 10, it is sigma 1 comma 0 and that signifies the change in the stress in wire 1. Similarly, I have epsilon$_1$ minus epsilon$_1$ comma 0 and that signifies change in the strain in wire 1. So, it is change in the stress in wire 1, this is the change in the strain in wire 1. And similarly, this is the change in the martensite volume fraction in wire 1, and they are related to this constitutive relation. Again, we know that this equation is a highly non-linear equation because this xi$_1$ is a non-linear function of sigma. So, xi$_1$ is non-linear function of sigma$_1$ and it we can use it Tanaka model or Liang and Rogers model or Brinson model, whatever model we use, this is a non-linear function of sigma$_1$. And we have seen how to solve this equation using an iterative techniques like Newton Raphson technique that we discussed in the last lecture of the previous week.

So, accordingly we have to find this out based on some initial and final strain. So, if I know the initial strain, if I know the final strain, we can find this quantity, find the stress out by solving the non-linear equation. So, that makes the entire solution of the finite element equation highly iterative. So, now, let us discuss how we can solve it.

(Refer Slide Time: 12:12)



So, here is the flow chart of the iteration procedure. So, at first we guess epsilon$_c$ because – Let us suppose that we are solving a static problem. So, for a given load, for a given static load, first we find out what is the amount of displacement? What is the deformation in the beam? So, for that condition, we do not know the epsilon$_c$. So, we guess – that is what we have to find out.

So, let us guess epsilon$_c$. Guess some value and based on the guess, we find out sigma$_1$. Find out sigma$_1$, sigma$_2$. So, in this case our epsilon$_p$ becomes the initial strain, sigma$_{1,0}$ and the final strain is epsilon$_1$ which is equal to epsilon$_p$ plus epsilon$_c$ for the wire 1, and the for the wire 2, it is epsilon$_2$ equal to epsilon$_p$ minus epsilon$_c$. And for both the wires our initial strain is epsilon$_p$. So, based on that, we find out our stresses and again this itself is a iterative procedure. So, to evaluate this block, itself we need to do the Newton Raphson based iteration and find out sigma$_1$ and sigma$_2$. Once we get our sigma$_1$ and sigma$_2$, we can find out the amount of forces in the wire, we can find out the amount of tensions in the wire.

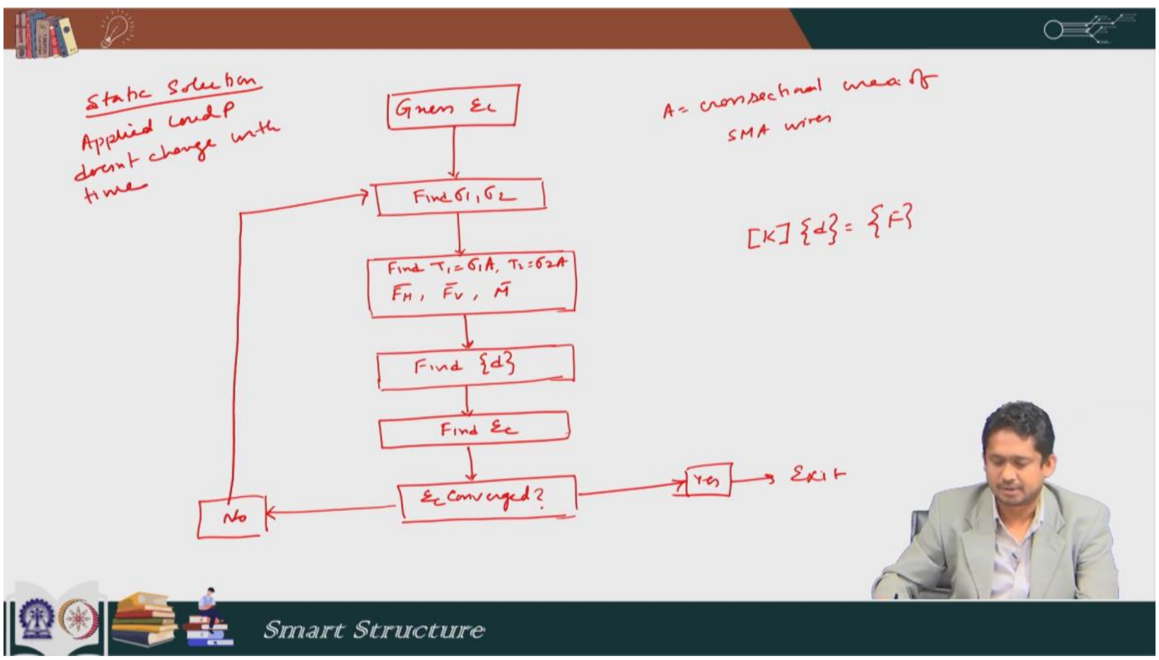
So, sigma$_1$ is that a stress in wire 1 and sigma$_2$ is the stress in wire 2. If I multiply those by the area of cross section of the wire, we get our tension T_1 and T_2. So, find T_1 equal to sigma$_1$ Aand T_2 equal to sigma$_2$ A, where A is the cross sectional area of SMA wires. If I get my T_1, T_2, I can find out the quantities F_H bar, F_V bar and M bar. Once I know those quantities, now my force matrix, force vector in the final element formulation is defined.

Now, for that known force vector I can find out d. So, now please understand here we are solving a static problem, we are assuming that the applied force is time independent. So, in this case we would just solve this equation K, d equal to F. Then once we have found out d, we know what our $d_{2N \text{ plus } 1}$ and what our $d_{2N \text{ plus } 2}$ and what is our $d_{2M \text{ plus } 1}$. And then based on that we find out again epsilon_c. Now, the question is whether this epsilon_c or the previously got epsilon_c is agreeing or not. If they are not converged, we have to repeat the procedure. If they have converged, we can stop the procedure.

So, now, epsilon_c converged, if the answer is yes, then we can exit. And if the answer is no, again we go back and feed that newly obtained value of epsilon_c and repeat the procedure until and unless the improvement of epsilon_c stops. So, until and unless the convergence of epsilon_c is obtained. Once the convergence of epsilon_c is obtained, the corresponding d is our obtained value of the nodal degrees of freedom at different node points and that gives us the solution.

So, it is a solution of the static problem. So, static solution. So, applied load P does not change with time.

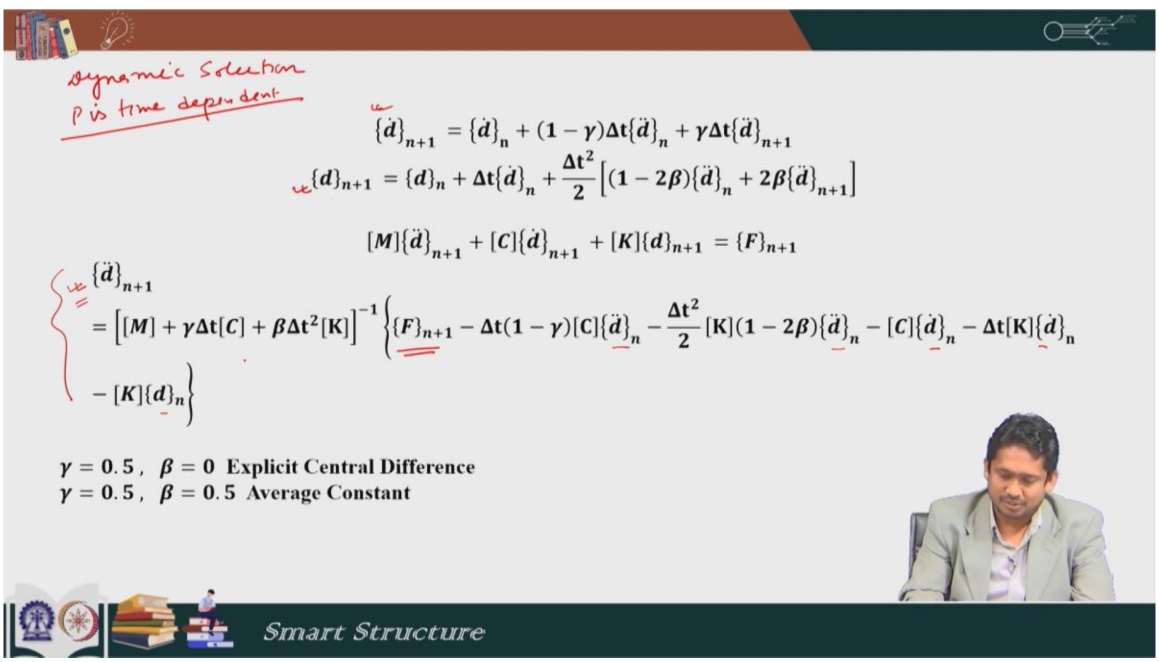
(Refer Slide Time: 17:18)



Now, what if we want to do the dynamic solution and that is the goal of this work, because we have added a damping and our force P is a time dependent force and the vibration caused due to that has to be dampened out. So, here let us go to the dynamic solution. P is time dependent. Now, for dynamic solution, we already discussed the various algorithms to do the time merging. One of them was the beta Newmark method. We did this discussion in week 5. So, here in the beta Newmark scheme. It goes like this. At every time step n plus 1, we know the quantities at the previous time step. So, at the n plus 1th time step, we know d double dot, at the time step n, we know d dot at time step n and we know d at time step n. So, based on those known quantities, we find out our d double dot at n plus 1, because in this expression all the quantities like d double dot and d are from the previous time step. So, we can find this out. After you find out d double dot at n plus 1, we can find out our $d_{n \text{ plus } 1}$. After you find out $d_{n \text{ plus } 1}$ and d double dot at n plus 1 we can find out d dot at n plus 1 and the and the time merging proceeds. Then we go to the n plus 2th time step and at that time step n plus 1 quantities are n plus 1 are known to me. So, I can do the same thing. However, here because this problem is highly non-linear, evaluation of this d double dot at n plus 1 involve the iteration. Because here, I have the force matrix here.

Now, to do this again we do the same procedure. So, whatever iteration we defined here, that entire iteration has to be done to solve this equation. So, for solution of this equation, this iteration has to be repeated. So, within each time step, we have to find out d double dot at n plus 1 by this kind of iteration. And after that we can find out d dot at d and $d_{n \text{ plus } 1}$. So, here if I want to find out this by iteration, again I make a guess of epsilon and based on guess of epsilon that we make, we can find out d dot at n plus 1, and that will give us d at n plus 1. And once we get that and we can find out d dot also. And then once we get that d at n plus 1, we can find out the new epsilon and again the procedure is repeated till my epsilon converges.

(Refer Slide Time: 25:30)



So, within each time step, the evaluation of these quantities they involved the iteration that was discussed in the previous slide and it continues. So, that is what that is what about the iterative solution of the finite element equation that we get.

Now, if you want to solve the same problem using the Ritz method, we can do it in this way. Again, we can write the variational form. So, let us write it down. We have the last quantity. Here the virtual work is at the point L_b, because that is where we are applying the externally applied load.

$$\int_0^L \rho A\ddot{w}\delta w dx + \int_0^L EIw_{,xx}\delta w_{,xx} dx - \int_0^{L_a} \bar{F}_H\, w_{,x}\delta\left(w_{,x}\right) dx - \bar{F}_v \delta w|_{x=L_a} - \bar{M}\,\delta\left(\frac{\partial w}{\partial x}\right)\bigg|_{x=L_a} - Pw|_{x=L_b} = 0$$

Now, here instead of having a localized basis functions, what we assume is w as a function of x and t is phi x multiplied by, phi_j x multiplied by q_j t, where phi_j is a known function of x and q_j is unknown function of time.

$$w(x,t) = \sum_{j=1}^{N} \phi_j(x) q_j(t)$$

So again, we follow the same procedure that we discussed for Ritz method, we put everything in the virtual work equation and then finally, we get the quantities as finally, we get the a set of ordinary coupled differential equations which looks like this.

$$[M]\{\ddot{q}\} + [K]\{q\} = \{F\}$$

And then, we have M_{ij} is equal to 0 to L, phi, I should have the rho A also, rho A phi_i phi_j dx. And then, we have k_j is equal to E I phi_i comma x x phi_j comma xx dx minus we have F_H bar, 0 to L_a. So, this quantity 0 to L_a, not L. Phi_i comma x phi_j comma x dx.

$$M_{ij} = \int_0^L \rho A \phi_i \phi_j dx$$

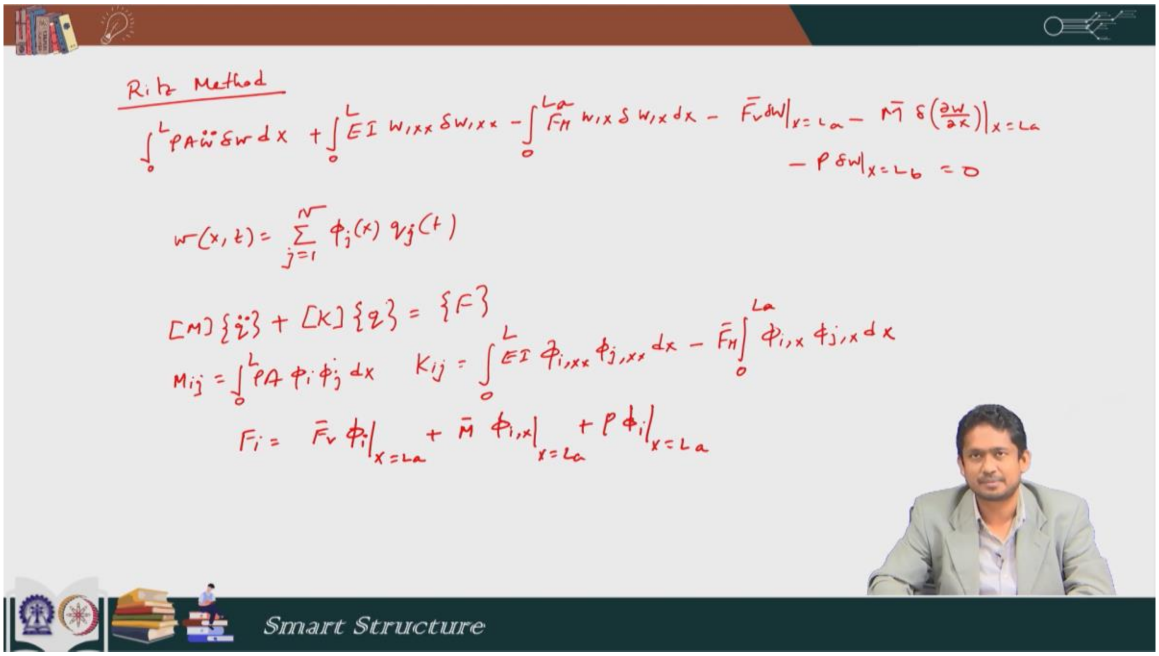
$$K_{ij} = \int_0^L EI\phi_{i,xx}\phi_{j,xx} dx - \bar{F}_H \int_0^L \phi_{i,x}\phi_{j,x} dx$$

And then we have the force as F_V bar multiplied by phi_i evaluated at x equal to L plus M bar multiplied by phi_i comma x evaluated at x equal to L, L_a, plus p multiplied by phi_i evaluated at x equal to L_a.

$$F_i = \bar{F}_V \phi_i|_{x=L_a} + \bar{M}\phi_{i,x}\big|_{x=L_a} + \rho\phi_i|_{x=L_a}$$

So, this is the set of ordinary differential equations that we get when we use the Ritz method. In the finite element problem our force matrix was mostly 0 except for few places here the force matrix is populated fully. And then the solution remains same. We have to iteratively find out q. So, there we are finding out d, here we will be finding out q. And the iteration has to be in such a way that epsilon$_c$ converges.

(Refer Slide Time: 25:10)



Now, if we want to simplify it, let us assume that we have a one term approximation for w and the approximation is x by L cube multiplied by q_1.

$$w(x) = \left(\frac{w}{l}\right)^3 q_1$$

So, that gives the equation as L by 7, q_1 double dot, plus 12 multiplied by EI, 12EI by L cube, minus 9 F_H bar, 5L multiplied by q_1 minus F_V bar, L_a by L. So, it is L suffix a, minus 3M bar L_a square by L cube, plus P into L_b equal to 0.

$$\frac{L}{7}\ddot{q}_1 + \left(\frac{12EI}{L^3} - \frac{4\bar{F}_H}{5L}\right) q_1 - \bar{F}_V \frac{L_a}{L} - \frac{3\bar{M}L_a^2}{L^3} + PL_b = 0$$

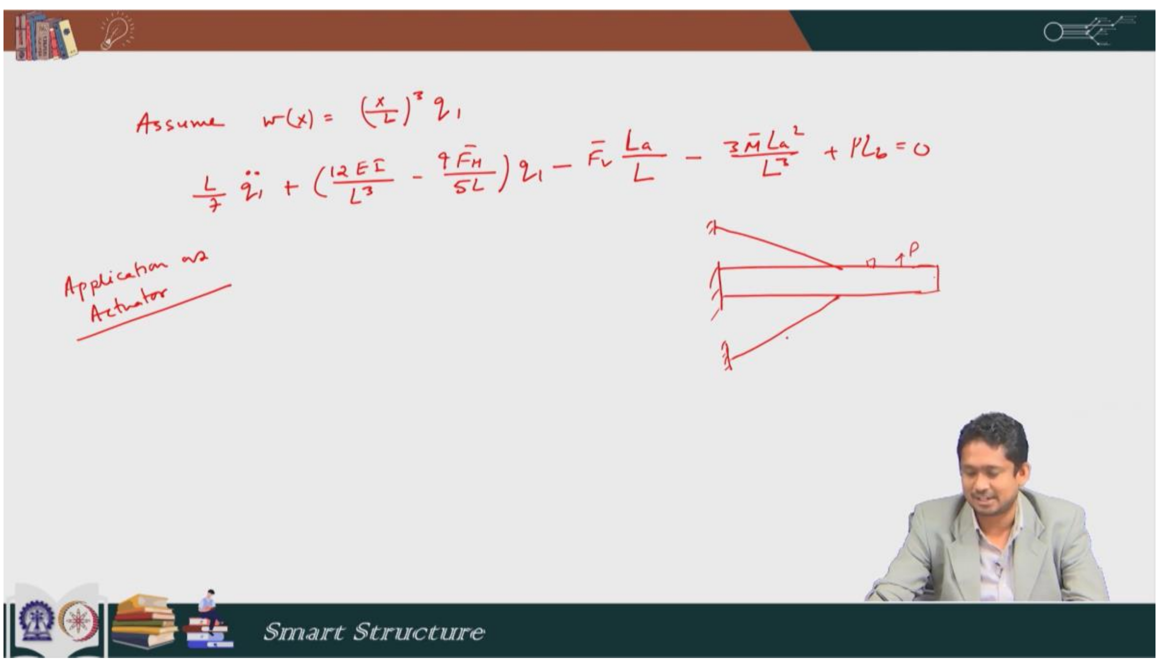
So, here I have the force terms directly written in terms of F_V bar and M bar. So, here I have just one equation and this equation now has to be solved again iteratively. So, we discussed. We have to do the time marching and within each time step. We have to iteratively find out this q's so that epsilon$_c$ converges. So, that is how this problem can be solved.

Now, shape memory alloys are used as actuators also in that case we have to give thermal actuation. So, for the same configuration, if it has to be used in actuation mode. So, application as actuator for the same problem, if we want to use the same configuration, I mean, the shape memory alloys put in the same configuration as actuators. In that case, let us assume that both the wires is in the martensite phase. So, martensite phase with some permanent deformation.

Now when due to the application of this force P, this beam wants to go up. In that case, what we can do is – because the upward moment of P, upward moment of the beam would try to extend this wire. Now, if I give a heat actuation here, then the heat actuation would try to take the wire to its previous shape which has a length less than the length that is here. So, that would generate an actuation force to the beam in the bottom direction and would try to prevent the upward moment. Similarly, when the beam tries to go downward, we can excite this actuator thermally and that would try to bring this shape memory alloy wire to the austenite phase. So, it would try to reduce the length of it and then the beam would experience an upward force. So, that is how it can be controlled by thermal actuation.

Now here, apart from the thermal apart from the temperature, the stress also comes into picture, because when the beam is going up or down the stress in the wires also changes. So, the same nonlinearity still remains which needs to be solved, but temperature also comes into picture because of the actuation. And the entire system can be designed in such a way that, it has to sense by a sensor, it has to sense whether the displacement is up, how the displacement is, how the velocity is, and accordingly, one of the actuators have to be excited to control the displacement.

(Refer Slide Time: 29:28)

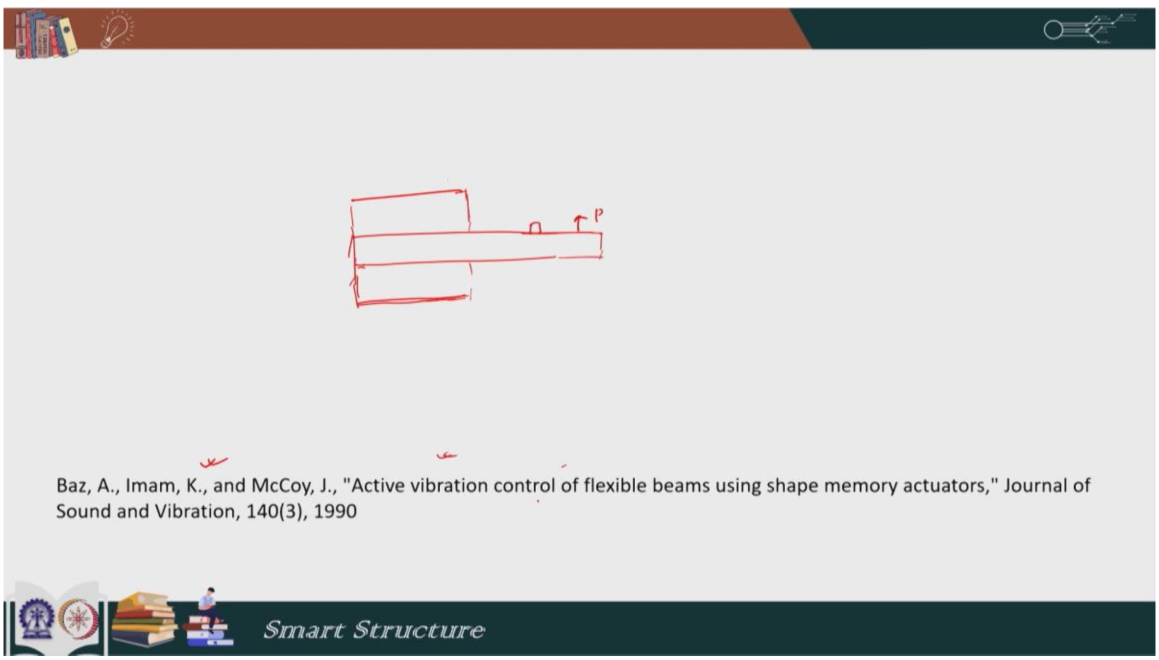


So, for this, this paper can be referred to see an application like this. There are various papers which talks about applications of shape memory materials for vibration control or shape control. However, in this paper it is one of the earliest papers in this area here the actuation equation is much more simplified. So, this can be a good paper to start with to read it read those things.

In this paper the system is little different from the paper that we saw before. Here, we have a beam in this paper and there are two parallel shape memory alloy rods and they are connected to the beam. And this entire beam is vibrating under the action of some load and there are some sensors which sense the displacement and velocity of the beam.

Now, the formulation given in this paper includes the bending stiffness of the shape memory alloy rods at the top and bottom also. And by giving thermal excitation to one of the shape memory alloy rods, based on what the sensors senses the vibration is controlled. The formulation can be seen in the paper.

(Refer Slide Time: 31:00)



So, with that I would like to conclude this lecture here.

Thank you.