Smart Structures
Professor Mohammed Rabius Sunny
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur
Week 10
Lecture No: 52
Analysis of a Beam with Shape Memory Alloy Wire
Part 03

In this lecture, we will discuss modeling of a beam, which has shape-memory alloy wire attached to it for damping application.

So, for that the problem that we discuss is taken from this paper by Gandhi and Chapuis, passive damping augmentation of a vibrating beam, using pseudoelastic shape memory alloy wires. Now, in this paper, the material modeling that is used is based on complex elastic modulus. So, that is a different model. However, in our discussion, we will use the constitutive relations of shape-memory alloy that we discussed in the last week. And also, many of our notations would be different from the paper.

Now, the structure that is considered here is a beam and it looks like this. So, we have a beam structure. It is a cantilever beam and the beam. So, the cantilever beam has an externally applied load P and this load P can be time dependent.

And let us assume that, the load P is applied at a distance L_b from the root of the beam and entire length of the beam is say L. And apart from that, we have two shape memory alloy wires attached to it in this way. One is at the top surface here and another is at the bottom surface here. And, they are symmetrically placed which means the angle here and the angle here is same and the both the alloys have same property. And, let us say that the location at which, the wires are fitted is at a distance L_a from the root of the beam.

So, at a distance L_b, we are applying the force. At a distance L_a, we are attaching the shape memory alloy wires for damping purpose. Now here, these two wires are prestressed which means this length, this length is more than original length of the wire, which means under this undeformed condition both the wires are under some tension, and the tension is equal because they are put symmetrically. So, if we look at the phase diagram of the shape-memory alloy, which looks like this. Now, these wires are under prestress, tensile prestress. So, we may say that in the undeformed configuration, it is somewhere here. And, let us also assume that the material is under austenite condition, in this prestress condition. So, it is in austenite condition. So, xi equal to 0 here. And, because it is under some prestress, we may say that, in the phase diagram, the material is somewhere here and here xi is equal to 0. So, when this beam tries to go up, when this beam tries to go up. So, this point moves up if the beam goes up, in that case, the length of the upper wire reduces, which means the tension is reducing here. So, it generates some amount of compression because of the upward movement. So, the net tension reduces. That means, in the phase plane diagram, the upper wire goes down. The stress reduces whereas, the bottom part of the wire experiences increasing length because this point is going up. So, because of that the tension here, it is under more tension.

So, the beam further goes up, the top wire, the stress reduces for the bottom wire stress increases. So, when the bottom wire goes up and crosses this M_s and M_f lines, it becomes martensite. And then, when the beam deforms in the downward direction, the reverse thing happens. So, the top part of it comes down the bottom part of the wire, which is here, it comes down and comes here. In that way, both the wires along vertical line, which is parallel to parallel to the sigma line, goes up and goes down. So that means, it shows some super elastic behavior. And we know that, when the behavior is superelastic, it is associated with some energy dissipation because a superelastic behavior looks like this. So, between the loading and unloading, we see this loop and that signifies deception of energy, and that is the source of damping generated due to the shape shape memory alloy wires.

Now here, we have to keep one thing in mind that depending on the amount of prestress these wires have and depending on the deformation this beam is showing, the wires can also cross this line and go to the compression region, that is also possible. If the compression is not very high, that is ok. If the compression is very high, then due to the compression also phase transformation may take place and that we have to incorporate. However, if the compression is not so high, if the pre stress and the beam deformation is such that it barely touches this line or just shoots little bit down in that case, we do not bother about the phase transformation due to compression otherwise, we have to take care of that into account. So, that is the entire mechanism of damping augmentation by this shape memory alloy wires.

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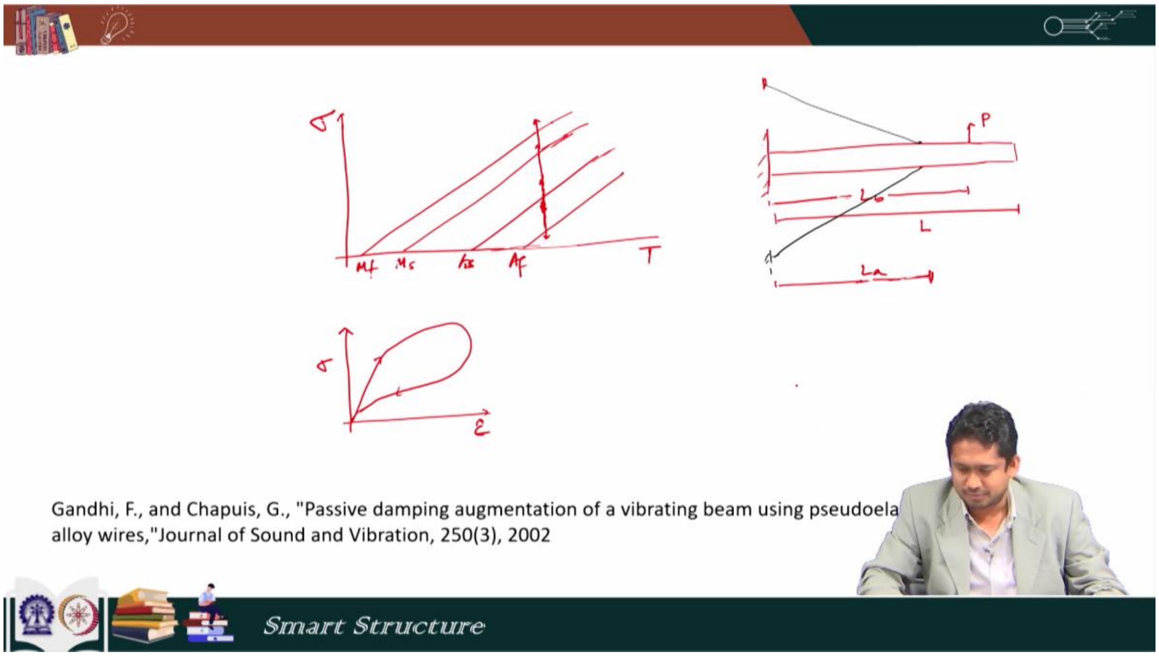

Now, if you want to go to the mathematical model, we need to first see, what are the forces acting on this beam. So, this is our beam fixed here and we have seen that we have a shape memory alloy wire here, and a wire here. So, we are just drawing the part which is attached to the beam. So, the top part has a tension T_1 and the bottom part has a tension T_2.

Now, because they are symmetrically placed, when the beam is undeformed T_1 and T_2 are same. Otherwise, T_1, T_2 are different. And we have a force applied P here, that is the externally applied force. So, because of the externally applied force the beam vibrates and the wire because of its pseudoelastic nature, it tries to dampen the vibration.

Now, if we draw the same diagram below, then these two forces T_1 and T_2, they give a horizontal force, which we can call F_H bar, and they give a vertical force which we can call F_V bar. And apart from that, there would be a moment that is induced and that we can call as M bar. Again, if T_1 and T_2 are same in the undeformed configuration the moment is 0 otherwise, there can be a moment. And the force P that was acting here that remains same. As per our consideration, this length is L_a and this is L_b. Now, also let us consider that the thickness of the beam is t_b. Now, the question is what is F_H bar? what is F_V bar and so on?

So, F_H bar. If I consider this angle as theta and also this angle as theta in the undeformed configuration, then F_H bar is minus of cosine theta and we have, T_1 plus T_2. And then, F_V bar is T_1 sin theta minus T_2 sin theta. And M bar, M bar is cosine theta t_b by 2 multiplied by T_1 minus T_2.

$$\bar{F}_H = -\cos\theta\,(T_1 + T_2)$$

$$\bar{F}_V = T_1 \sin\theta - T_2 \sin\theta$$

$$\bar{M} = \cos\theta \frac{t_b}{2}(T_1 - T_2)$$

Because if, I look at the horizontal component of the force T_1, it is T_1 cosine theta and also the horizontal component of force is T_2 cosine theta. So, T_1 cosine theta and T_2 cosine theta if they are not equal, they are going to give rise to a moment, which is of amount this, and P remains as it is. Now, considering all these forces that are applied, the variational form of the beam looks like this. The variational form is 0 to L, rho A w double dot, delta w, the variation of w, plus we have, the internal virtual work, which comes as w comma xx multiplied by w comma xx, here w comma xx means, derivative of w with respect to x done twice. Now comes the external virtual works due to these four forces.

So, two forces F_{v1}, F_v bar, F_{v1}, F_v bar, and F_H bar, and the moment M bar, and also this force P. Now, the virtual work due to the force F_H bar is integral within the domain 0 to L_a of F_H bar multiplied by w comma x into variation of w comma x. And then, we have applied force F_v bar. So, the virtual work here is F_v bar multiplied by the displacement here. So, that is F_v bar and the displacement at that point can be written as w at x equal to L_a. And then, we have moment M bar. And again, we can write the displacement, because its moment, we have to multiply by the virtual slope at that point. So, we have to put variation symbol here delta because it is a virtual displacement. And here it will be virtual slope. So, delta of del w by del x at x equal to L_a. And then we have externally applied force P. So, it is P multiplied by w at x equal to L_b. So, this is the total virtual work equation for this problem.

$$\int_0^L \rho A \ddot{w} \delta w dx + \int_0^L EI w_{,xx} \delta w_{,xx} dx - \int_0^{L_a} \bar{F}_H\, w_{,x} \delta\left(w_{,x}\right) dx - \bar{F}_v \delta w|_{x=L_a} - \bar{M}\,\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{x=L_a} - Pw|_{x=L_b} = 0$$

Now, the question is – why this expression is so, why is the virtual work done by F_H bar looks like this. So, at this point, the force F_H bar is applied and it is applied horizontally. Now, the virtual work done by this force is – this force multiplied by the virtual displacement at this point. Now, we are assuming that the beam is extensionally stiff. So, there is no extensional contraction in this direction. However, because the beam bends, and because of the deflection in the vertical direction. So, if the part of the beam from x equal to 0 to x equal to L_a in the undefined configuration was like this then, because of the deflection, it deforms in this fashion.

Now, the length of the neutral axis remains same. So, this length is same as this length which means that, there should be some difference in x coordinate of this point from this point to keep the length same, because this is a curve and this is a straight line. And this displacement is half into w comma x square 0 to L_a dx.

So, the virtual work done by that force is F_H bar multiplied by half into variation of w comma x square integrated from 0 to L_a. And then, we know by the property of this operator that, this is equal to twice w comma x multiplied by variation of w comma x. So, finally, the expression becomes – w comma, finally, the expression becomes w comma x multiplied by variation of w comma x dx. So, that is here.

$$\int_0^{L_a} \bar{F}_H \frac{1}{2}\left(w_{,x}\right)^2 dx = \int_0^{L_a} \bar{F}_H w_{,x} \delta(w_{,x}) dx$$

Now, F_H is this. So, once we take this negative sign inside this entire expression becomes positive. So, F_H is a negative quantity that's why it works in this direction.

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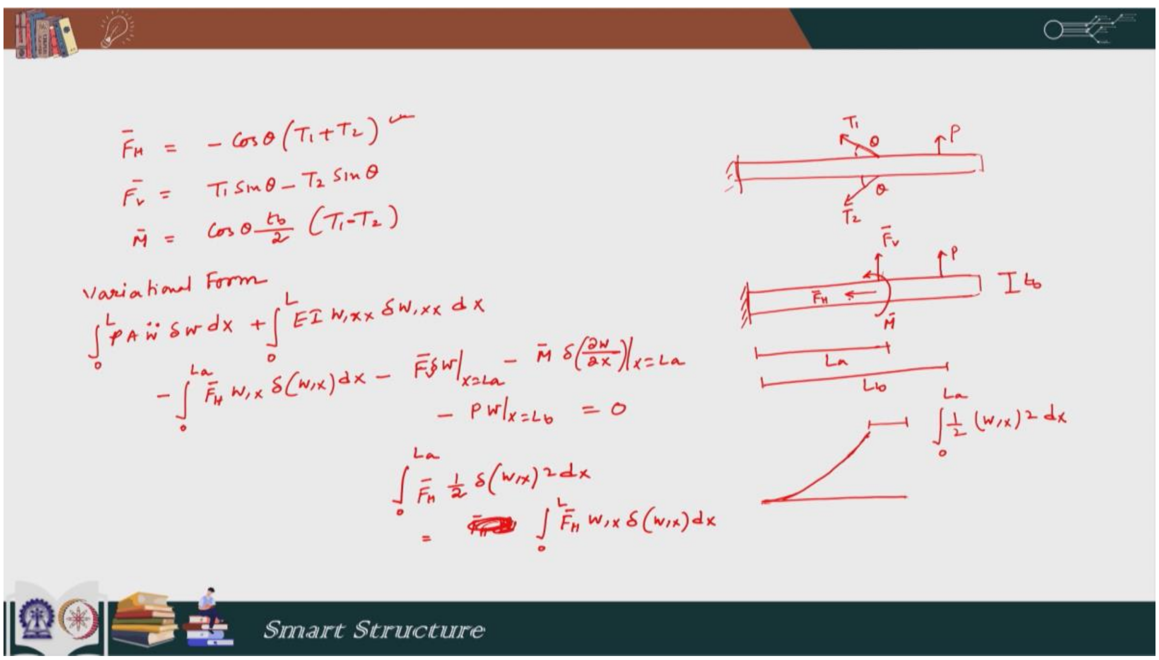

So, this is the virtual work expression for this.

Next is, we have to do the finite element formulation. So, if we discretize this entire beam into a set of elements, it looks like this. So, we can call it element number 1, and this is a generic element maybe element number e and then let us call an element number N. So, at the second node of element number N say, we have the shape memory alloys wires attached. So, here the forces due to the shape memory alloy wire is experienced. So, I have F_v bar here, I have F_H bar acting here, and I have the moment M bar acting here. And let us say we have element number M and at the end of the element number M, we have a force P, externally applied force P acting here. So, while discretizing, we have to make sure that, the point at which the loads are applied, this point loads are applied, they lie on some node. So, for this case, the first point where I have the shape memory alloy forces acting, that is at the second node of the Nth element. And the point at which the force P is acting that is at the second node of Mth element.

Now, we have to do the finite element formulation. So, after doing the formulation as described in last two lectures, the elemental equation that is obtained is m e multiplied by d e plus the elemental stiffness vector multiplied by d equal to 0. So, in this case we do not have any distributed force.

$$[m]^e\{\ddot{d}\}^e + [k]^e\{d\}^e = \{0\}$$

So, we are keeping the force vector 0. There are some discrete forces that will apply later on after assembling all the equations. Now, as we saw before the m_{ij} has the same definition rho A multiplied by N_i e N_j e dx. So, each element has a local coordinate system going from x equal to 0 to x equal to l_e, where l_e is the length of the element. We can write here once more.

$$m_{ij}^e = \int_0^{l_e} \rho A N_i^e N_j^e \, dx$$

And k, k has two parts because if we look at, if we look at the virtual work equation here, k comes from this equation as well as this equation. So, finally, k has two parts, we can call it k_1 e plus k_2 e, k_1 e has contribution from this expression, and k_2 e has contribution from this expression.

$$[k]^e = [k]_1^e + [k]_2^e$$

We know what is the contribution from this expression we discussed it in last two lectures. So, k_1 e ij is as seen before E I N comma xx, N_i comma xx, and N_j comma xx, dx. And k_2 ij is negative of F_H bar, multiplied by 0 to L_e, into N_i comma x, multiplied by N_j comma x.

$$k_{1\,ij}^e = \int_0^{l_e} EIN_{i,xx}^e N_{j,xx}^e \, dx$$

So, if you follow the same procedure that was described while discussing finite element method in last two classes from this expression, we get this as a matrix. Now, this quantity is valid only between x equal to 0 to L_a, because this integral is from x equal to 0 to L_a. Beyond x equal to L_a, this quantity is 0. So, we can may be define a quantity here eta, which is multiplied here and we can say that eta equal to 1, when e is less than equal to N.

$$k^e_{2\,ij} = \left[-\bar{F}_H \int_0^{l_e} N^e_{i,x} N^e_{j,x}\, dx \right] \eta$$

That means, any element where the element number is less than equal to N, which means any element which lies between x equal to 0 to L_a have eta equal to 1. That means, this quantity is as defined, otherwise eta equal to 0, when e is greater than N. So, anything beyond the element number N, those elements do not have any contribution from k_2.

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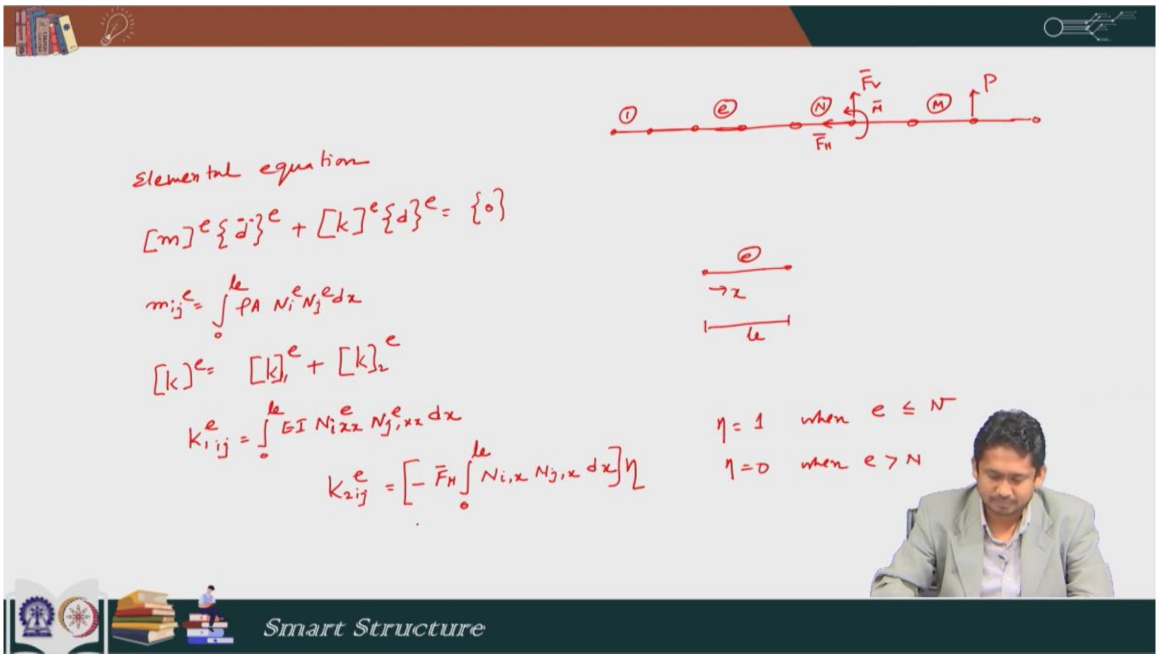

And the force matrix is 0 for the time being. And then, after we assemble it, after assembling the entire system looks like this. So, after assembling, this is our globally assembled matrix. The assembling process we discussed in the last two lectures. So, this is my global mass matrix and this is my global stiffness matrix.

Mass matrix is multiplied with the global degrees of freedom, the second order derivatives of the global degrees of freedom and the stiffness matrix is multiplied with these d's. And then, what we do here is we apply the essential boundary condition. So, as per the essential boundary condition, the beam is fixed at the root, which means the first two degrees of freedom d_1 and d_2 is 0, because we know that d_1 means, our d_1 means displacement at the first point and d_2 means slope at the first point. So, they are 0. So, we can just strike out these corresponding rows and columns. And whatever is remaining is our reduced set of equations by solving which we can get the solution. Now here, we have a globally assembled force matrix and this force matrix we are calling F.

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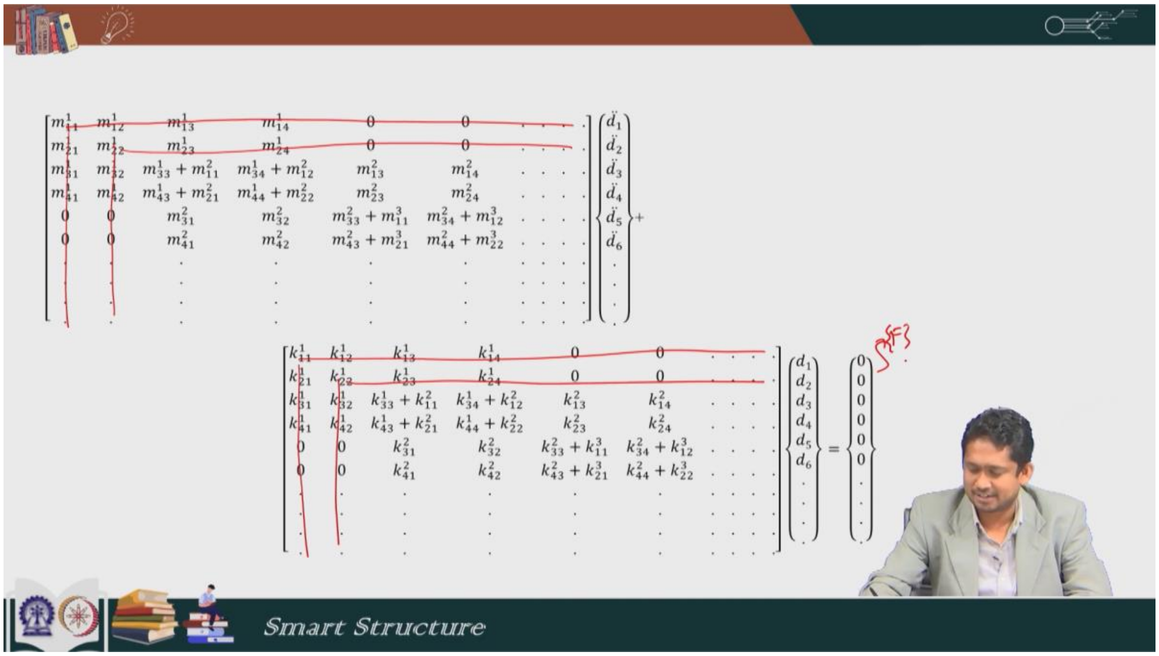

Now, this force matrix is not entirely 0, because we have to incorporate the contributions from the point loads that are applied here. So, how would it come? to discuss that let us first look at the numbering of the global degrees of freedom.

So, we have d_1, d_2. So, d_1 is displacement at the first node. d_2 is the second slope at the first node. And then d_3, d_4 it goes on. Here the degrees of freedom become $d_{2N \text{ plus } 1}$ and $d_{2N \text{ plus } 2}$. Similarly, at the end of the Mth element, it becomes $d_{2M \text{ plus } 1}$, $d_{2M \text{ plus } 2}$. Now, the effect of F_H is already incorporated, because my k_2 matrix has F_H here. I need to incorporate F_v bar, F_v bar is the vertically applied force here. So, that corresponds to the degree of freedom $d_{2N \text{ plus } 1}$, because $d_{2N \text{ plus } 1}$ is the displacement. So, force in the virtual work equation, force is multiplied with the virtual displacement, here the displacement is d of 2N plus 1.

So, F_v bar corresponds to the degree of freedom $d_{2N \text{ plus } 1}$. Similarly, this is moment and the moment is multiplied by the virtual slope.

And the slope is here $d_{2N \text{ plus } 2}$. So, M bar is multiplied with the variation of $d_{2N \text{ plus } 2}$. And here, I have force here, that corresponds to $d_{2M \text{ plus } 1}$.

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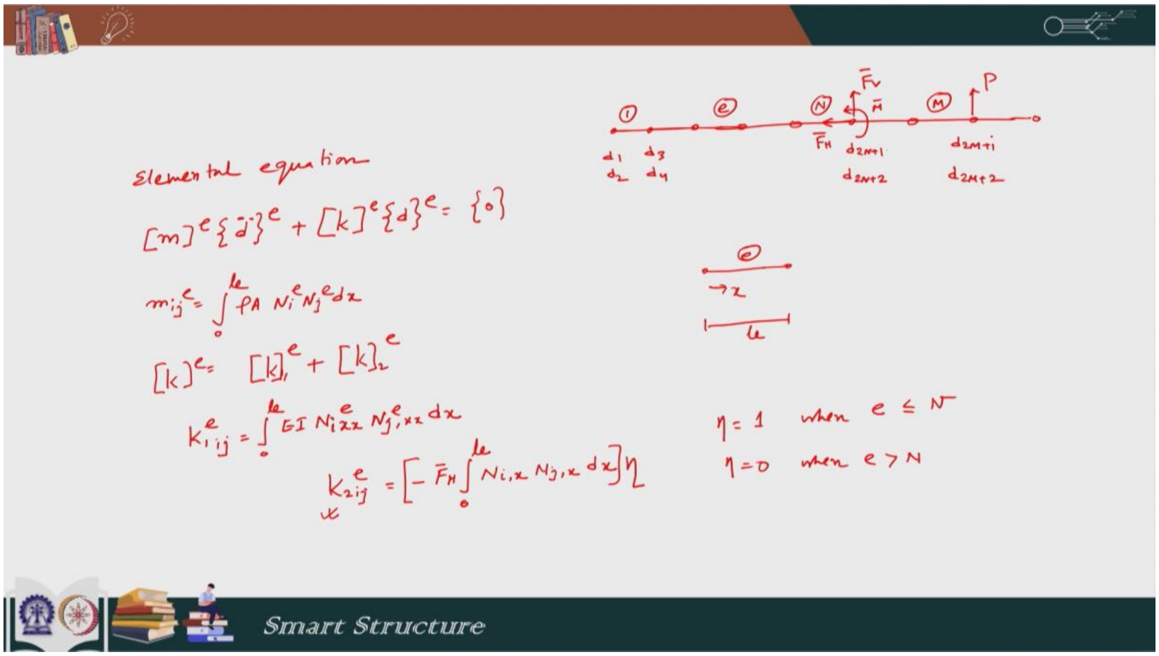

So, accordingly in the force matrix we have to put – the definition of the force matrix goes like this: F_i equal to 0, if i is not equal to 2N plus 1, and i is not equal to 2N plus 2, and also i is not equal to 2M plus 1. So, other than 2Nplus 1, 2N plus 2 and 2M plus 1, for all other indices F_i is 0. And F_i, F_i at 2N plus 1, which we can write as $F_{2N \text{ plus } 1}$ is the vertically applied force, because we have seen that the vertical force due to those shape memory alloy wires corresponds to the degree of freedom $d_{2N \text{ plus } 1}$. And $F_{2N \text{ plus } 2}$ is accordingly M bar. And $F_{2M \text{ plus } 1}$ is accordingly P. So, the force matrix for these three indices has some values otherwise they are 0. So, here it is not that everything is 0, these three quantities are non-zero, rest of them are 0.

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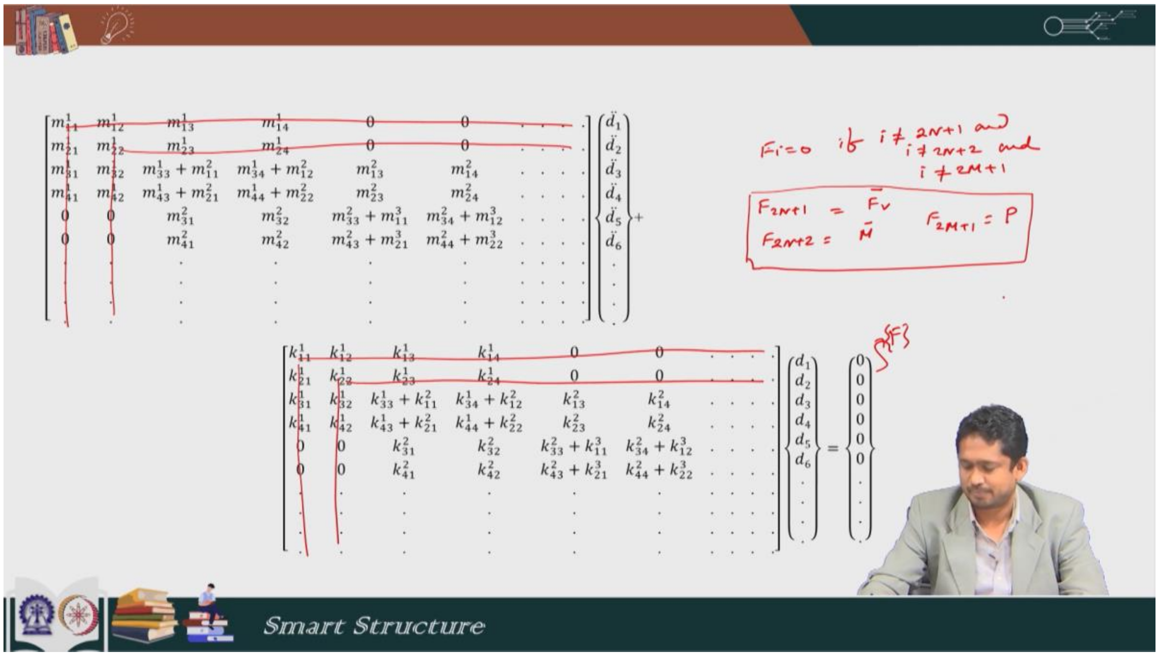

Now, after we strike out everything, we are left with rest of the system, and we solve rest of it and we get our solution.

So, the rest of the system, the global F equation, global finite element equation after applying essential boundary condition. Let us write as M_d double dot, plus K global K, d equal to F. So, if we solve this equation, we will get our d. But here again, we have to understand few things that, the components of the force vector, if we go back which is which are F bar, M bar and P and also in the k_2 matrix, we have F_H bar, ok, not the P, P is the externally applied load. So, the other components F bar and M bar and this F_H, they come from the contribution due to the shape memory alloy. And, the shape memory alloy wires have stresses and those stress are dependent on the strain in the shape memory alloy wire, and those strain depend on the displacement at this point.

So, the force F bar, moment M bar and this force F_H bar, they are function of the displacement at that point itself. So, here force is dependent on the displacement. So, based on the force we can solve and get the displacement, but again the displacement influences the force.

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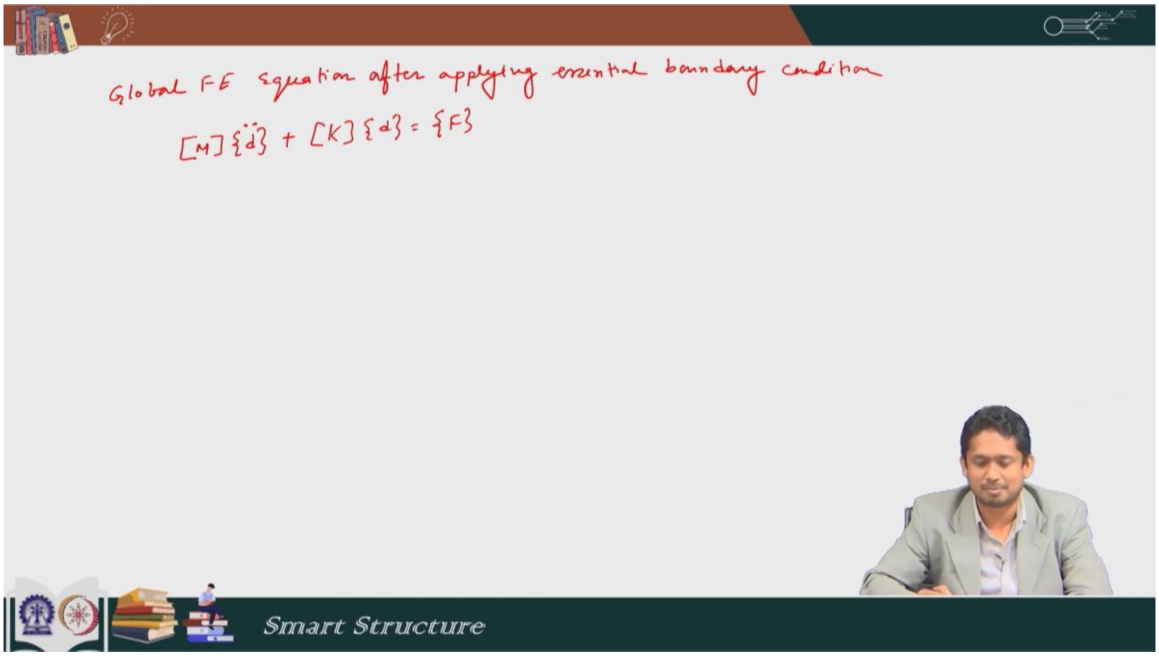

So, it is a highly non-linear problem and that needs to be solved by iteration and that we will discuss in the next class.

Thank you.