Smart Structures
Professor Mohammed Rabius Sunny
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur
Week - 07
Lecture No - 36
Finite Element Formulation of Euler - Bernoulli Beam - Continued

In the previous lecture we were discussing about the finite element formulation of Euler Bernoulli beam. We discretize the beam into a set of elements total number of elements was N e and then we did the elemental formulation and we ended up getting a set of coupled ordinary differential equations which we call elemental finite element equations and that equation consists of mass matrix, stiffness matrix and force vector. So, today we will start from there. Now, for this problem for the case of Euler Bernoulli beams the mass matrix the explicit expression for mass matrix looks like this. So, we have and then we have 6 11 L e square by 210 9 L by 70 minus 13 L e by 420 and here we have 11 L e square by 210 L e cube by 105 13 L e square by 420 minus L e cube by 140. So, if I do the integrations over the elemental length that I get as an expression for the mass matrix and then we have 9 L e by 70 13 L e square by 420 and then 13 L e by 35 and minus 11 L e square by 210 and then here we have minus 13 L e square by 420 minus L e cube by 140 minus 11 L e square by 210 and the final diagonal term is L e cube divided by 105.

$$[m]^e = \rho A \begin{bmatrix} \frac{13l_e}{35} & \frac{11l_e^2}{210} & \frac{9l_e}{70} & -\frac{13l_e^2}{420} \\ \frac{11l_e^2}{210} & \frac{l_e^3}{105} & \frac{13l_e^2}{420} & -\frac{l_e^3}{140} \\ \frac{9l_e}{70} & \frac{13l_e^2}{420} & \frac{13l_e}{35} & -\frac{11l_e^2}{210} \\ -\frac{13l_e^2}{420} & -\frac{l_e^3}{140} & -\frac{11l_e^2}{210} & \frac{l_e^2}{105} \end{bmatrix}$$

So, this we get after doing the integration and as we can see that this matrix that we get is a symmetric matrix. Now, the symmetry is there in stiffness matrix also the stiffness matrix that we get is this we have the stiffness matrix that comes is we have E I sitting outside then 12 by L e cube 6 by L e square minus 12 by L e cube and then again 6 by L e square and then we have 6 by 6 divided by L e square. In the diagonal term we have 4 by L and then minus 6 by L e square and 2 by L e L and then minus 12 by L e cube minus 6 by L e square and then in the diagonal term we have 12 by L e cube and minus 6 by L e square and in this term again it is symmetric. So, it is 6 by L e square this is 2 this is 2 by L e and again minus 6 by L e square and finally, in the diagonal term it is 4 by L e.

$$[k]^e = EI\begin{bmatrix} \frac{12}{l_e^3} & \frac{6}{l_e^2} & -\frac{12}{l_e^3} & \frac{6}{l_e^2} \\ \frac{6}{l_e^2} & \frac{4}{l_e} & -\frac{6}{l_e^2} & \frac{2}{l_e} \\ -\frac{12}{l_e^3} & -\frac{6}{l_e^2} & \frac{12}{l_e^3} & -\frac{6}{l_e^2} \\ \frac{6}{l_e^2} & \frac{2}{l_e} & -\frac{6}{l_e^2} & \frac{4}{l_e} \end{bmatrix}$$

So, this is our expression for the stiffness matrix and again this is possible only when our rho A and E I they are not varying across the domain. So, at any x, so, E I and rho A if they are functions of x, then they also get integrated and I may not get this expression.

Then if I assume that my p z is constant, then the... so, instead of calling it K E i we just call it the entire K matrix and there is a generic definition for K i j which will see and similarly this is also f we look into the generic definition of f. So, f, the force vector becomes p z and that is multiplied with L e by 2, L e square by 12, L e by 2 minus L e square by 12.

$$\{f\} = \begin{Bmatrix} \frac{l_e}{2} \\ \frac{l_e^2}{12} \\ \frac{l_e}{2} \\ -\frac{l_e^2}{12} \end{Bmatrix}$$

So, here m i j in general is defined as 0 to L e rho A N i e multiplied by N j e d x and K i j is equal to 0 to L e by 2 L e square by 12. So, here m i j in general is defined as 0 to L e E I N i comma x x, N j comma x x d x and f i j e is 0 to L e p z N i e d x.

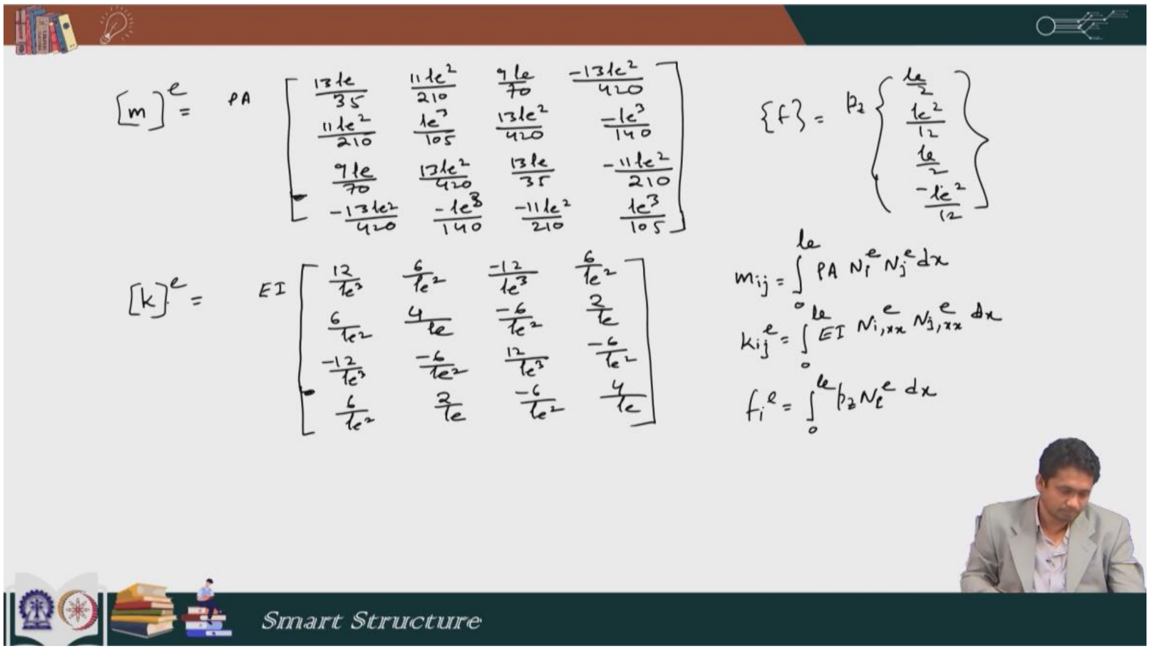
$$m_{ij}^e = \int_0^{l_e} \rho A N_i^e N_j^e dx$$

$$k_{ij}^e = \int_0^{l_e} EI N_{i,xx}^e N_{j,xx}^e dx$$

$$f_{ij}^e = \int_0^{l_e} p_z N_i^e dx$$

So, when p z E I and rho A they are not functions of x they can come out of the integral sign and rest of the term on being integrated give me this this matrix this matrix and this vector.

(Refer Slide Time: 07:46)



So, now that we have found out the elemental equations, we need to find out the global set of equations. So, to do that again let us remember that we had divided the integral sign by the integral sign of the term. So, we have to find out the global set of equations. So, let us say that the integral sign of the term is 0 to L e i minus L e i minus L e i equations. So, to do that again let us remember that we had divided the entire domain into a set of elements going from x equal to 0 to x equal to L and each of these elements has two nodes the nodes that are at the two end of the structure they are the two end not shared by more than one element whereas, the intermediate nodes are shared by two elements.

Accordingly, we defined our degrees of freedom. We had d 1 1, d 2 1, d 3 1, d 4 1 then d 1 2, d 2 2. Now this point cannot have more than one displacement otherwise the displacement continuity would not be there, which means that d 1 1 and d 1 2 they should be same. So, d 1 2 is equal to d 3 1 2, d 3 1 2 is equal to some value we call it d 2 we will see that and similarly d 4 1 is equal to d 2 2 is equal to some value. So, that brings us to another nomenclature of the degrees of freedom.

So, all these nomenclatures where I have a superscript they are with respect to some element. Now I put another nomenclature which is with respect to the entire global system. So, globally this is the first degree of freedom. So, it is d 1 first node first degree of freedom. So, it is d 1 and then d 2 then globally I have after that d 3, d 4, d 5, d 6, d 7, d 8 and we can go on. So, finally, I will have d 2 Ne plus 1 and d 2 Ne plus 2. So, we can say that ah d 1 2 is equal to d 3 1 is equal to d 3 and d 2 2 is equal to d 4 1 is equal to d 4 and accordingly this node also will have nomenclature with respect to elemental definition the degrees of freedom at this node. So, we have ah d 3 2 and d 1 3 they are same and they are equal to d 5 similarly d 4 2 is equal to d 2 3 they are same that is equal to d 6.

$$d_1^2 = d_3^1 = d_2$$

$$d_4^1 = d_2^2 = d_4$$

Now, we have seen the force terms also. So, if I look at the force terms, this, we are drawing it once again here.

So, we have the node specified. So, that one place does not get crowded. So, we are drawing it separately. So, here we have f 1 1, f 2 2 and f 3 1, f 4 1 the second one is not f 2 2 it is f 2 1. Similarly, f 1 2, f 4 1, f 5 1, f 6 1, f 2 2, f 3 2, f 4 2 and again this has f 1 3 and f 2 3 and the nomenclature can go on.

We have seen that there are 4 degrees of freedom and accordingly there are 4 components of the force vector also. So, that is what we are writing here. So, each element has 4 component of the force vector and again they are local numbering of the force vector. So, globally I can call them f 1, f 2, f 3, f 4, f 5, f 6 and so on. Now, f 1 1 is globally f 1, f 2 1 is globally f 2.

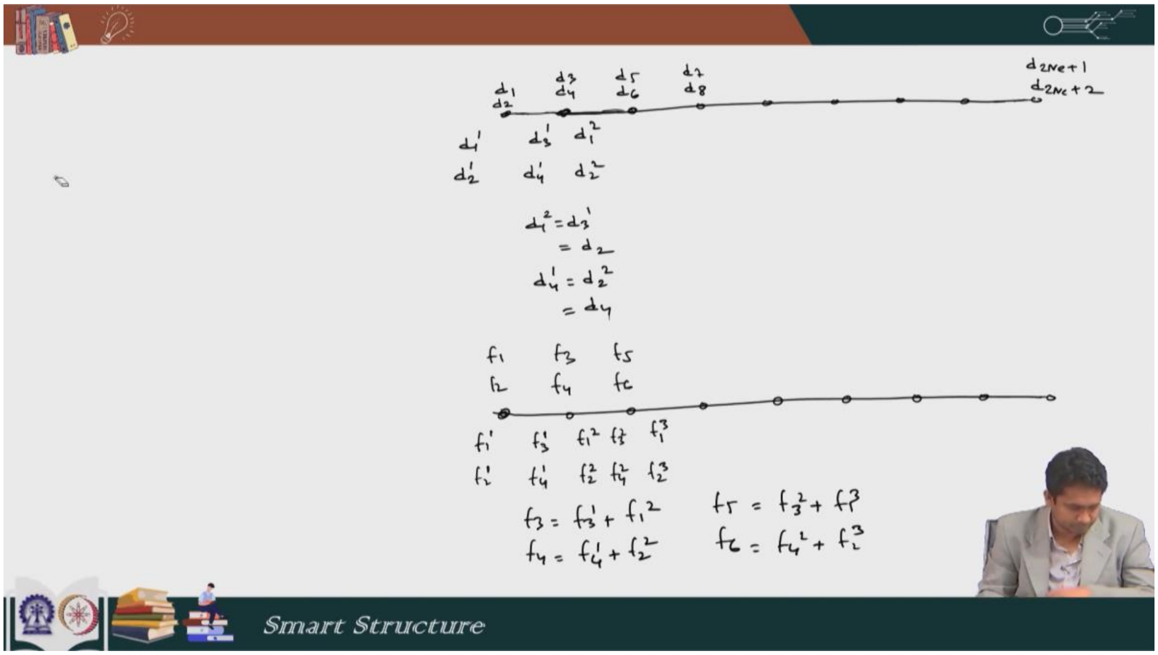
However, f 3 is globally a combination of f 3 1 plus f 1 2. Similarly, f 4 globally is the combination of f 4 1 plus f 4 2. So, here this force that I get from the element number 1 and this force that I get from element number 2 the adjacent element on being added they give me the force f 3. So, f 3 is f 3 1 plus f 1 2. Similarly, f 4 is f 4 1 plus f 2 2 and accordingly we can say f 5 is equal to f 3 2 plus f 1 3 and f 6 is equal to f 4 2 plus f 2 3.

$$f_3 = f_3^1 + f_1^2 \qquad f_5 = f_3^2 + f_1^3$$

$$f_4 = f_4^1 + f_2^2 \qquad f_6 = f_4^2 + f_2^3$$

So, now combining all the equations. So, now we have the finite element equation the 4 by 4 system from the first element and the 4 by 4 system from the second element. Then in the equation for the first element we have terms like d 3 1, d 4 1 we can instead of calling them as d 3 1, d 4 1 we can call them by their global number d 3, d 4. Similarly, in the second equation also we have in the second finite element equation set also we have d 1 2, d 2 2 and instead of calling them as d 1 2, d 2 2 we can call them as d 3 and d 4. And similarly, we have force and we can add up the forces and finally, after adding up.

(Refer Slide Time: 14:10)



So, if I look into the first elemental equation, we have m 1 1 or if I look into the first elemental equation, we have m 1 1 of 1 multiplied by d 1 double dot. So, we are writing the degrees of freedom in terms of the global number and then we have m 1 2, d 2 double dot and then plus we have m 1 3, d 3 double dot plus m 1 4, d 4 double dot plus k 1 1, d 1 these are all mass matrix corresponding to the first element and that is equal to f 1 of 1. Similarly, I have the second equation which I am not writing then the third equation is f 3 1 multiplied by d 1 double dot plus m 3 2 multiplied by d 2 double dot plus m 3 3 of 1, d 4 multiplied by d 4 multiplied by d 5 multiplied by d 6. So, f 1 multiplied by d 3 double dot plus m 3 4 of 1 multiplied by d 4 double dot plus we have k 3 1 d 1 plus k 3 1 of 1 d 2 plus k 3 3 of 1 d 3 multiplied by d 5 multiplied by d 5 plus k 3 4 of 1 d 4 is equal to f 3 of 1. And similarly, m 4 1 d 1 double dot m 4 2 d 2 double dot plus m 3 3 of 1 d 5 multiplied by d 5 multiplied by d 5 multiplied by d 5 m 4 3 of 1 d 4 double dot plus m 4 4 plus, we have m 4 4 of 1 d 4 double dot and then we have k 4 1 d 1 and then we have k 3 4 of 1 d 5 multiplied by d 5 multiplied by d 5. So, k 4 2 of 1 d 2 plus k 4 3 of 1 d 3 plus k 4 4 of 1 d 4 is equal to f 4. Please understand the second equation was not written, but that equation exists.

$$m_{11}^1\ddot{d}_1 + m_{12}^1\ddot{d}_2 + m_{13}^1\ddot{d}_3 + m_{14}^1\ddot{d}_4 + k_{11}^1 d_1 + k_{12}^1 d_2 + k_{13}^1 d_3 + k_{14}^1 d_4 = f_1^1$$

$$m_{21}^1\ddot{d}_1 + m_{22}^1\ddot{d}_2 + m_{23}^1\ddot{d}_3 + m_{24}^1\ddot{d}_4 + k_{21}^1 d_1 + k_{22}^1 d_2 + k_{23}^1 d_3 + k_{24}^1 d_4 = f_2^1$$

$$m_{31}^1\ddot{d}_1 + m_{32}^1\ddot{d}_2 + m_{33}^1\ddot{d}_3 + m_{34}^1\ddot{d}_4 + k_{31}^1 d_1 + k_{32}^1 d_2 + k_{33}^1 d_3 + k_{34}^1 d_4 = f_3^1$$

$$m_{41}^1\ddot{d}_1 + m_{42}^1\ddot{d}_2 + m_{43}^1\ddot{d}_3 + m_{44}^1\ddot{d}_4 + k_{41}^1 d_1 + k_{42}^1 d_2 + k_{43}^1 d_3 + k_{44}^1 d_4 = f_4^1$$

So, that was the four equations from the first element.

So, first element now if I write the second equation from the first element the equation would look like this m 1 1 of the second element multiplied by d 1, it is not d 1 now it is d 3 because d 1 of second element is globally d 3 plus m 1 2 of second element multiplied by d 4 plus m 1 2 of second element multiplied by d 4 dot plus m 3 of second element multiplied by d 5 dot plus m 4 of second element multiplied by d 6 double dot, these are all double dots, plus k 1 1 of second element multiplied by d 3 plus k 1 2 of second element multiplied by d 4 plus k 1 of second element multiplied by d 5 plus k 1 3 of second element plus k 1 3 of second element plus k 1 4 of second element multiplied by d 6 and that is equal to f 1 of second element. Now, we can see that this comes from the second element.

$$m_{11}^2\ddot{d}_1 + m_{12}^2\ddot{d}_2 + m_{13}^2\ddot{d}_3 + m_{14}^2\ddot{d}_4 + k_{11}^2 d_1 + k_{12}^2 d_2 + k_{13}^2 d_3 + k_{14}^2 d_4 = f_1^2$$

Now, we can see that if I look at this equation with the third equation from the first element and the first equation from the second element these two equations have d 3 and d 4 as common terms. And if I add these two equations in the right-hand side f 3 1 and f 1 2 gets added and they give me a force term f 3. So, if I add third equation of first element and first equation of second element that gives me a m 3 1 d 1 double dot plus a m 3 1 d 1 double dot plus a m 3 1 d 1 double dot plus a m 3 2 1 d 2 double dot then when I am adding this term and this term the mass terms gets added and I get m 3 3 1 of 1 m 3 3 of 1 plus m 1 1 of 2 they gets added and they are multiplied with d 3 double dot. Similarly, here I have d 4 double dot and here I have d 4 double dot. So, m 3 4 of first element and m 1 2 of second element gets added and that gives me m 3 4 of 1 plus m 1 2 of 2 and that summation multiplied by d 4 double dot.

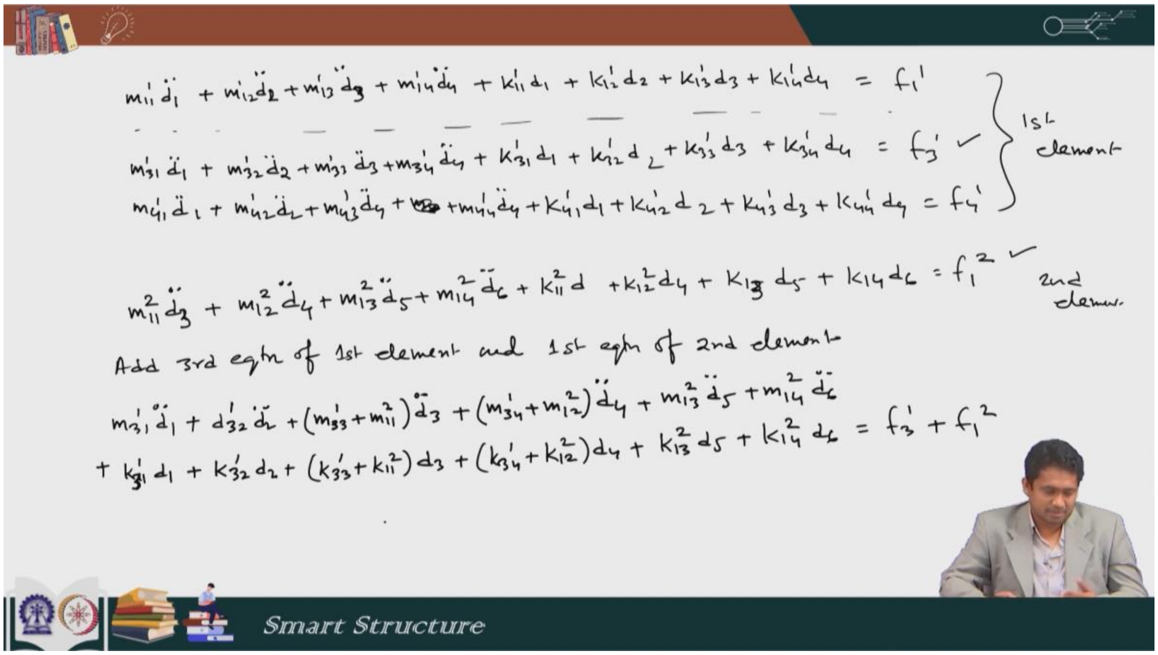
Then I have these two terms they do not gets added to anything m 3 2 d 5 double dot m 1 3 of 2 d 5 double dot, m 1 4 of 2 d 6 double dot. Likewise I have the stiffness terms here and they also gets added and that gives me k 1 1 of 1 multiplied by d 1 k 3 1 of 1 plus k 3 2 of 1 multiplied by d 2 plus k 3 3 of 1 plus k 1 1 of 2 multiplied by d 3 plus k 3 4 of 1 plus k 1 2 of 2 multiplied by d 4 plus k 3 3 of 1 multiplied by d 5 plus k 1 3 of 2 multiplied by d 5 plus k 1 4 of 2 multiplied by d 6 at the right hand side I have f 3 of 1 and f 1 of 2 added. So, I have f 3 of 1 plus f 1 of 2.

$$m_{31}^1\ddot{d}_1 + d_{32}^1\ddot{d}_2 + (m_{33}^1 + m_{11}^2)\ddot{d}_3 + (m_{34}^1 + m_{12}^2)\ddot{d}_4 + m_{13}^2\ddot{d}_5 + m_{14}^2\ddot{d}_6$$

$$+k_{31}^1 d_1 + k_{32}^1 d_2 + (k_{33}^1 + k_{11}^2) d_3 + (k_{34}^1 + k_{12}^2) d_4 + k_{13}^2 d_5 + k_{14}^2 d_6 = f_3^1 + f_1^2$$

So, likewise, I can add the fourth equation from the first element and the second equation from the second element and they also show equation like that.

(Refer Slide Time: 22:37)



In that way the intermediate equations can be added and a global system of equations can be formed and that system would be of size 2 N plus 2.

So, finally, the global system of equation is formed and the form system the global system of equation looks like 2 N e plus 2 multiplied by 2 N e plus 2. Then the global displacement vector which is of size 2 N e plus 2 multiplied by 1, I have a double dot here, then plus the global stiffness matrix again 2 N e by 2 multiplied by 2 N e plus 2 multiplied by 2 N e plus 2 and that is multiplied with a d vector that degree of freedom vector which is of size 2 N e plus 2 multiplied by 1 and that is equal to a right hand size 4 force vector which is of size 2 N e plus 2 multiplied by 1. So, the right-hand size 4 force vector which is of size 2 N e plus 2 multiplied by 1.

$$[M]_{\substack{(2N_e+2)\\ \text{x}\,(2N_e+2)}}\{\ddot{d}\}_{\substack{(2N_e+2)\\ \text{x}\,1}}+[K]_{\substack{(2N_e+2)\\ \text{x}\,(2N_e+2)}}\{d\}_{\substack{(2N_e+2)\\ \text{x}\,1}}=\{f\}_{\substack{(2N_e+2)\\ \text{x}\,1}}$$

So, that is our global finite element equation and then after that we can put the essential boundary conditions. So, for example, if I have a cantilever beam like this and in that case, this is the fixed end fixed end means the displacement and slope is 0 at this end which means our d 1 and d 2 the global boundary conditions d 1 and d 2 are 0 because we saw that... we have discretized into a lot of elements the first degree of freedom global is d 1 and d 2 the first one is the displacement the second one is the slope. So, these two quantities are 0 and that is our essential boundary condition. So, in this case I have to then apply the essential boundary conditions. To apply the essential boundary condition, what I do is... I remove the first row and first column from both a matrix and k matrix and that is how the system is reduced to a 2 N e by 2 2 N e system because our d matrix is of size 2 N e plus 2 and the first two degrees of freedom are already known to me, they are 0. So, I do not need to calculate them. So, I have to deal with only rest of it. So, starting from the degree of freedom d 3 to the last degree of freedom I have to deal with. So, I cancel the first row and first column of the mass matrix of the stiffness matrix and the force vector.

Then finally, I get a reduced set of linear simultaneous ordinary differential equations. After applying the essential boundary conditions and the boundary condition is d 1 equal to d 2 equal to d 1 of 1 equal to d 2 of 1 equal to 0 and if they remain 0 always that means, that d 1 equations and the boundary condition is d 1 equal to d 2 equal to d 1 of 1 equal to d 2 of 1 equal to 0 and if they remain 0 always that means, that d 1 double dot d 2 double dot d 1 of 1 double dot d 2 of 1 double dot always remains 0.

$$d_1 = d_2 = d_1^1 = d_2^1 = 0$$

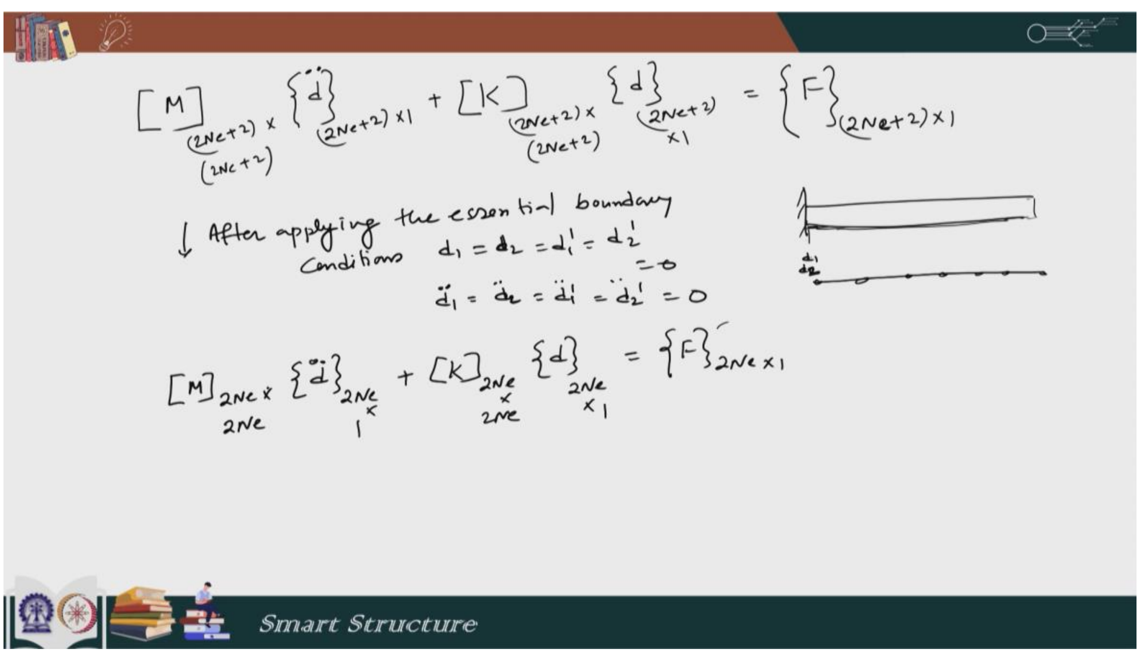
$$\ddot{d}_1 = \ddot{d}_2 = \ddot{d}_1^1 = \ddot{d}_2^1 = 0$$

If we set them 0 and we remove the corresponding rows and columns from the mass and stiffness matrix and then we get a reduced set of equations of size 2 N e by 2 N e and that gets multiplied with a d matrix of size 2 N e by 1 plus k 2 N e by 2 N e and that gets multiplied with the d vector of size 2 N e by 1 and then I have the force vector 2 N e by 1.

$$[M]_{2N_e \times 2N_e} \{\ddot{d}\}_{2N_e \times 1} + [K]_{2N_e \times 2N_e} \{d\}_{2N_e \times 1} = \{f\}_{2N_e \times 1}$$

Now, this set of equations I can solve. So, if my force is a time dependent force then we can solve this ordinary differential equation.

(Refer Slide Time: 27:05)



We have already discussed various techniques to solve the ordinary differential equations and within using that we can find out d at each and every time step. If we are interested in finding out the Eigen value of the system then we can set then we do not need f then this then the force vector is 0 and then we can solve the Eigen value problem considering m and k and that would give me the natural frequency and mode shape of the system. So, this is about the finite element formulation of the beam and so, this formulation was useful to understand the formulations that are given in the paper that we are going to discuss for analyzing a beam which has shape memory alloy fitted to it.

So, with that I would like to conclude the lecture here.

Thank you.