Smart Structures

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Week - 10
Lecture No - 50

Finite Element Formulation of Euler - Bernoulli Beam

Welcome to the 10th week of this course.

In this week we will look into analysis of structures which has shape memory alloy as a component. Now for this we will refer to some research papers. However, the formulations have been done in the papers using the finite element method. So, we need to spend some time in understanding the basics of finite element method. Now finite element method itself is a can be one course and there are lots of topics within it.

Here we will look into only a small part of it which we will find relevant to understand the mathematical formulation. Now the example structures that will be that we will see are beam structures. So, we will see how we do finite element formulation of beam with Euler Bernoulli approximation. So, these beams are all Euler Bernoulli beam.

While dealing with piezoelectric materials we have looked into mathematical formulations using the energy principle or the Galerkin technique and there all the formulations were based on Ritz method or the Galerkin method. So, there the approximation functions that we take are valid throughout the domain. So, when I solve a beam problem if the beam has a length of l then our approximation functions, they were also valid throughout the domain from x equal to 0 to x equal to l. However, in finite element method the basic difference is that all the approximations are piece wise valid. So, the forms that we saw the virtual work equations energy equations they all remain same.

However, when we put the approximations, the approximations are piece wise approximations. So, while analyzing a beam we saw the variational form as rho multiplied by u 0 dot minus z into del 3 w by del 2 del t 2 del x and then we have the variation of this term u 0 minus z into del w by del x minus z into del w by del x and the entire thing is valid throughout the domain plus we had rho multiplied by w double dot w and then d v. So, these are the contribution from the inertia terms. So, here we are solving a time dependent problem and then plus we have E multiplied by epsilon 0 minus z into kappa minus epsilon p that is the free strain. So, I am just writing whatever we had for the Euler Bernoulli beam in the piezoelectric case and then we will drop the terms which I do not need here.

$$\int_{v} \rho\left(\ddot{u}_0 - z\frac{\partial^3 w}{\partial z^2 \partial x}\right)\delta\left(u_0 - z\frac{\partial w}{\partial x}\right)dv + \int_{v} \rho\ddot{w}\delta w dv$$
$$+ \int_{v} E\left(\varepsilon_0 - zk - \varepsilon_p\right)\delta(\varepsilon_0 - zk)dv - \int_0^L p_x \delta u_s dx - \int_0^L p_w \delta w dx = 0$$

So, we are quite familiar with this formulation.

Now, here we are analyzing just a beam this formulation was written based on the fact that it has some piezoelectric component in it. Now, here we do not have any piezoelectric component. So, epsilon p we do not have. So, epsilon and then in this case I had distributed force across x direction and across z direction, but here we are assuming that all the forces are along z direction only.

So, we have p z only and p x equal to 0. So, no force along x direction and as we do not have any asymmetry with respect to the neutral axis. So, we do not have any coupling between the in plane and the out of plane terms. So, under a pure transverse load I mean the load which is applied along the z direction this beam will deflect only in the z direction. So, there will not be any u 0 component induced.

So, we do not have these as well and we do not have epsilon 0 the mid plane strain as well. And apart from that if I look into this contribution of this term whatever is left here the contribution of this term is also much less as compared to the contribution of the other terms because when these two terms get multiplied z square. So, z square on being integrated gives me I, the moment of inertia and I is much less as compared to the cross-sectional area A and density is also much less as compared to the elastic modulus. So, this quantity is also we are going to neglect to keep the formulation simple. So, finally, we have here if I integrate this with respect to the modulus of the beam.

So, integrate this term I will just write it once then we have rho, I multiplied by del w double dot by del x multiplied by variation of del w by del w by del x. So, this is the dx and then we have here rho A multiplied by w double dot multiplied by variation of w dx and then our kappa is nothing, but minus of z into kappa is sorry, kappa is just del 2 w by del x 2 and kappa is del 2 w by del x 2. So, on being integrated I have z square that will give me a moment of inertia and I is multiplied. So, E I del 2 w by del x 2 into variation of del 2 w by del x 2 dx and then minus I have p z into del w dx. So, this rho x so, will do 0.

$$\int_0^L \rho I \frac{\partial \ddot{w}}{\partial x}\delta\frac{\partial w}{\partial x}dx + \int_0^L \rho A\ddot{w}dx + \int_0^L EI\frac{\partial^2 w}{\partial x^2}\delta\left(\frac{\partial^2 w}{\partial x^2}\right)dx - \int_0^L p_z\delta w dx = 0$$

Now the contribution of this term is much less as compared to this term because i is much less as compared to A because this beam has much less thickness. So, because of the low thickness I is much less than A and then if I compare this term with this term although I have I in both the terms, but rho is much less than E. So, finally, we drop this term. So, after dropping this term we are only left with this. So, this is equal to this rho A w double dot del w dx plus I have integral of E I del w dx plus I have integral of E del 2 w by del x 2 multiplied by variation of del 2 w by del x 2 dx minus 0 to L p z del w dx.

$$\int_0^L \rho A\ddot{w}dx + \int_0^L EI\frac{\partial^2 w}{\partial x^2}\delta\left(\frac{\partial^2 w}{\partial x^2}\right)dx - \int_0^L p_z\delta w dx = 0$$

So, we have neglected this term. So, this is our integral of E I del w dx plus variation form that will deal with.

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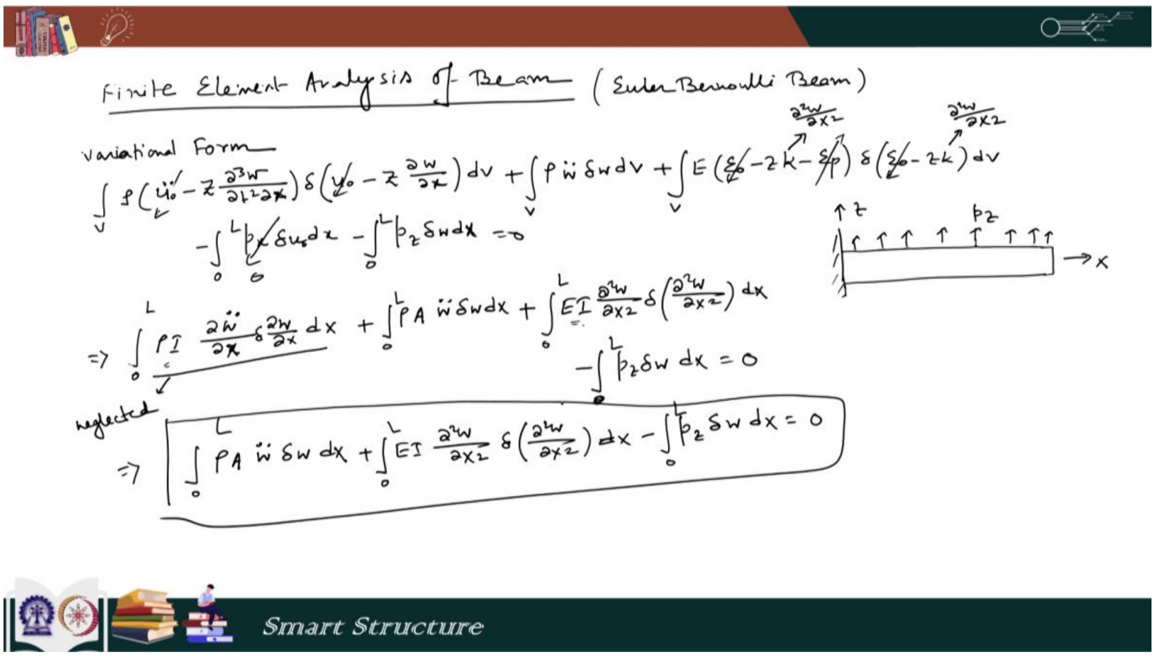

Now, in this form we would put our piece wise approximation and go for the finite element solution. So, to do that our entire solution domain is x equal to 0 to L that is the length of the beam. So, we have x equal to 0 to x equal to L and that is our x.

So, what we do is we divide the solution domain into a set of elements, element means a set of sub domains and then we can name this each of the sub domains as some elements. So, its element number 1, element number 2, these are generic element number E and this is the last element number N e. So, N e is the total number of elements. Now these points which are which are the two sides of the element at the two ends of each element they are called nodes. So, these are node these are node these are node and so on.

So, except the first and the last node rest of the nodes are shared between two elements. For example, this element number 1 and element number 2 shares this node which is the second node. Now let us look into element number E let us look at an enlarged view. So, we are drawing it little bigger. So, element number E let us assume that it has a length of l e and it has two nodes.

So, this is the first node of element number e second node of element number e and our solution that we want is w, w is the displacement in the transverse direction. So, the value of w at the first node of the e th element we call it w 1 e and the value of e at the second node of element number e we call it w 2 e. So, the superscript denotes element number and the subscript denotes the node number. Similarly, I define the slope here. So, del w by del x the slope of w at node number 1 of element number e is del w by del x 1 e and this is del w by del x 2 e.

So, and these values that we are defining here at the nodes are called the degrees of freedom. So, we say this is our first degree of freedom at node number element number e this we call as a second degree of freedom at element number e this we call as third degree of freedom at element number e and this we call fourth degree of freedom at element number e.

$$d_1^e = w_1^e \qquad w_2^e = d_3^e$$

$$d_1^e = \left(\frac{\partial w}{\partial x}\right)_1^e \qquad \left(\frac{\partial w}{\partial x}\right)_2^e = d_4^e$$

So, we have two degrees of freedom per node and those are the value of the solution w and its slope and so, there are total four degrees of freedom. And also let us define a local coordinate system small x running from this node to the to this node. So, each element has its local coordinate denoted as small x.

Now, if I put the degrees of freedom number here. So, we can say we have d 1 1 here because this node is the first node of element number 1 and we can also say that we have d 2 1 here, yes d 2 1 here and this we can call it d 1 3 sorry, its d 3 1 this we can call d 3 1 and this we can call d 4 1. Similarly, this node is shared by element number 2 also. So, this d the displacement here the value of w here can be written as can also be called d 1 2 and we can call it d 2 2. Similarly, d 3 2 and d 4 2.

So, thus we have the nodal degrees of freedom at the nodes of each element and because this node, the intermediate nodes are shared by 2 elements. So, I get 2 degrees of freedom here d 3 1 when I consider it for the first element, d 4 1 when I consider it for the first element. Similarly, for the second element it is d 1 2 and d 2 2 later on we will see that d 3 1 and d 1 2 are same d 4 1 2 and d 2 2 are same to maintain the continuity we will come to that later on. Now, let us again ah focus on a generic element e and we have defined a local coordinate system x. Now, let us assume the w displacement at element number E as a 0 plus or we can call it a 1 plus a 2 x plus a 3 x square plus a 4 x cube.

$$w(x) = a_1 + a_2x + a_3x^2 + a_4x^3$$

So, our approximation is such that I have 4 unknowns here a 1 a 2 a 3 a 4, they are 4 unknowns and if the problem is time independent problem these are the just constants and if they are time dependent problem then a 1 a 2 a 3 a 4 are time dependent, they have time dependent value. So, here are I have 4 unknowns a 1 a 2 a 3 a 4 and I have the 4 degrees of freedom in the formulation and the approximation is such that it has at least the second order derivative non zero because in the variational form we have seen that the highest order derivative appearing for w is 2. In fact, it has a derivative of the third order also. Now, if this is w, then what we do is we say that at x equal to 0 which means at the first node we have w 1 e which is d 1 e is equal to a 1 because at the first node x equal to 0 and then we say at that same node del w by del x 1 e is equal to d 2 e is equal to a 2 because if I differentiate it once with respect to x I get just a 2 and the value of that rest of the terms are x terms after doing the derivative after finding out the derivative. So, this term vanish only I am left with a 2.

$$\text{at } x = 0, \qquad w_1^e = d_1^e = a_1$$

$$\left(\frac{\partial w}{\partial x}\right)_1^e = d_2^e = a_2$$

And then, at x equal to l e, I have del 2 which means at w 2 of e which means this is d 3 of e and that is equal to a 1 plus a 2 l e which is equal to l e plus a 3 plus a 3 l e square plus a 4 l e cube and then again I can differentiate this and evaluate at x equal to l e and that gives us del w by del x at the second node at the second node that is equal to d 4 e and that becomes a 2 plus 2 a 3 l e plus 3 a 4 l e square. So, here I have 4 equations 1 2 3 4 and using these 4 equations I can represent a 1 a 2 a 3 a 4 in terms of d 1 d 2 d 3 d 4.

$$x = l_e, \qquad w_2^e = d_3^e = a_1 + a_2l_e + a_3l_e^2 + a_4l_e^3$$

$$\left(\frac{\partial w}{\partial x}\right)_2^e = d_4^e = a_2 + 2a_3l_e + 3a_4l_e^2$$

So, let us define this as equation 1 this as equation 2 this as equation 3 this as equation 4 this as equation 4 this as equation 5. So, find a 1 a 2 a 3 a 4 in terms of d 1 d 2 d 3 d 4 using equations 2 to equation 4. So, this is equation 2 this is equation 3 and this is equation 4 and then we put that in equation 1 then writing equation 1 in terms of d 1 d 2 d 3 d 4 we can write that.

So, we have found out a 1 a 2 a 3 a 4 in terms of d 1 d 2 d 3 d 4 we can write that. So, we have found out a 1 a 2 a 3 a 4 in terms of d 1 d 2 d 3 d 4 from equation 2 3 4 5 and then we put that in equation 1. If I do that I get the entire expression as w as a function of small x we have 1 minus 3 x by l e square plus 2 x by l e whole cube this entire thing multiplied by d 1 e whole cube plus we have we have x multiplied by 1 minus x by l e square this multiplied by d 2 e then plus we have 3 multiplied by x by l e whole square minus 2 into x by l e whole cube multiplied by d 3 e and then plus we have again x multiplied by x by l e square minus x by l e and this multiplied by d 4 e.

$$w(x) = \left[1 - 3\left(\frac{x}{l_e}\right)^2 + 2\left(\frac{x}{l_e}\right)^3\right] d_1^e + \left[x\left(1 - \frac{x}{l_e}\right)^2\right] d_2^e + \left[3\left(\frac{x}{l_e}\right)^2 - 2\left(\frac{x}{l_e}\right)^3\right] d_3^e + x\left[\left(\frac{x}{l_e}\right)^2 - \left(\frac{x}{l_e}\right)\right] d_4^e$$

Now, this quantity is called we give a name to it we denote this as N 1 of x and this we denote as N 2 of x this entire thing and this entire quantity is denoted as N 3 of x and this one, we denote that as N 4 of x.

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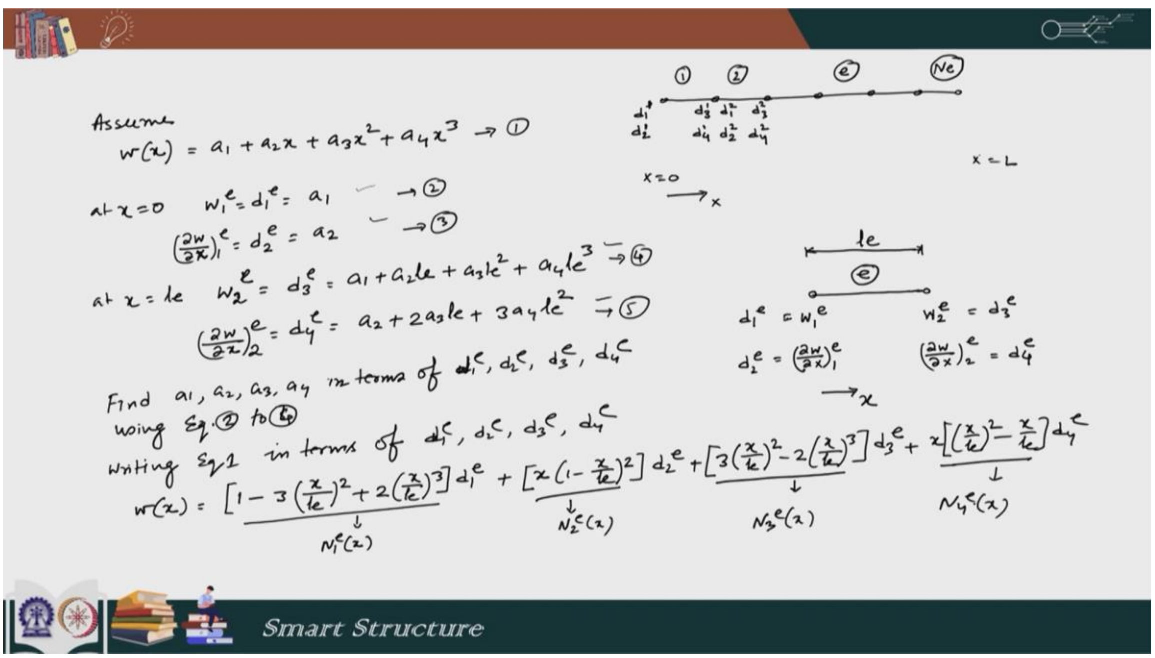

So, finally, again we write w x within each element within element e as N 1 of x multiplied by d 1 N 2 multiplied by d 2 with of course, the superscripts for element e d 3 plus N 4 multiplied by d 4 and these n's are called shape functions.

$$w(x) = N_1^e(x)d_1^e + N_2^e(x)d_2^e + N_3^e(x)d_3^e + N_4^e(x)d_4^e$$

So, N 1 N 2 N 3 N 4 are called shape functions. Now, what we do is we put this approximation in the variational form. So, before doing that I need to define del w also.

So, we have del w of x and that is if I take variation here this N 1 is a known quantity which is defined by us. So, we cannot vary that whereas, d 1 is an unknown quantity. So, it becomes N 1 e multiplied by variation of d 1 plus N 2 e multiplied by variation of d 2 plus N 3 e multiplied by variation of d 3 e plus N 4 e multiplied by variation of d 4 e.

$$\delta w(x) = N_1^e(x)\delta d_1^e + N_2^e(x)\delta d_2^e + N_3^e(x)\delta d_3^e + N_4^e(x)\delta d_4^e$$

Then we put everything in the variational form. So, if I put it in the variational form the variational form would look like this. So, our entire domain was x equal to 0 to capital L the length of the beam. However, we have divided the beam into set of elements and total number of elements is N e. So, we should do the integration now piece wise and then add up for all the elements.

Now, this shape functions as we just now derived, they are derived for each and every element separately. So, N 1 e N 2 e N 3 e N 4 e this means that they are N 1 N 2 N 3 N 4 of element e. So, accordingly the first element would have N 1 1 N 2 1 N 3 1 N 4 1 and so on. So, as the definitions are confined within each element we need to do the integration within each element and then add them up. So, for that we write it in this way we have a summation going from e is equal to 1 to e and then we have N 1 e d 1e plus N 2 e d 2 e.

So, we have N 1 e d 1 E double dot plus all the way up to N 4 e d 4 e double dot because we have a term w double dot in the variational form. So, if my w is this then w double dot is N 1 multiplied by d 1 double dot because N 1 is not a function of t, but d 1 can be a function of t as we discussed. Similarly, N 2 into d 2 double dot plus N 3 into d 3 double dot plus N 4 into d 4 E double dot.

$$\ddot{w} = N_1^e\ddot{d}_1^e + N_2^e\ddot{d}_2^e + N_3^e\ddot{d}_3^e + N_4^e\ddot{d}_4^e$$

So, that is what we are writing here and then we have del w. So, for that we write N 1 e multiplied by a variation of d 1 e and then all the way up to N 4 multiplied by the variation of d 4 e and then d x. So, that is the first term the inertia term. And then we have the next term which gives us E I then we have N 1 comma x x into E multiplied by d 1 E because if this is my w and in the second term in the variational form, we have del 2 w by del x 2. So, if I want to differentiate this, I cannot differentiate d because d is not a function of x if I differentiate that will give me 0. So, I have to differentiate N and after differentiating twice I get N 1 comma x x and I get this as this. Similarly, here I have N 1 comma x x then multiplied by the variation of d 1 and all the way up to N 4 comma x x multiplied by variation of d 4 d x and this integral is from 0 to l e and then we have 0 to l e p z multiplied by just w and w is N 1 e multiplied by del w and that is N 1 E multiplied by delta d 1 e till N 4 e multiplied by delta d 4 e and d x and that is equal to 0.

$$\sum_{e=1}^{N_e}\left[\int_0^{l_e} \rho A\left(N_1^e \ddot{d}_1^e + \cdots\cdots + N_4^e \ddot{d}_4^e\right)\left(N_1^e \delta d_1^e + \cdots\cdots + N_4^e \delta d_4^e\right) dx \right.$$
$$+ \int_0^{l_e} EI\left(N_{1,xx}^e d_1^e + \cdots\cdots + N_{4,xx}^e d_4^e\right)\left(N_{1,xx}^e \delta d_1^e + \cdots\cdots + N_{4,xx}^e \delta d_4^e\right) dx$$
$$\left. - \int_0^{l_e} p_z\left(N_1^e \delta d_1^e + \cdots\cdots + N_4^e \delta d_4^e\right) dx \right] = 0$$

So, this is the variational form that I get after I put the elemental approximations.

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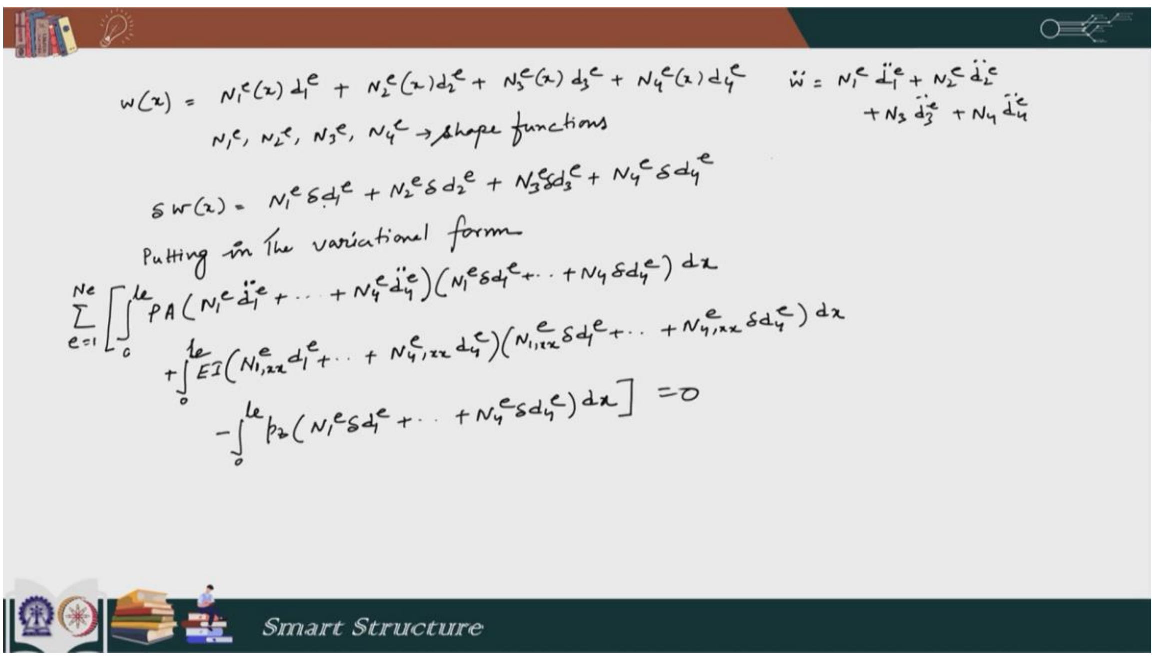

Now, again we follow a similar logic our variations are all arbitrary now the del w is an arbitrary variation and it can be arbitrary only when my del d del d 1 e del 2 e del d 2 e del 3 e and del 4 e are arbitrary. So, del w is arbitrary means the variations of d are also arbitrary. So, what I would do is I would write the entire expression in terms of delta d 1 multiplied by few terms plus delta d 2 e multiplied by few terms and so on.

So, if we do that again it looks like this. So, we have delta d 1 multiplied by 0 to l e rho A multiplied by N 1 e and then we have N 1 into d 1 plus up to N 4 e into d 4 e d x. Similarly, I have again 0 to l e E I multiplied by N 1 comma x x and then multiplied by N 1 comma x x N 1 e comma x x multiplied by d 1 e plus up to N 4 e comma x x multiplied by d 4 e d x and then we have 0 to l e p z multiplied by N 1 E d x. We keep doing it. So, we do this for delta d 2 delta d 3 and delta d 4.

We will just write the last one. So, we write till the last term which is delta d 4 and here we have rho A N 4 e multiplied by N 1 e d 1 e plus again N 4 e d 4 e d x and then 0 to l e E I. We again multiply N 1 e x x with these terms and then we have here p z N 4 e d x and that is equal to 0.

$$\sum_{e=}^{N_e}\left[\delta d_1^e\left\{\int_0^{l_e}\rho A N_1^e(N_1^e d_1^e+\cdots\cdots+N_4^e d_4^e)\,dx\right.\right.$$
$$\left.+\int_0^{l_e} EIN_{1,xx}^e\left(N_{1,xx}^e d_1^e+\cdots\cdots+N_{4,xx}^e d_4^e\right)dx-\int_0^{l_e} p_z N_1^e\,dx\right\}+\cdots\cdots$$
$$+\delta d_4^e\left\{\int_0^{l_e}\rho A N_4^e(N_1^e d_1^e+\cdots\cdots+N_4^e d_4^e)\,dx\right.$$
$$\left.\left.+\int_0^{l_e} EIN_{4,xx}^e\left(N_{1,xx}^e d_1^e+\cdots\cdots+N_{4,xx}^e d_4^e\right)dx-\int_0^{l_e} p_z N_4^e\,dx\right\}\right]=0$$

So, here we see that we have delta d 1 delta delta d 2 delta d 3 delta d 4 and delta d is d 1 is multiplied with a big expression and so is delta d 2 delta d 3 and delta d 4. Now, the entire summation is 0, but we are saying that delta d 1 delta d 2 delta d 3 and delta d 4 are independent variations arbitrary variations. So, the entire summation can only be 0 when each of the term that they are multiplied with are separately 0, which means whatever is delta d 1 e multiplied with that is 0 whatever is delta d 2 is multiplied with that is 0 whatever is delta d 3 multiplied with that is 0 and whatever is delta d 4 is multiplied with that is 0.

So, I get so each of those terms are 0 so that gives me 4 equations for each element. So, we get 4 equations for each element. So, whatever the number of degrees of freedom per element we get that many equations. In this case, we have 4 degrees of freedom. So, we get 4 equations for each element and the equations is of the form this k e that is a 4 by 4 matrix multiplied by d that is a 4 by 1 vector is equal to f e that is a 4 by 1 vector and also, we have inertia term.

So, we will get m multiplied by d double dot e. So, this is a 4 by 1 this is 4 by 4. So, each element gives me a set of coupled ordinary differential equation of size 4. Now, this is called mass matrix, this is called stiffness matrix, this is called force vector. So, this we call mass matrix, this we call stiffness matrix and this we call force vector.

$$[m]_{4\text{x}4}^e\{\ddot{d}\}_{4\text{x}1}^e+[k]_{4\text{x}4}^e\{d\}_{4\text{x}1}^e=\{f\}_{4\text{x}1}^e$$

This is elemental these are all elemental matrices. So, it is elemental mass matrix, elemental stiffness matrix and this is elemental force vector.

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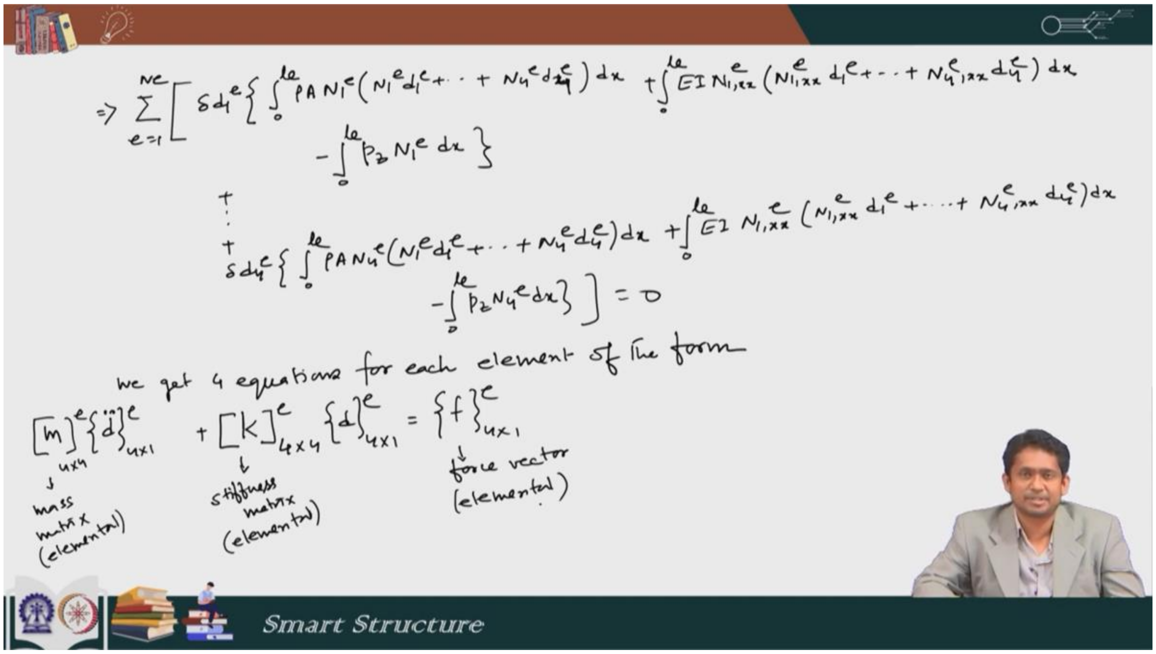

So, now that we have defined this for each element now, we have to combine the contributions from all the elements and then we have to add them up and make a global system of equations which we can solve and get my fine response of the structure.

So, that we will do in the next lecture I would like to conclude this lecture here.

Thank you.