Smart Structures
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Week 09
Lecture No: 48
Constitutive Relations of Shape Memory Alloys - Continued
Part 04

Today we will solve some numerical problems for shape memory alloys. So, here we will be using the constitutive relations that we have dealt with so far.

There is a first problem, it says that, we have a shape memory alloy wire and the wire is fixed at one end and free at other end and its dimensions are given. So, we have a wire like this it is free at one end and fixed at other end its dimensions are given and the free end had a weight and the weight is 1 kg, the weight is also given. Now, the wire is at a temperature of 15 degree centigrade in martensite phase and then we start heating the wire up to 77 degrees centigrade. And then, we cool down to a temperature of 20 degree centigrade. So, we have to find out the tip position of the wire at 74 degrees centigrade during heating and 48.5 degree centigrade during cooling. Now, the weight of the wire remains same and there is no constraint here. So, because the weight of the wire remains same the stress is also constant. So, at a constant stress all that heating and cooling is happening.

So, first we have to find out the stress. So, the stress here, we can call it sigma 0, initial stress and that is the load, the load is 1 kg multiplied by 9.81. So, that becomes Newton divided by the area and the area is pi by 4 multiplied by the diameter square and the diameter is 0.31 into 10 to the power minus 3. So, if I square it, I get the area and it is in Newton per meter square. So, finally, the stress that comes is 86.046 Mega Pascal. Now, at this stress, if I want to find out the tip stress, I have to just find out the strain and accordingly or, if I want to find out the tip position, I have to just find out the strain and accordingly I can find out the tip position. Now, before that let us see few things. These our phase diagram that is how it looks like this. This is the temperature axis and this is the stress axis and the entire diagram looks like this. We can see, at the lowest temperature is M_f, then we have M_s and then we have A_s and then we have A_f and the slopes of the lines are given. So, we have C_A and C_M and please note that here C_A and C_M are different although the as per the drawing it may look same, but please ignore the fact and please understand that they are different here. Now, we are operating at a stress of 86.046 Mega Pascal. So, which means we are somewhere here may be. So, at this constant temperature constant stress all the heating and cooling is happening. So, everything is happening along this dashed line and this stress is 86.046 Mega Pascal. So, we have 86.046 Mega Pascal.

$$\sigma_0 = \frac{1 \text{ x } 9.81}{\frac{\pi}{4}\text{x}(0.381\text{x}10^{-3})^2} = 86.046 \text{ MPa}$$

So, we have to find out the characteristic temperatures at this stress. So, let us find out M_f star. So, M_f star is M_f plus sigma by C_M. M_f is given to me, sigma I have found out. C_M is also given to me. So, if I put all the values, the M_f value comes to be 47.87 degree centigrade. And then we have M_f star which means M_s at that value of sigma. So, we may denote this as M_f star, M_f star, A_f star, A_f star. So, M_f star is again M_s plus sigma by C_M and the value comes to be 50.67 degree centigrade.

$$M_f^* = M_f + \frac{\sigma}{C_M} = 47.87^0C$$

$$M_s^* = M_s + \frac{\sigma}{C_M} = 56.67^0C$$

Similarly, I can find out the value of A_s star, A_s star is A_s plus sigma by C_A and this is 62.756 degree centigrade. And then we have A_f star which is A_f plus sigma by C_A and that comes as 75.756 degree centigrade.

$$A_s^* = A_s + \frac{\sigma}{C_A} = 62.756^0C$$

$$A_f^* = A_f + \frac{\sigma}{C_A} = 75.756^0C$$

Now, the temperature at which the material is at the beginning is 50 degrees centigrade. So, that is much less than M_f star as well as M_f. So, it is in this zone, and we have sigma critical s, which means the stress at which the stress induced martensite formation takes place, I mean start taking place, which means that detwining start taking place, that is 138 Mega Pascal and its finish at 172 Mega Pascal. So, our stress is 86 which is much less than sigma cr s. So, sigma cr s may be somewhere here and sigma cr f may be somewhere here. So, sigma cr s and sigma cr f. So, it is in the twinned zone. The stress is less than sigma cr s and the temperature is less than M_f or M_f star. So, the material is in martensite condition.

Now, if I want to find out the strain in the material at this temperature and at this stress, all I do is I just divide this sigma 0, the stress by the Young's modulus at the martensite phase. So, epsilon at 15 degrees centigrade is sigma 0 divided by E_M, E_M is given to us 20.3 Giga Pascal and sigma 0 we have found out 86.046 Mega Pascal. So, finally, the value comes to be 4.24 into 10 to the power minus 3. And then if I want to find out the deflection, the deflection at this strain is the original length multiplied by strain. So, original length is given to us and the original length is 0.3 meter. So, let us say this length we call it may be L. So, it is L into sigma. So, the delta L is 1.272 millimeter, 1.272 millimeter. So, the total length is L plus delta L, and that comes to be 0.301272 meter. So, the tip position is at a distance 0.301272 meter from the fixed end at the initial stage when the material is at a temperature of 15 degree centigrade.

$$\varepsilon \text{ at } 15^0C = \frac{\sigma_0}{E_M} = 4.24 \text{ x } 10^{-3}$$

$$\Delta l = L\varepsilon = 1.272\ mm$$

$$(L + \Delta l) = 0.30127\ mm$$

Now, everything takes place along this dashed line. So, when the heating is taking place and the temperature is between A_s star and A_f star that time martensite formation process is going on. So, we are supposed to find out the tip position of the wire at a temperature of 74 degree centigrade and 74 is between A_s star and A_f star because our A_s star is 62 and A_f star is 75.756. So, 74 is in between that. So, when I want to find out my strain or the tip position, I need to take care of the fact that at that time austenite formation is taking place. Then when we reverse it when we cool down, we are supposed to find out the tip position at 48.5 degree centigrade, which means when we are between M_f star and M_s star because M_f star is 47.87 degree centigrade and M_s star is 56.67 degree centigrade. That means, in between these two we have to find out the strain again. So, at that time we have to take care of the fact that at that time martensite formation is taking place.

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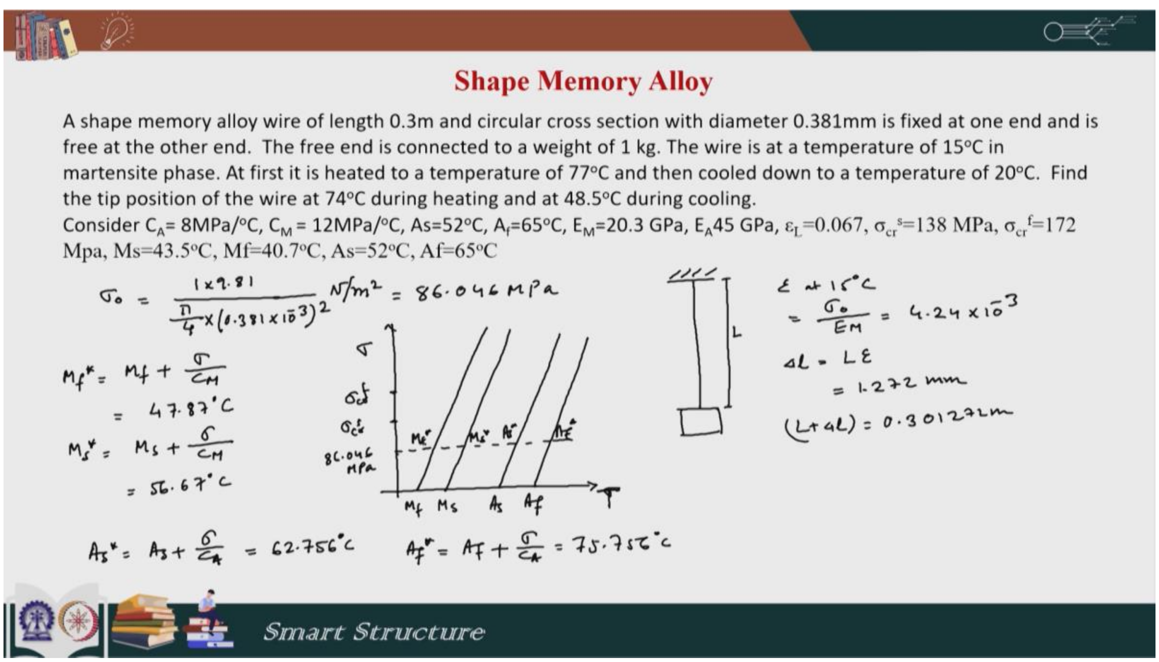

So, now we will first find out what is happening when we are at temperature 74 degree centigrade. So, to do that we have to find out the amount of martensite fraction and if I know that will give me rest of the quantities and this, we will do using both the Tanaka model and the Liang and Rogers model. Now, in Tanaka model we model the variation of xi during the transformation as an exponential function and in Liang and Rogers model it is modeled as a cosine function. Now, in both the cases the starting and the end conditions are same. So, for example, when the austenite formation is taking place, the starting condition is xi equal to 1 and after the formation it is xi equal to 0. Similarly, during the martensite formation, at the beginning of the formation, xi is equal 0 and at the end it is xi equal to 1. So, both the models give the value of xi at the beginning and end, I mean the value of xi given by both the model agree to a great extent, but in between them because one is modeling the variation as exponential function and the another is modeling the variation as cosine function, in between them the xi obtained from these two models are quite different, significantly different.

So, let us first do it using the Tanaka model. So, as per Tanaka model, so, at 74 degrees centigrade using Tanaka model what we have is xi T is equal to e to the power a_A multiplied by A_s minus T, plus b_A sigma 0. Now a A as per Tanaka model is natural log of 0.01 divided by A_s minus A_f and that comes to be 0.3542 per degree centigrade and b_A as per Tanaka model is a_A by C_A, C_A is given to us and a_A, we just calculated. So, b_A is 0.0443 per Mega Pascal of stress. So, if I find out xi here and if I put all the values here. So, at T is equal to 74 degrees centigrade, our xi comes to be 0.01867 and in percentage it comes to be 1.867 percent, which means that austenite formation is about to finish. Initially, during the martensite the martensite content was 100 percent now it has reduced to 1.867 percent.

$$\xi(T) = e^{a_A(T-A_s)+b_A\sigma_0}$$

$$a_A = \frac{\ln(0.01)}{A_s - A_f} = 0.3542\ /^0C$$

$$b_A = \frac{a_A}{C_A} = 0.0443\ /\mathrm{MPa}$$

$$\xi(T = 74^0C) = 0.01867 = 1.867\%$$

So, at this same temperature, now if I find out the martensite content using the Liang and Rogers model, then it comes to be as per the cosine rule that we know from the Liang and Rogers model. Now here as per that model xi T is equal to 1 by 2 multiplied by cosine of a_A, T minus A_s, plus b_A sigma 0 plus 1. Now here a_A and b_A differ because as per the Liang and Rogers model it is pi by A_f minus A_s and that comes out to be 0.242 per degree centigrade. And b_A comes to be minus a_A by C_A and that is for this model minus 0.03021 per MPa. So, if I put these values at and I find out the xi at a temperature of 74 degree centigrade, then xi comes to be 0.0428 which in percentage is 4.28 percent.

$$\xi(T) = \frac{1}{2}\{\cos[a_A(T - A_s) + b_A\sigma_0] + 1\}$$

$$a_A = \frac{\pi}{A_f - A_s} = 0.242\ /^0C$$

$$b_A = -\frac{a_A}{C_A} = -0.03021\ /\text{MPa}$$

$$\xi(T = 74^0C) = 0.0428 = 4.28\%$$

So, you can see that there is some difference between the xi predicted by the Tanaka model, and the Liang and Rogers model.

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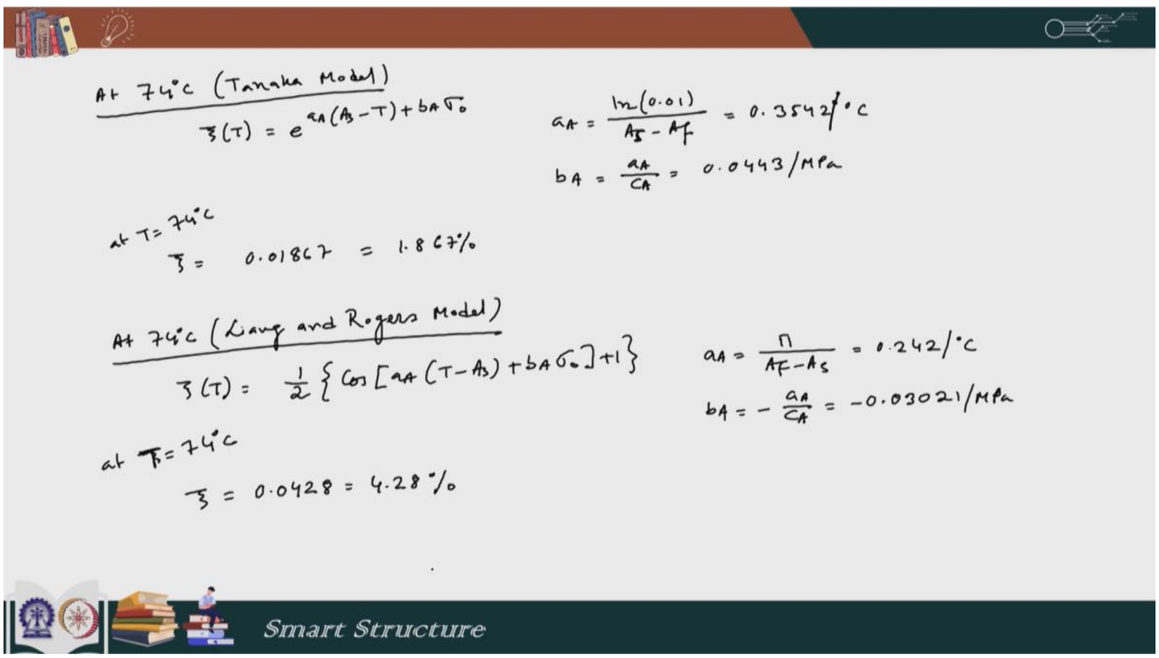

Now that we have got the xi's, we have to now put the constitutive relation, put the xi in the constitutive relation and find out the strains.

So, we know as per the constitutive relation that sigma minus sigma 0 is equal to E of xi multiplied by epsilon minus epsilon 0, plus omega of xi multiplied by xi minus xi_0 plus this capital phi multiplied by T minus T_0. Now, this we neglect because the effect is much smaller as compared to the other two effects because of E and omega. Now in this entire process, sigma and sigma 0 are same because we have seen that our load remains same and there is no constant to it. So, sigma and sigma 0 are same. So, the left hand side is 0 and that leaves us with E as a function of xi multiplied by epsilon minus epsilon 0, plus omega as a function of xi multiplied by xi minus xi_0. And then we have E again as a function of xi multiplied by xi minus xi_0, minus epsilon L into E as a function of xi, because we know that omega is equal to minus epsilon L multiplied by E and that entire thing multiplied by xi minus xi_0. So, and that right hand side is 0.

$$\sigma - \sigma_0 = E(\xi)(\varepsilon - \varepsilon_0) + \Omega(\xi)(\xi - \xi_0) + \Theta(T - T_0)$$

$$\Rightarrow E(\xi)(\varepsilon - \varepsilon_0) + \Omega(\xi)(\xi - \xi_0) = 0$$

$$\Rightarrow E(\xi)(\varepsilon - \varepsilon_0) - \varepsilon_L E(\xi)(\xi - \xi_0) = 0$$

Now, E as a function of xi is – E_A plus xi into E_M minus E_A that we get from the rule of mixture.

$$E(\xi) = E_A + \xi(E_M - E_A)$$

So, all I have to do is using the Tanaka's model, I have to find out xi at the desired temperature for the given constant stress. And once I get my xi, I can get my E. Once I get my E, I can get this quantity which is omega. And then I put in this equation that will give me epsilon. Then the similar thing we can do it Liang and Rogers model. Using that model, we will find out xi and again we will get E, we will get omega. We will put in this equation and that will give us epsilon.

So, let us do it using Tanaka's model. So, we can say strain calculation using Tanaka model. To do this we have to find out our xi at that particular stress at that particular temperature and stress. So, for that is already done by us and the value that we got is 0.01867. So, for that particular xi, if I find out the E, then E comes to be E xi is so, when xi is equal to the value that I got 0.01867. So, for that xi our E is so, we do not need to write it E xi anymore and that is 44.54 GPa. We are just using this equation and we are putting xi is equal to 0.01876. So, naturally omega becomes minus epsilon E and that comes as minus 2.984 GPa. Then if we put this in this equation, if I put this quantity and this quantity in this equation with all the initial conditions taken care of, then we have 44.54 in GPa. Strain is unknown to me. The initial strain was 4.24 multiplied by 10 to the power minus 3 and then, plus we have omega which is minus 2.984. And then, we are multiplying that with xi minus xi_0. The xi at present is 0.01867 and xi_0 is minus 1 because initially the material was martensite now it is becoming austenite gradually. Then if I solve this equation then E comes to be minus 0.0615.

$$E = 44.54 \text{ GPa}$$

$$\Omega = -\varepsilon_L E = -2.984 \text{ GPa}$$

$$44.54(\varepsilon - 4.24\text{x}10^{-3}) + (-2.984)(0.01867 - 1) = 0$$

$$\varepsilon(T = 74^0 C) = -0.0615$$

So, that is the amount of strain it is experiencing at a temperature of 74 degree centigrade. So, it is at T is equal to 74 degrees centigrade.

Now that we know our epsilon, we can find out the elongation or contraction which is delta L, and delta L is L multiplied by epsilon and that gives me a contraction of 0.01845. So, and that is it meter and then we can find out the final length, we can call it the L_{final} and that is L plus delta L and then we get 0.2815meter.

$$\Delta l = L\varepsilon = -0.01845\ m$$

$$L_{final} = L + \Delta l = 0.2815\ m$$

So, we can say that now the tip position is at a distance 0.2815 from the fixed end.

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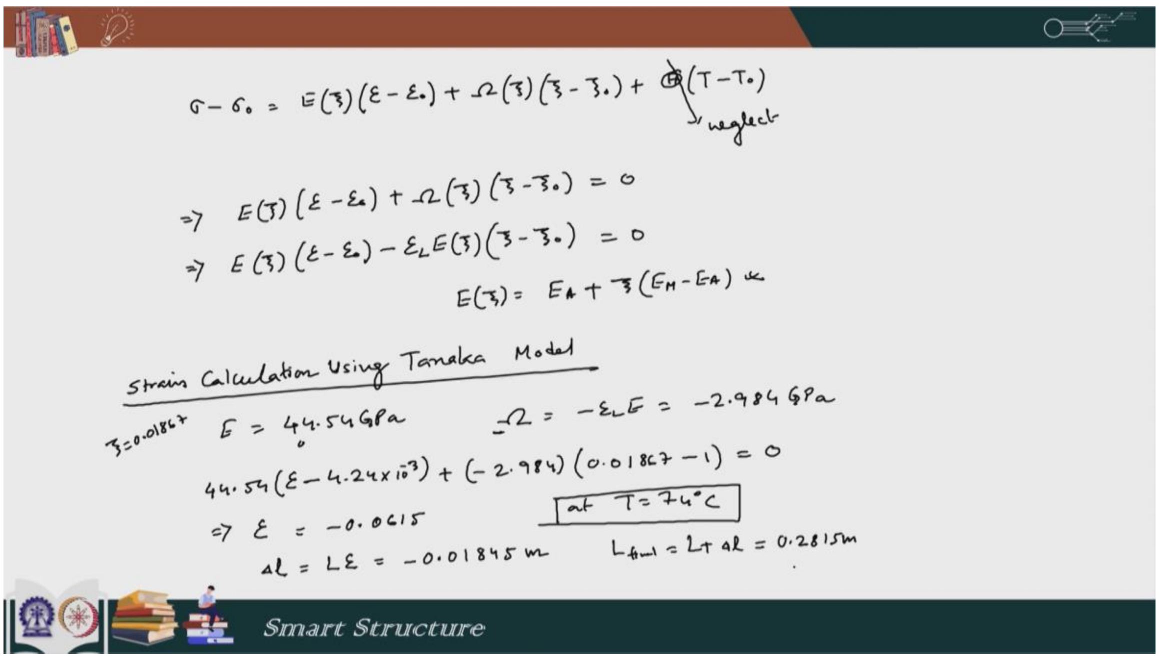

Now, we have to do the same thing using the Liang and Rogers model. So, now we will do strain calculation using Liang and Rogers model. So, here again we got xi equal to, from this model, the xi that we got is 0.0428 and then again, if we apply the same formula, our stress is not changing. So, it is 0 and then we have, ok. So, at that xi we need to find out the E. So, E as per the same formula E_A plus xi into E_M minus E_A that comes to be 43.94 GPa. And omega comes to be minus 2.944 GPa. Then if we apply the same procedure, same constitutive relation the equation looks like this and then we have plus omega. So, it is minus 2.944 and then multiplied by 0.0428 minus 1. So, by solving we get our strain as minus of 0.0599. And then, that gives me a delta L as epsilon multiplied by L and we get that to be minus 0.01797 meter. So, L_{final} which means L plus delta L is 0.282 meter and everything is at a temperature of 74 degree centigrade.

$$\xi = 0.0428 \qquad E = 43.94 \text{ GPa} \qquad \Omega = -2.944 \text{ GPa}$$

$$0 = 43.94(\varepsilon - 4.2\text{x}10^{-3}) + (-2.944)(0.0428 - 1)$$

$$\varepsilon = -0.0599$$

$$\Delta l = \varepsilon L = -0.0797\ m$$

$$L_{final} = L + \Delta l = 0.212\ m$$

So, as per the Liang and Rogers model. At temperature of 74 degree centigrade while the material is being heated the tip position of the wire is at a distance of 0.282 meter from the fixed end.

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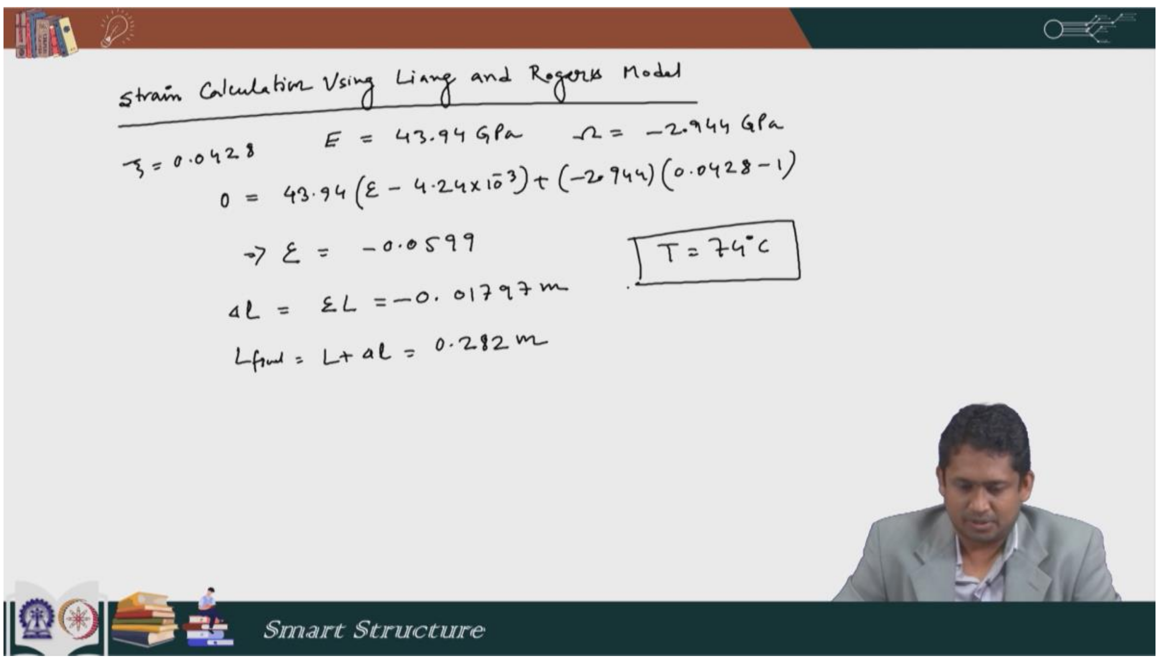

So, these are the two results predicted by these two models. And as we go closer to the austenite finish temperature, A_f star, the model will come to more and more agreement. And similarly at the beginning closer to austenite star temperature the results predicted would be very similar. In between them, there would be some variation because as we said just now, one of the models is modeling the xi variation using exponential function and another model is modeling the xi variation using cosine function.

Now, the next job is to find out the variation of xi and then find out the strains and tip displacement during the cooling operation.

So, that we will do in the next lecture with this I would finish this lecture here.

Thank you.