Smart Structures

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Week - 09
Lecture No - 47

Constitutive Relations of Shape Memory Alloys (continued)

We started the discussion of Brinson's model in the previous lecture and today we will continue with that. So we saw that Brinson's model incorporates the stress and temperature induced martensite separately. So it has two variables xi S which is stress induced martensite and there is xi T which is temperature induced martensite. Now that means that there is one more state vector here we have temperature we have strain and we have two martensite volume fractions. So xi S plus xi T should give me the total martensite fraction and that should be less than equal to 1 and it should be greater than equal to 0. So now we can write the constitutive relation as this sigma minus sigma 0 equal to E multiplied by epsilon minus epsilon 0 plus we have to write separately for omega S is equal to omega S minus omega 0 plus omega T is equal to omega T is equal to xi T minus xi 0 and plus capital phi is equal to T minus T0.

$$0 \leq \xi_s + \xi_T \leq 1$$

$$\sigma - \sigma_0 = E(\varepsilon - \varepsilon_0) + \Omega_S(\xi_S - \xi_0) + \Omega_T(\xi_T - \xi_0) + \Theta(T - T_0)$$

$$d\sigma = Ed\varepsilon + \Omega_s d\xi_s + \Omega_T d\xi_T + \Theta dT$$

$$E(\xi) = \frac{\partial\sigma}{\partial\varepsilon}, \Omega_S = \frac{\partial\sigma}{\partial\xi_S}, \Omega_T = \frac{\partial\sigma}{\partial\xi_T}, \Theta = \frac{\partial\sigma}{\partial T}$$

$$\xi_{s_0} = 0, \xi_{T_0} = 0, \varepsilon_0 = 0, \sigma_0 = 0$$

$$\xi_S = 1, \xi_T = 0, \varepsilon = \varepsilon_L, \sigma = 0, M_S > T > A_s$$

$$0 = E\varepsilon_L + \Omega_S = 0 \Rightarrow \Omega_S = -E\varepsilon_L$$

Now because our xi S and xi T we have taken them separately so according to we have omega S and omega T separately. In the differential form we can write okay please excuse me this should have been sigma minus sigma 0. Now in the differential form the same thing can be written as E multiplied by d epsilon plus omega S multiplied by d xi S so this is dS 0 and this is dT 0 sorry this is this is xi S 0 that means, xi S at the initial time step initial condition and xi T 0 means xi T at the initial condition and xi S minus xi S 0 is dS d xi S and plus we have omega omega T multiplied by d xi T and plus we have capital phi multiplied by dT. And then we can write that E and this we can take as a function of xi so this is del xi by del xi by del epsilon omega S equal to del xi by del xi S and omega T is equal to del xi by del xi T and phi equal to del xi by del T and phi can also be a function of xi.

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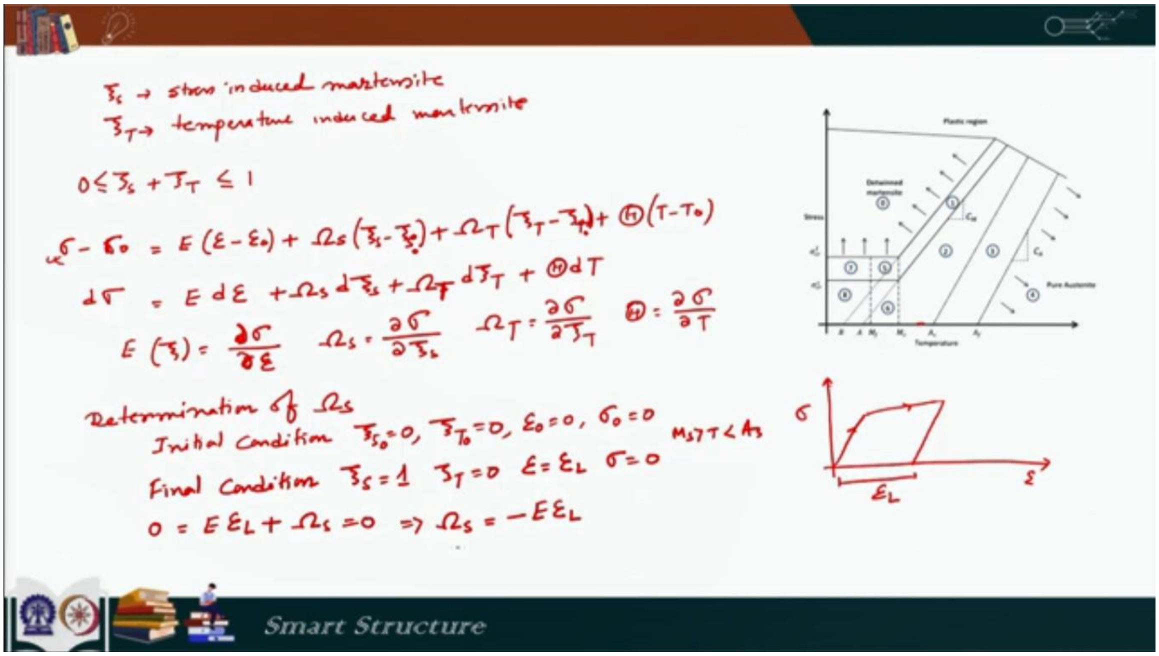

Now we have to determine the relation we have to express these quantities omega S omega T in terms of some other quantities. So, again we will do it in such a way that we will take a condition we will take some initial condition we take some final condition which are known to us and then we impose the condition and that gives us this variable this quantities this constants in terms of some other constants. So, initially we get those constants as something which does not depend on xi and then again we will do some mathematical manipulation and get those quantities as a function of xi. So, first determination of omega S so the initial condition is our initial condition is xi S equal to 0 xi T equal to 0 epsilon 0 equal to 0 sigma 0 equal to 0 and our temperature is below the austenite star temperature and it is more than the martensite star temperature and we do not change the temperature so at that remains fixed which means we are somewhere here in this zone and xi S 0 and xi T 0 means there is no martensite martensite fraction which means the material is entirely in the austenite condition. Now what we do is from here we increase the stress that means, go up now as we go up the transformation takes place the entire material becomes a detweened martensite or stress induced martensite and then we come back and after again we come back to the initial position the material still remains detweened martensite which means xi S becomes 1 there is no temperature induced martensite.

So, xi T is 0 epsilon 0 it becomes epsilon L because we are unloading it after it fully becomes detweened martensite and then if you unload it the residual strain that remains is epsilon L which is the maximum recoverable strain and sigma again come back comes back to where it started so sigma becomes 0. So, we are in the stress strain plane this is just this which we saw several times and we are coming back here and this is our epsilon L. So, the final condition is so we should put 0s here because these are initial conditions. So, final condition is epsilon xi S equal to 1 xi T equal to 0 epsilon equal to epsilon L and sigma equal to 0 and then if we put those in the and throughout this process temperature is maintained to be more than M s and less than A s. Now, if we write the constitutive relation I mean this relation with this set of initial and final condition we get the equation as E multiplied by epsilon L plus omega S equal to 0 and this tells me that omega S equal to minus of E into epsilon L.

In the previous case in the Liang and Rogers model or in Tanaka model we had the similar equation, but that was for omega now here because we are separately treating omega S and omega T. So, this relation is now for omega S. Now, we have to see what omega T is. So, now let us come to determination of omega T. So, for this we take initial condition as xi S 0 equal to 0 xi S xi T 0 equal to 1 and then we have sigma 0 equal to 0 and epsilon 0 equal to 0 and the final condition is xi S equal to 1 xi T equal to 0 sigma equal to 0 and epsilon equal to epsilon L and that is possible if we are in this side of the diagram that means, if our temperature is below mf we are in somewhere in zone 8.

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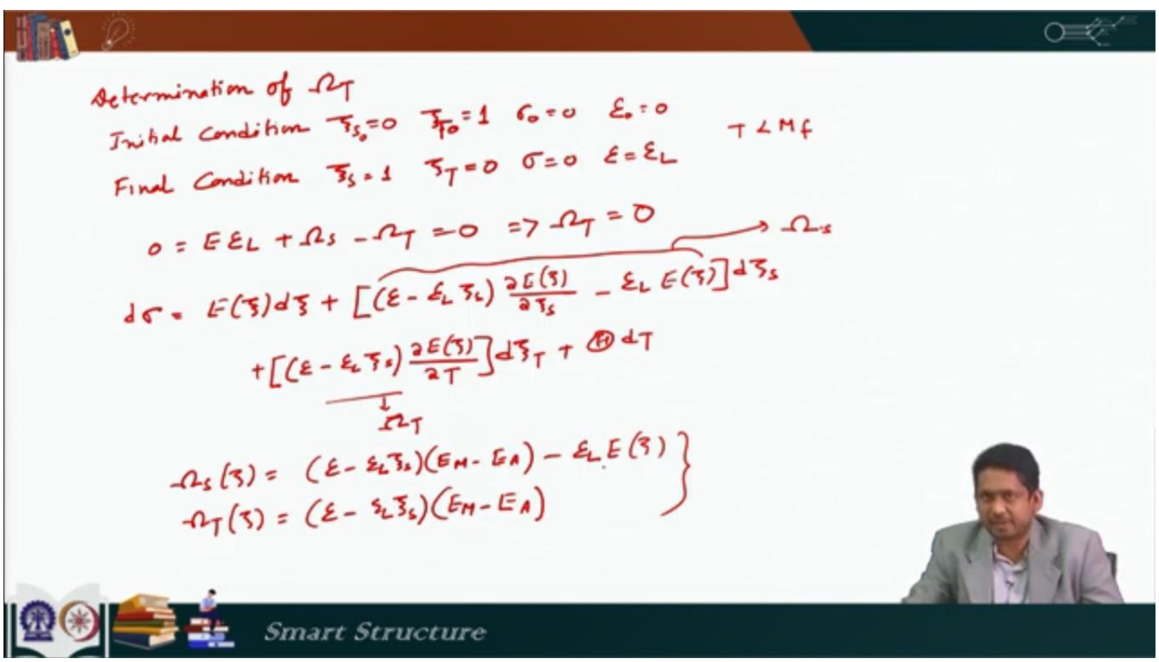

$$\xi_{S_0} = 0, \xi_{T_0} = 1, \sigma_0 = 0, \varepsilon_0 = 0$$

$$\xi_S = 1, \xi_T = 0, \sigma = 0, \varepsilon = \varepsilon_L, T < M_f$$

$$0 = E\varepsilon_L + \Omega_S - \Omega_T = 0 \Rightarrow \Omega_T = 0$$

$$d\sigma = E(\xi)d\xi + \left[(\varepsilon - \varepsilon_L\xi_L)\frac{\partial E(\xi)}{\partial T_S} - \varepsilon_L E(\xi)\right]d\xi_s + \left[(\varepsilon - \varepsilon_L\xi_S)\frac{\partial E(\xi)}{\partial T}\right]d\xi_T + \Theta dT$$

$$\Omega_S(\xi_S) = (\varepsilon - \varepsilon_L\xi_S)(E_M - E_A) - \varepsilon_L E(\xi)$$

$$\Omega_T(\xi_T) = (\varepsilon - \xi_L\xi_S)(E_M - E_A)$$

So, initially if everything is stress in temperature induced martensite and then we are gradually increasing thus we are increasing the stress after we cross sigma crf it becomes between martensite and then if we unload the detween martensite remains same it remains 1 and stress induced martensite become 0 and the residual strain is epsilon L. So, this is possible when T is less than Mf and that is maintained. Now, if we put everything in the constitutive relation the equation that we get is 0 equal to E multiplied by epsilon L plus omega S minus omega T equal to 0. Now, these 2 quantities cancel each other. So, we are left with omega T equal to 0.

So, we have seen we have got omega S and omega T. Now, we will get them as functions of xi. Now, we will get this omega S and omega T as functions of xi and that derivation is given in the paper by Brinson. So, for that the differential form of the constitutive relation is written based on the relations that we got so far and this becomes E multiplied by d xi plus epsilon minus epsilon L multiplied by xi S and that entire thing is multiplied by del E of del xi S minus epsilon L E xi into d xi S plus epsilon minus epsilon L into xi S multiplied by del of E by del T multiplied by d xi T plus capital phi multiplied by d T. Now, this quantity is denoted as omega S and this quantity is denoted as omega T.

So, omega S as a function of xi becomes epsilon minus epsilon L into xi S multiplied by E M minus E A minus epsilon L into E at xi and omega T can be rewritten as epsilon minus epsilon L xi S multiplied by E M minus E A. So, if we look at these two expression when we look at only this and if E is also a constant then xi is a constant, but if you look at the entire expression xi S is a variable and similarly if we look at omega T and if we neglect this expression then omega T is 0 and if we look at the entire expression then omega T is also a function of xi T which makes it a function of xi S also because xi S and xi T on being added should be 1 because xi S and xi T. Now let us look into the relations which defines the variation of xi S and xi T. So, for xi S when it A to M when it A transforms from A to M the relation is if my T is more than M S and our stress is between these two quantities. So, we have sigma f C R now ok.

So, let us put the critical quantity at the denominator and whether it is S or F in the sorry let us put the critical quantity in the subscript and whether it is S or f in the superscript. So, we have C M multiplied by T minus M S. Now in Brinson's model also the cosine functions are used, but those have to be modified because the phase diagram is quite different here our M S M F are different and also we have two variables xi S and xi T. So, accordingly the modified cosine functions look like this by 2 and then xi T is equal to xi T 0 minus xi T 0 by 1 minus xi S 0 and xi S minus xi S 0. Now when I have T less than M S and sigma S C R between sigma is between sigma S C R and sigma f C R then we have xi S equal to 1 minus xi S 0 by 2 multiplied by cosine of pi by sigma C R S minus sigma C R f multiplied by sigma minus sigma C R f plus 1 plus xi S 0 by 2 and xi T is equal to xi T 0 minus xi T 0 by 1 minus xi S 0 multiplied by xi S minus xi S 0 plus delta T xi and delta T xi now have to be defined.

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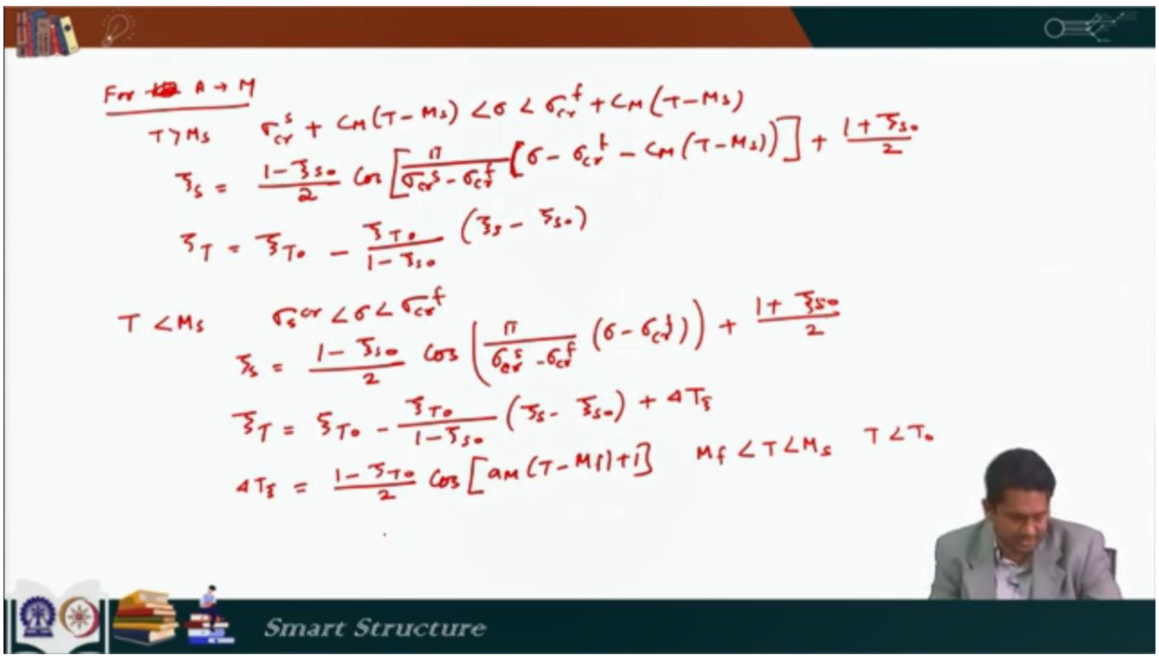

$$T > M_S, \sigma^s{}_{cr} + C_M(T - M_S) < \sigma < \sigma_{cr}{}^f + C_M(T - M_S)$$

$$\xi_S = \frac{1-\xi_{S0}}{2} cos\left[\frac{\pi}{\sigma_{cr}{}^s - \sigma_{cr}{}^f} \sigma - \sigma_{cr}{}^f - C_M(T - M_S)\right] + \frac{1+\xi_{S0}}{2}$$

$$\xi_T = \xi_{T0} - \frac{\xi_{T0}}{1-\xi_{S0}}(\xi_S - \xi_{S0}) + \Delta T_\xi$$

$$\Delta T_\xi = \frac{1 - \xi_{T0}}{2} cos\left[a_M\left(T - M_f\right) + 1\right], M_f < T < M_s, T < T_0$$

So, delta T xi is defined as 1 minus xi T 0 by 2 multiplied by cosine of a M T minus M f plus 1 when I have my temperature between M f and M S and T is less than T 0 that means, temperature is reducing or we can have delta T is equal to 0 otherwise for other condition. So, when temperature is when temperature is reducing and the temperature is between M S and M f we have some value of delta T xi otherwise it is 0. So, that is the transformation from austenite to martensite. Now, if you want to go from martensite to austenite transformation then based on the condition that T is more than A S and we have C A T minus f a F and C a multiplied by T minus a S defining the range of sigma if sigma is in between these two then we have xi equal to xi 0 by 2 multiplied by cosine of a A multiplied by T minus A S minus sigma by C A plus 1 and xi S equal to xi S 0 multiplied by xi S 0 by xi 0 multiplied by xi 0 minus xi and we have xi T equal to xi T 0 multiplied by xi T 0 by xi 0 multiplied by xi 0 minus xi. So, these functions define the evolution of our martensite volume fractions for both stress induced and temperature induced for varying sigma and T during transformation.

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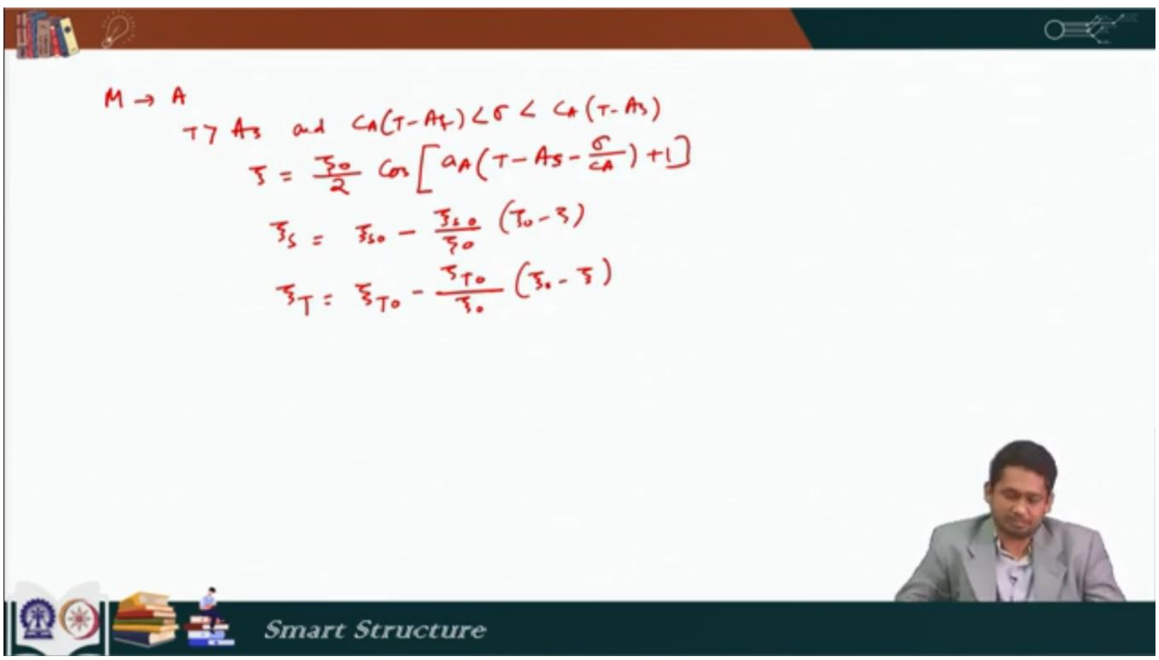

$$T > A_s, C_A\left(T - A_f\right) < \sigma < C_A(T - A_s)$$

$$\xi = \frac{\xi_0}{2} cos\left[a_A\left(T - A_s - \frac{\sigma}{C_A}\right) + 1\right]$$

$$\xi_s = \xi_{S0} - \frac{\xi_{S0}}{\xi_0}(\xi_0 - \xi)$$

$$\xi_T = \xi_{T0} - \frac{\xi_{T0}}{\xi_0}(\xi_0 - \xi)$$

Now, this model has some limitations and this limitation can be felt when we look very closely at these zone 5. In these zone various transfer of transformation various types of transformation are possible austenite to martensite temperature induced martensite to austenite and so on. And it has been seen that there can be some cases when the total volume fraction of martensite can be more than one which is nonphysical specially if temperature is reduced and simultaneously if stresses increased which means if I traverse along this line from this diagonal to this diagonal. In that case it has been noticed that there is a possibility of xi S plus xi T becoming more than 1. So, to address this researchers Chang and his co-authors Kundel 1 and his co-authors they worked on overcoming the limitations of these models.

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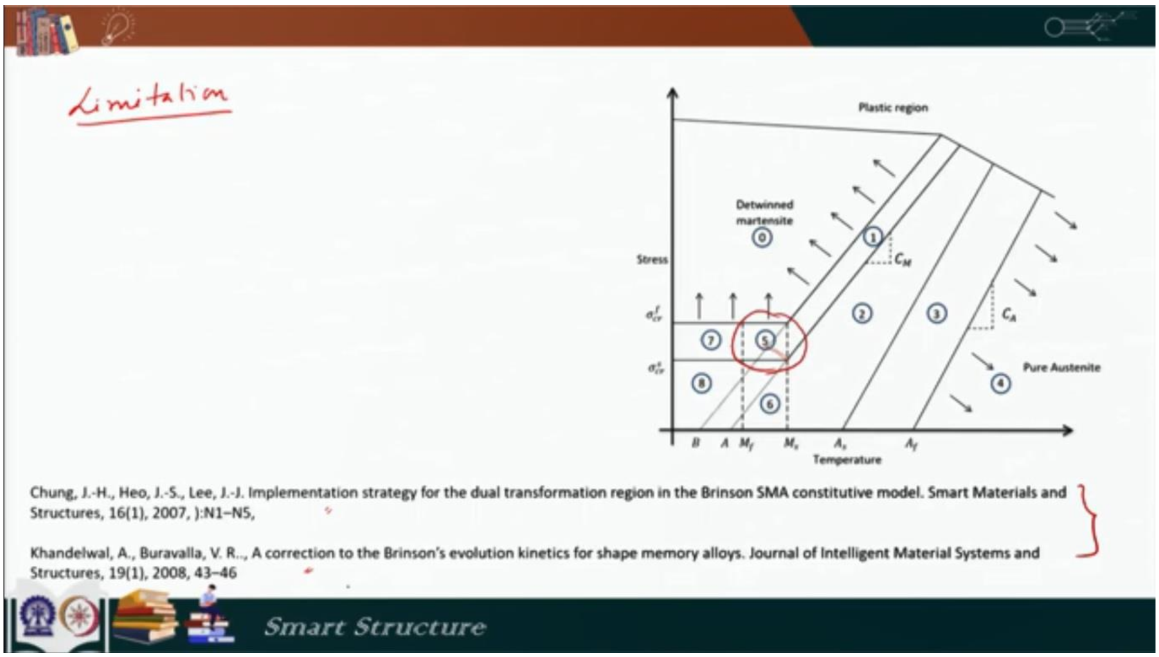

So, for example, if you look at the paper by Chang we can see some modified equations and same thing we can find here also. So, they try to impose some limitations here. So, while modifying they made sure that the main requirements that is total of xi S plus xi T is always less than equal to 1. When the stress is more than sigma CRF xi S is equal to 1 and if the temperature is less than or equal to mf total xi equal to 1 and importantly any transformation that it that is taking place along this line that means, there is a continue of continuity of transformation between zone 1 and zone 5. So, with that they try to overcome the limitation and some of modified equations are proposed in this paper.

And then later on there has been many work where the constitutive relations have been extended to 3D forms also. So, that was that is what that was the full discussion on the constitutive relation of shape memory alloys. Now, we will talk about few things. So, far we have seen the effect of stress on the transformation temperatures, but we did not say whether the stress is tensile or compressive. Now, in most of the cases if we use a shape memory alloy actuator as a wire it will take only tension compression does not come there, but in some of the cases where shape memory alloys have been used as tubes washers or any many different forms in those cases compression can also arise.

So, there has been some work to differentiate that tensile and compressive behavior. These two are notable work from the research group of Professor Inman. So, I will just highlight some basic findings of their some basic findings by them. So, in a they perform test in an alloy of nickel and titanium and they saw that E m becomes 50 GPa when it is under compression, but for same amount of tension it can be 42 GPa, E A is 23 GPa for compression and it is 25 GPa for tension. So, they have highlighted the difference in behavior difference in material constants under compression and tension.

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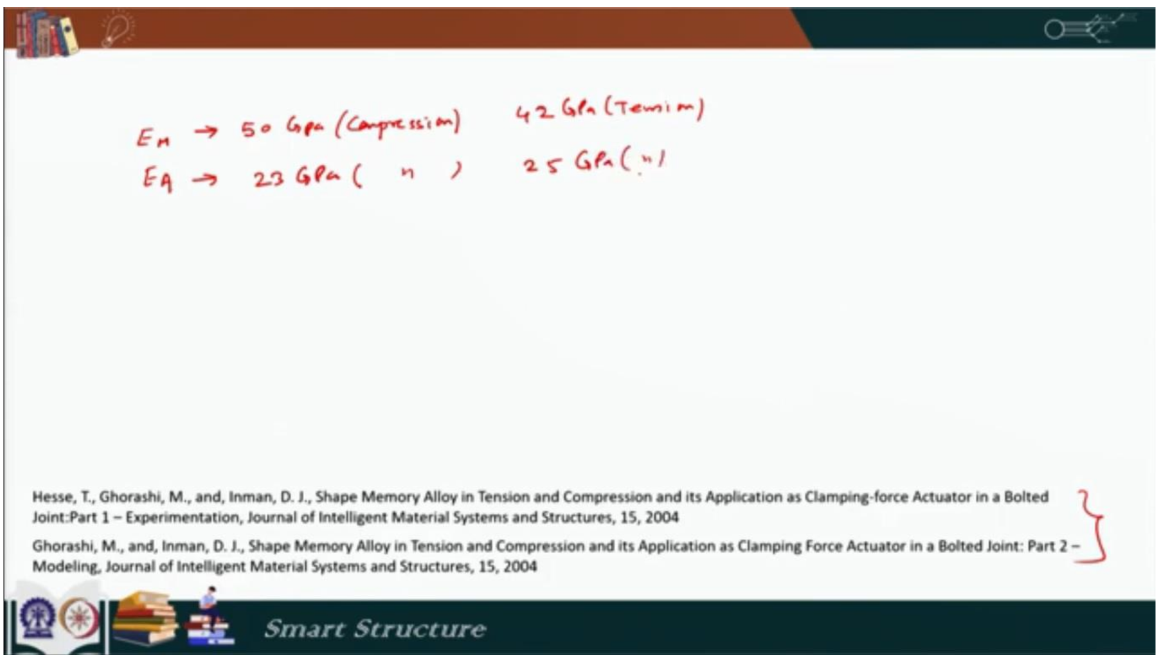

So, these two papers can be referred to if you are interested in looking at how these properties vary depending on the type of the stress. Now, we look into one more concept that is free recovery versus constraint recovery. Let us imagine that we have a shape memory alloy wire that is fixed at one side and this wire is stressed and it is stressed to a to a limit from which full recovery is possible with the by temperature and then from here it is unloaded. And again it does not have to be stressed to that final limit it can be to a lesser value also. And as we know that if we unload it we get epsilon L as the epsilon L as the residual strain and then if we heat it up the strain becomes 0 and it follows this path.

So, it is because of temperature. So, this line is because of temperature we may put a black arrow here just to denote that that going back is induced by temperature. And that is possible because this wire is set free it is only constraint at what at one end other end is kept free. Now suppose the same wire is constrained here also we do not set it free and that constraining is done after loading it and then unloading it. So, suppose when the strain is epsilon L this is constant.

Now if we heat it up it will not be able to go to the 0 strain because I have already constrained it. In that case instead of strain becoming 0 stress will increase because I am not allowing it to go to its natural length at that temperature because it is prevented from doing so, it will show an increase in stress because of the temperature effect. So, that is called a constraint recovery. So, there is some stress here sigma we may call it some sigma c may be constant recovery stress. So, if it is set free it goes back to its natural length and there is no stress if it is fully constrained it cannot go back to its natural length at that temperature that is why stress generates.

And this is quite similar to what we saw in case of piezoelectric materials in a piezoelectric material we saw that if I apply voltage to it and if it is free it goes to the free strain, but if it is constrained because it cannot go to the free strain configuration it generates stress same thing is happening here and that is we can call it a actuation stress. Now suppose the same thing instead of putting a full constraint at the end we put a spring in that case it is neither a fully free recovery it is neither a fully constrained recovery. So, it is a partially constrained recovery in that case the stress will not be totally 0 sorry strain will not be 0 and stress will not be this full stress. So, it will be somewhere in between. So, it will follow an angular line from here in that case some strain will remaining here will remain here and some stress will also be generated depending on the stiffness of this k.

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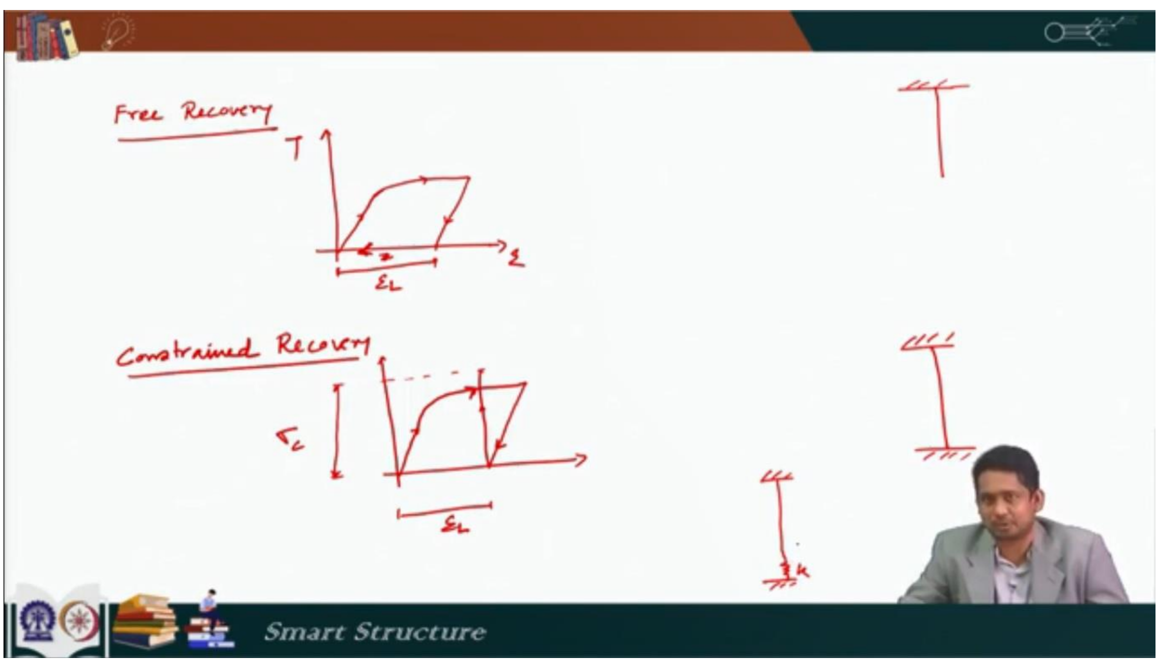

So, this case the fully constrained recovery case can be thought of as a special case of in fact in both the cases the free recovery case and the fully constrained recovery case can be thought of as a special case of this partial recovery case. When k is 0 it is a full recovery this and when k is infinite that means, it is fully constrained in that case it is a fully constrained recovery. So, with this I would like to conclude this lecture here in the next lecture we will solve some numerical problems and we will see how we can use the constitutive modeling models that we have developed so far.

Thank you.