Smart Structures
Professor Mohammed Rabius Sunny
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur
Week 07
Lecture No: 39
Analysis of composite laminate with piezoelectric patches(continued)
Part 04

Today, we will continue our discussion on Analysis of Composite Laminates which has piezoelectric patches.

In the previous lecture, we saw a case where the deformation was purely in plane, it was because our actuators were actuated in such a way that it generates only in plane force and also the composite laminate was symmetric.

Now, today we will see a pure bending case. Now again the structure is same, we have a composite laminate and the laminate is symmetric, dimensions are same, this is our x axis, this is y. So, this dimension is L and this is c by 2 and this is c by 2 and it is fixed at one end. And if we look at the view along the x z plane, it looks like this.

So, it has a piezoelectric patch and again we can assume that it is throughout, but that does not matter even if we keep it at a small part of it, the analysis remains same. So, let us assume that our thickness of the laminate is t_b and thickness of the two patches are t_c and the laminate is symmetric. So, this is the differential equation that ordinary differential equation, that we derived considering the Rayleigh-Ritz technique. So, we are solving it using Rayleigh-Ritz method.

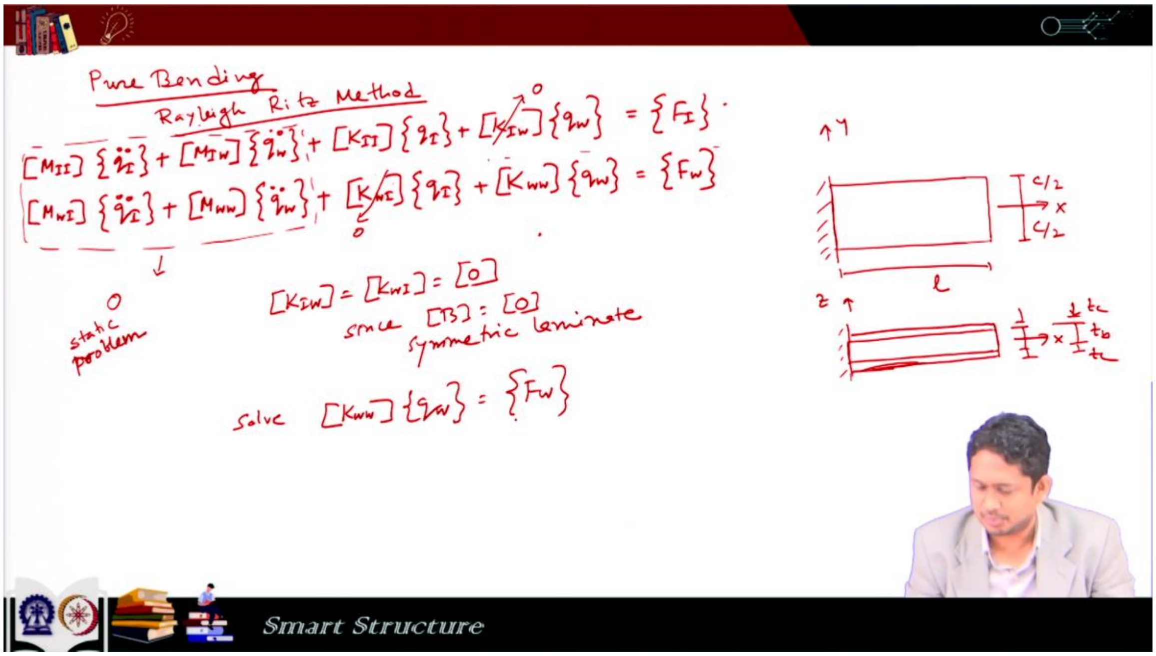
$$[M_{II}]\{\ddot{q}_I\} + [M_{IW}]\{\ddot{q}_W\} + [K_{II}]\{q_I\} + [K_{IW}]\{q_w\} = \{F_I\}$$

$$[M_{WI}]\{\ddot{q}_I\} + [M_{WW}]\{\ddot{q}_W\} + [K_{WI}]\{q_I\} + [K_{WW}]\{q_w\} = \{F_W\}$$

Now, within this differential equation ah these terms are 0, these terms do not come into picture because this is 0 because it is a static problem. The actuation is static. So, there is no acceleration and K_I, K_I, K_{IW} equal to K_{IW} equal to 0 as before since B matrix is 0, symmetric laminate. So, we are left with only this equation. So, these two equations i.e., the first equation and the second equation can be solved separately.

Now, in this case the actuation is purely a bending actuation. So, we need to solve only the only this equation. So, solve K_{WW} q_W is equal to F_w. So, now as before we need to make some approximations. So, let us go to next slide.

(Refer Slide Time: 4:00)



Our approximation was a function of x y is the approximation that we make is as x square by l square multiplied by q_{WI}. So, this is we can call it phi$_{WI}$ and this is our this is q_{WI}.

$$w(x,y) = \frac{x^2}{l^2} q_{WI}$$

Now, this satisfies the essential boundary condition. So, at x equal to 0, w is 0 and at x equal to 0 dw by dx is equal to 0. So, the cantilever boundary condition is satisfied slope and displacements at x equal to 0 is 0.

Now, we need to find out our K_{WW}. So, K we are just writing it as K because other components of K are not there anymore and to do that, we have the expression as B_W transpose, multiplied by the D matrix, multiplied by B_W and it is integral. Now, we need to find out this B_W transpose and to find out the B_W transpose, we need our quantities like kappa.

$$K = \int_{\Omega} [B_W]^T [D][B_W] \, d\Omega$$

So, kappa as we know is del 2 w by del x 2 this is del 2 w by del y 2 and this is 2 del 2 w del x del y and as per the approximation, this can be written as del 2 phi$_{WI}$ by del x 2. This quantity is 0 because phi is a function of only x. So, again it is a very simplified approximation. We are assuming that the bending i.e, the w is function of only x. So, bending is with respect to only the y axis. And this quantity is 0 and this quantity is 0. And this is equal to q_{WI}. And also, del 2 phi$_{WI}$ by del x 2 is: if I just differentiate it twice, it becomes 2 by l square. And this is our B_W matrix we can say.

$$K = \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial^2 \phi_{WI}}{\partial x^2} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{2}{l^2} \\ 0 \\ 0 \end{Bmatrix}$$

And also, we need the M_P's and M_P's have only one component and which is this. Again, that is an approximation. So, we have only one component of M_P which is M_{XP} and 0 0. And we know how to find out M_{XP}, we have done it before.

$$M_P = \begin{Bmatrix} M_{PX} \\ 0 \\ 0 \end{Bmatrix}$$

So, finally, if we put everything in this expression this becomes an integral over x and y 0 0. And then, we have D matrix here and here we have 0 0. Let us write this as an integral from x and y. So, this quantity reduces to del 2 phi_{WI} by del x 2, D_{11}, del 2 phi_{WI}, del x 2 dx dy. And finally, on being integrated, this gives us 4 c. So, this is D_{XX} generally as per our convention. So, this D matrix many a times we write it as D_{11}, D_{12}, I mean all these A B or D matrix, sometimes we write it from write it as A_{11} A_{12} A_{16} or we can write it as D_{XX} D_{XY} and D_{XS}. So, some books you will find out these subscripts to be 12 or 6 and in some of the books you may find it to be x y or s and both are same.

$$K_{11} = \int_{-c/2}^{c/2} \int_{0}^{l} \begin{Bmatrix} \dfrac{\partial^2 \phi_{WI}}{\partial x^2} & 0 & 0 \end{Bmatrix} [D] \begin{Bmatrix} \dfrac{\partial^2 \phi_{WI}}{\partial x^2} \\ 0 \\ 0 \end{Bmatrix} dxdy$$

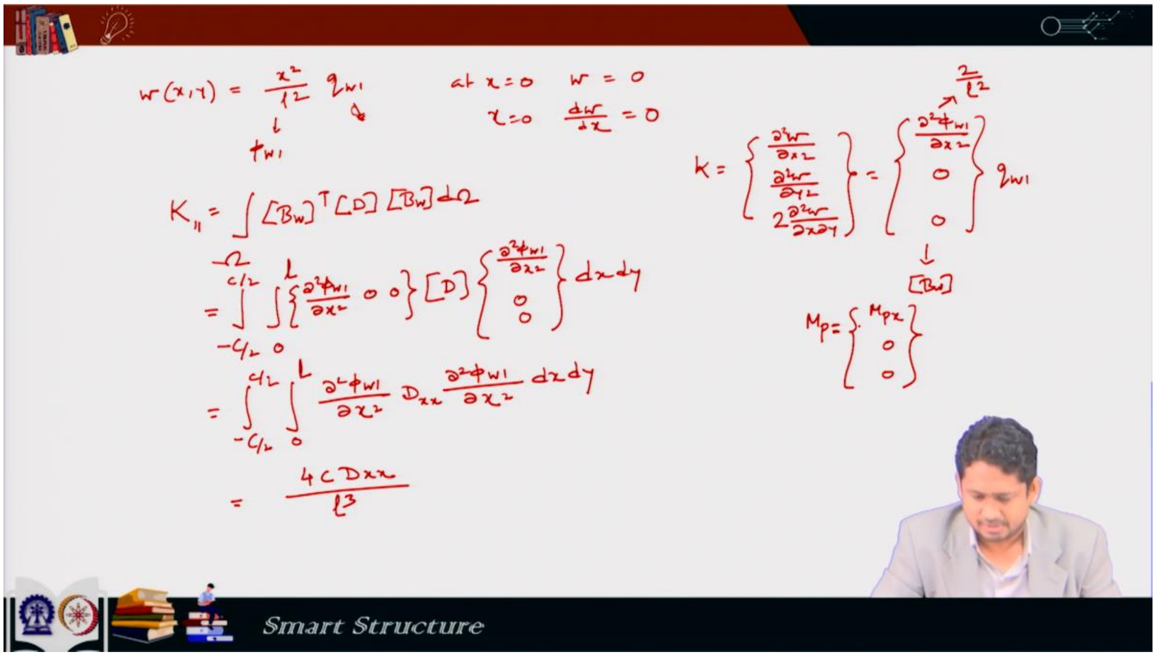
$$K_{11} = \int_{-c/2}^{c/2} \int_{0}^{l} \frac{\partial^2 \phi_{WI}}{\partial x^2} [D] \frac{\partial^2 \phi_{WI}}{\partial x^2} dxdy$$

$$K_{11} = \frac{4CD_{XX}}{l^3}$$

So, this is our expression for K and K has only one component. So, let's call it K_{11}.

Now we have to find out the right-hand side the force term which comes due to the piezoelectric actuation. Now, just to note one thing this we are we have been calling M_{PX} not M_{XP} same thing.

(Refer Slide Time: 10:00)



Now, we have to find out the force term. So, we have F_P is equal to minus c by 2, c by 2, 0 to l and then we have del 2 phi_{WI} by del x 2 0 0. And this is multiplied with this vector M_{PX} 0 0 and we have this.

$$F_P = \int_{-c/2}^{c/2} \int_{0}^{l} \left\{ \frac{\partial^2 \phi_{WI}}{\partial x^2} \quad 0 \quad 0 \right\} \begin{Bmatrix} M_{PX} \\ 0 \\ 0 \end{Bmatrix} dxdy$$

So, finally, this turns out to be this.

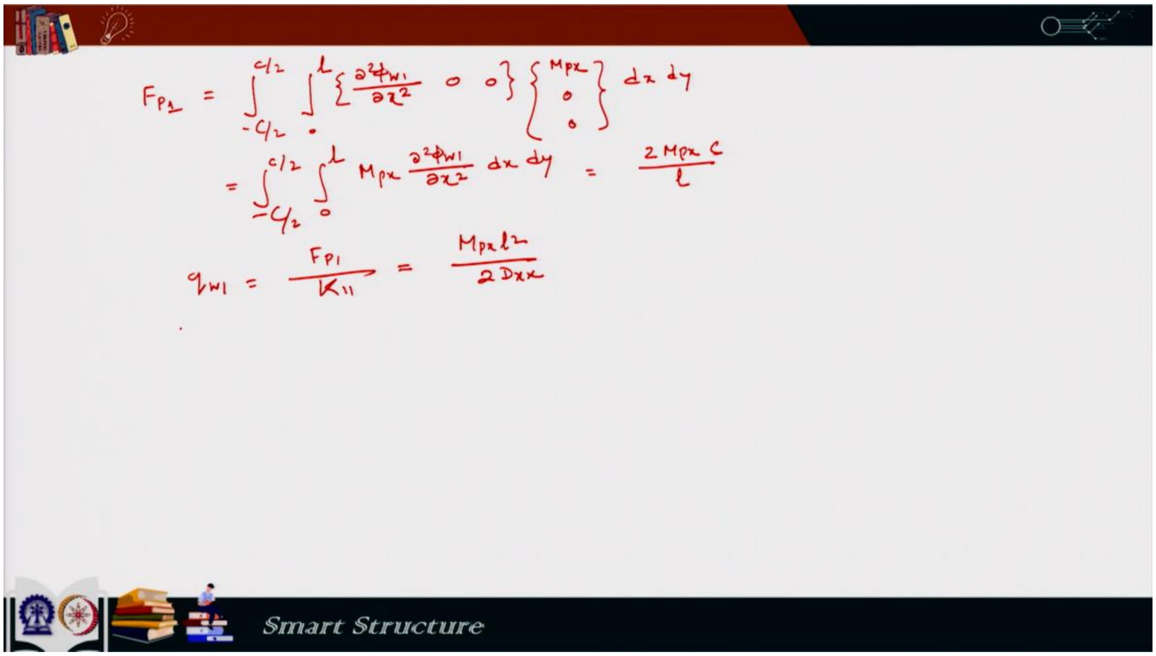
$$F_P = \int_{-c/2}^{c/2} \int_{0}^{l} M_{PX} \frac{\partial^2 \phi_{WI}}{\partial x^2} dxdy = \frac{2M_{PX}c}{l}$$

And then on being integrated the value comes to be 2 M_{PX} c by L, and then we can solve the equation and q_{w1} can come to. So, this is again it has only one component let us call it F_{P1} by K_{11} and that gives me q_{w1} to be M_{PX}, l square by 2 D_{XX}. So, this is our q_{w1} and then we can substitute q_{w1} to the original approximation and that gives us w as a function of x.

$$q_{WI} = \frac{F_{PI}}{K_{11}} = \frac{M_{PX} l^2}{2D_{XX}}$$

Now, we will solve this similar problem using the Galerkin technique.

(Refer Slide Time: 11:47)



To do it using Galerkin technique, we need to again make a different approximation because in Galerkin technique when we are having a purely bending problem, the highest order derivative appearing is 4. So, accordingly the approximation has to be changed. So, let us approximate w x as 6 x by l square, minus 4 x by l cube, plus x by l to the power 4 multiplied by q_{w1}. So, this is our approximation now. So, this is our phi_{W1}. Now this satisfies both essential and natural boundary condition which means the geometric and force boundary conditions.

$$w(x,y) = \left[6\left(\frac{x}{l}\right)^2 - 4\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4\right] q_{WI}$$

So, satisfies geometric and force boundary conditions. So, those conditions are w 0 y is 0, now del w by del x at x equal to 0 is 0. And then we have del 2 w by del x 2, it is a cantilever beam. So, at x equal to l is 0 and then we have del 3 w by del x 3 at x equal to l is 0. It has the derivatives up to fourth order existing. So, we need to make sure that minimum fourth order derivative exists because that is that is the highest order of derivative that would appear in our equation.

$$w(0,y) = 0 \quad \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=l} = 0$$
$$\left.\frac{\partial w}{\partial x}\right|_{x=0} = 0 \quad \left.\frac{\partial^3 w}{\partial x^3}\right|_{x=l} = 0$$

So, the equation is D_{XX} del 4 w by del x 4 is equal to del 2 M_{PX} by del x 2.

$$D_{XX}\frac{\partial^4 w}{\partial x^4} = \frac{\partial^2 M_{PX}}{\partial x^2}$$

And then, we define an error as we have been doing and the error is D_{XX} multiplied by the fourth order derivative of w with respect to x minus this.

$$\epsilon = D_{XX}\frac{\partial^4 w}{\partial x^4} - \frac{\partial^2 M_{PX}}{\partial x^2}$$

And then we say that we have to multiply this error with phi_{w1} and on being integrated over the domain that should be 0. So, let us do that. Now we multiply phi_{w1} minus del 2 M_{PX} by del x 2 dx dy is equal to 0. So, this quantity would give us a term where I have q_{W1} and this would give us a pure force term and by solving that we can find out our q_{W1}.

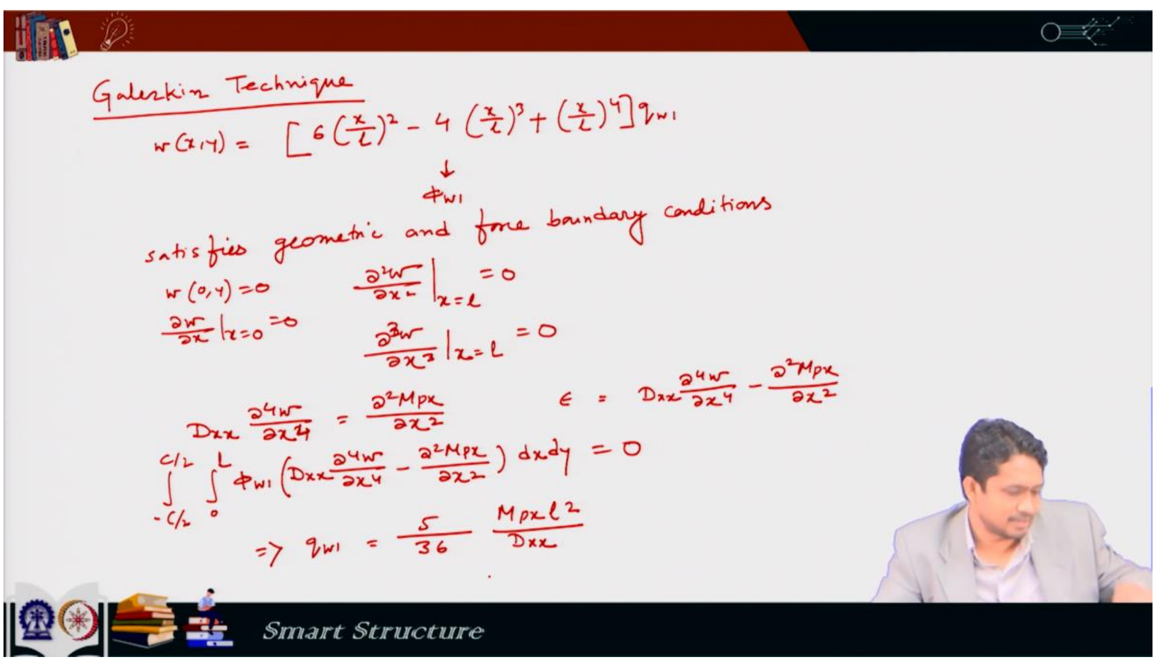
$$\int_{-c/2}^{c/2}\int_{0}^{l} \phi_{WI}\left(D_{XX}\frac{\partial^4 w}{\partial x^4} - \frac{\partial^2 M_{PX}}{\partial x^2}\right) dxdy = 0$$

So, if these entire expressions are evaluated finally, q_{W1} comes to be 5 by 36 multiplied by M_{PX} l square by D_{XX}.

$$q_{WI} = \frac{5}{36}\left(\frac{M_{PX}l^2}{D_{XX}}\right)$$

And then it can be put in the original approximation and our solution is done.

(Refer Slide Time: 16:23)



Now, we will solve another problem where we have a anti symmetric laminate.

Now, we know that, in an anti symmetric laminate our B matrix is: so, with this anti symmetric laminate is angle ply. It is not a cross ply. Because depending on whether the anti laminate is angle ply or cross ply our B matrix changes. So, this is B_{XS} this is 0 0, B_{YS}, this is B_{XS} B_{YS} 0.

$$[B] = \begin{bmatrix} 0 & 0 & B_{XS} \\ 0 & 0 & B_{YS} \\ B_{XS} & B_{YS} & 0 \end{bmatrix}$$

And then we have A matrix and we know that for any balance laminate, our A matrix has A_{XS} as 0, 0 0 0, A_{SS}.

$$[B] = \begin{bmatrix} A_{XX} & A_{XY} & 0 \\ A_{XY} & A_{YY} & 0 \\ 0 & 0 & A_{SS} \end{bmatrix}$$

And then we have D matrix and because this laminate is anti symmetric. So, our D_{XS} D_{YS} are also 0, 0 0 D_{SS}.

$$[D] = \begin{bmatrix} D_{XX} & D_{XY} & 0 \\ D_{XY} & D_{YY} & 0 \\ 0 & 0 & D_{SS} \end{bmatrix}$$

And again, our structural details remain same. We have a plate laminate which has a dimension of l along the x axis and a dimension of c along the y axis. And if we look at the x z plane again, we have piezoelectric patch at the top and bottom and the patches are of same property. Both material and geometric property are same.

So, this is t_b this is tc and this is our x axis, this is z axis. Now, in this laminate, the piezoelectric patches which are put here, if they are actuated even in the in purely in plane mode, suppose I give same voltage at the top and bottom, then also it would induce some out of plane deformation because our B matrix is non-zero. So, now because in the B matrix, we have B_{XS} and B_{YS} present. So, it basically a coupling between the in plane deformation and the twisting. Now, please understand that B_{XS}, B_{XY} and B_{YY} are 0 and that is why any in plane loading, any in plane actuation is not going to cause any bending, but it would it is going to cause twisting. So, here twisting and in plane deformations are coupled.

Now, there is no coupling between bending and twisting, but there is a coupling between in plane normal strain components and torsional strain. So, we need to approximate both u and w and solve it accordingly. So, now let us make some approximation. So, assume u_0 as a function of x and y is the approximation that we take is x by l, q_{u1}.

So, this is our phi_{u1}, and for w, let us make the approximation to be x square y by l square c, q_{w1}. And this is called phi_{w1}. And we have v_0 as 0.

$$u_0(x,y) = \frac{x}{l} q_{u1} \qquad w(x,y) = \frac{x^2 y}{l^2 c} q_{u1}$$

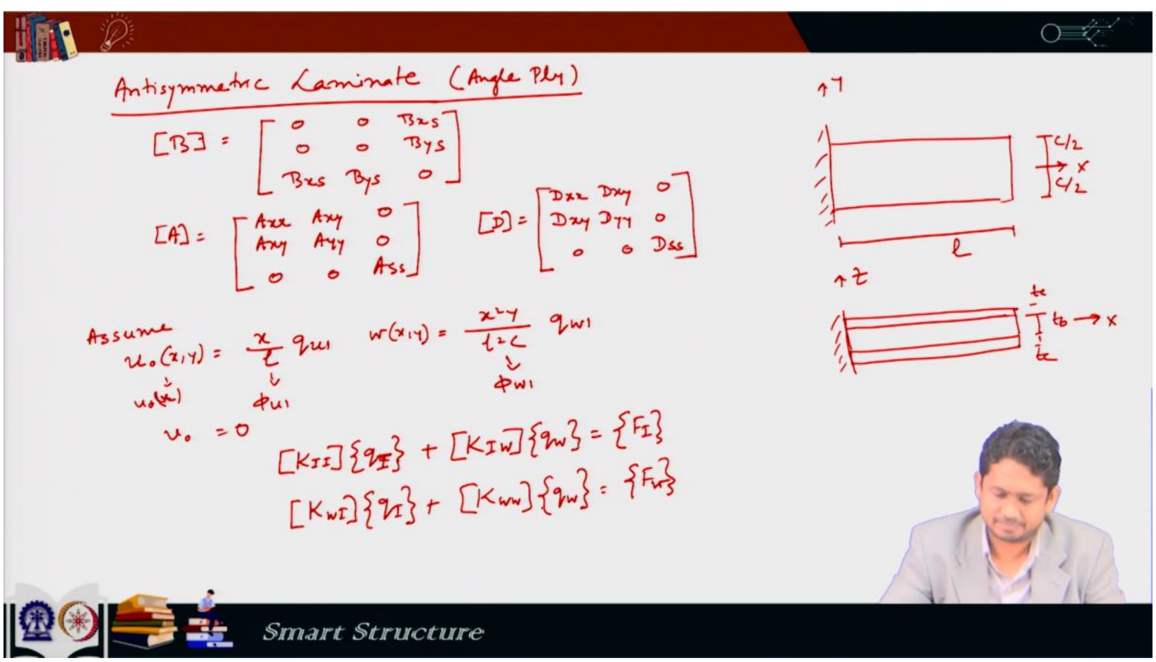
Now in our approximation for w, we took a y component also because we expect a twisting to be induced that is why we do not have a purely x component, I mean, w x y is not just a function of x it is a function of y as well. And u_0 is function of x only. Now all we need to do is we need to solve that solve two equations. So, finally, in this case the equation would look like this.

We will have K_{IW} multiplied by q_I plane plus: we just wrote the equation in the, ok. So, we have to solve two equations here. We have K_{II} multiplied by q_I which we call in plane plus K_{IW} multiplied by q_w is equal to F_I. And here we have K_{WI} multiplied by q_I plus K_{WW} multiplied by q_W is equal to F_W. So, we have to solve these two equations. So, we need to evaluate these matrices.

$$[K_{II}]\{q_I\} + [K_{IW}]\{q_W\} = \{F_I\}$$

$$[K_{WI}]\{q_I\} + [K_{WW}]\{q_W\} = \{F_W\}$$

(Refer Slide Time: 23:36)



So, finally, this entire thing looks like this if we write it in the, I mean, as per their expressions. So, this is B_I T multiplied by A matrix, multiplied by B_I transpose dx dy and we have q_{u1} plus minus c by 2, c by 2, 0 to L, then we have B_I T multiplied by the B matrix, multiplied by B_W (transpose should not be there) and then we have dx dy. And finally, at the right-hand side we have this is multiplied by q_{w1} and at the right-hand side we have minus c by 2, c by 2, 0 to l. And then we have B_I transpose into N_P dx dy.

$$\int_{-c/2}^{c/2}\int_0^l [B_I]^T[A][B_I]dx\,dy\,q_{u1} + \int_{-c/2}^{c/2}\int_0^l [B_I]^T[B][B_W]dx\,dy\,q_{w1} = \int_{-c/2}^{c/2}\int_0^l [B_I]^T\{N_P\}dx\,dy$$

So, again if we follow the same procedure, this comes to be minus c by 2, c by 2, 0 to l. And then, we have del phi_{u1} by del x, 0 0. And we have the A matrix here and this becomes del phi_{u1} by del x, 0 0 dx dy, q_{u1} plus we have same thing del phi_{u1} by del x, 0 0 and here we have the B matrix and here we have del 2 phi_{w1} del x 2, 0 twice of del 2 phi_{w1} del x del y. Now please understand if we look at the ah approximations for w this phi_{w1}, if we differentiate it twice with respect to x there is a non zero derivative. If we differentiate it twice with respect to y there is a zero derivative. And if we differentiate it with respect to x once and y once again there is a non-zero derivative. So, that is why the middle term is 0 and the and the other two terms are non-zero. dx dy. And this is multiplied by q_{w1} minus c by 2, c by 2, 0 to l. And this quantity is del phi_{u1} by del x, 0 0. Here we have N_{PX} 0 0 and dx dy.

$$\int_{-c/2}^{c/2}\int_0^l \left\{\frac{\partial \phi_{u1}}{\partial x} \quad 0 \quad 0\right\}[A]\left\{\begin{matrix}\frac{\partial \phi_{u1}}{\partial x}\\ 0\\ 0\end{matrix}\right\}dx\,dy\,q_{u1}$$

$$+\int_{-c/2}^{c/2}\int_0^l \left\{\frac{\partial \phi_{u1}}{\partial x} \quad 0 \quad 0\right\}[B]\left\{\begin{matrix}\frac{\partial^2 \phi_{w1}}{\partial x^2}\\ 0\\ 2\frac{\partial^2 \phi_{w1}}{\partial x \partial y}\end{matrix}\right\}dx\,dy\,q_{w1}$$

$$= \int_{-c/2}^{c/2}\int_0^l \left\{\frac{\partial \phi_{u1}}{\partial x} \quad 0 \quad 0\right\}[\left\{\begin{matrix}N_{PX}\\ 0\\ 0\end{matrix}\right\}dx\,dy$$

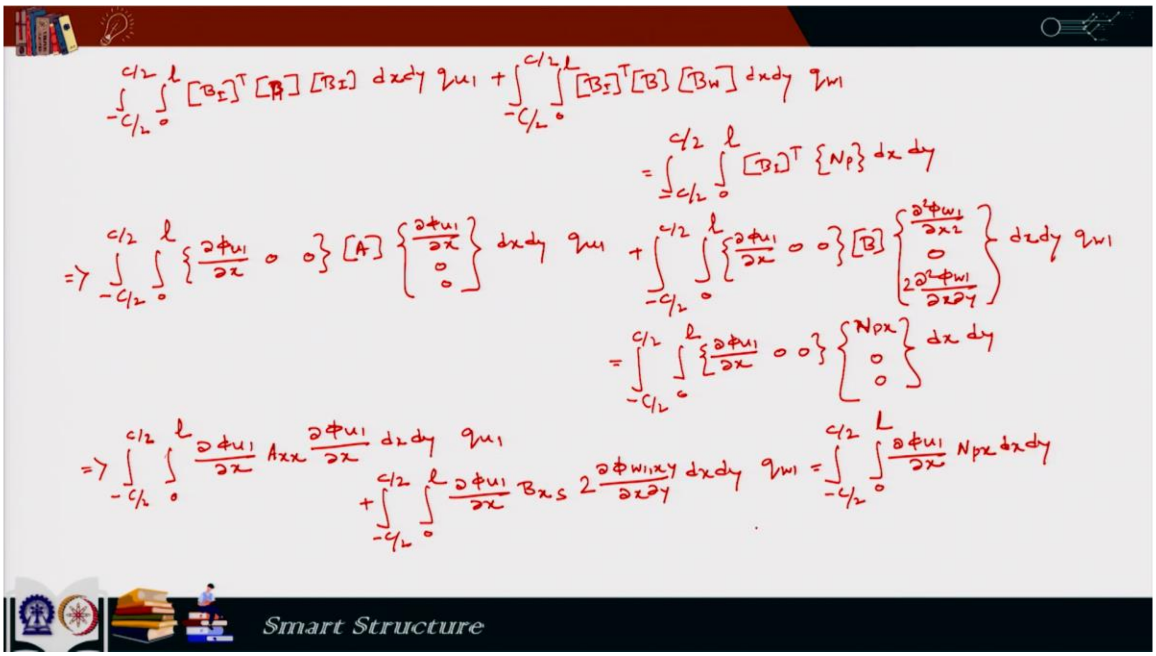
So, after all these evaluations this quantity comes to be minus c by 2, c by 2, 0 l. So, here we have del phi_{u1} by del x, A_{XX} multiplied by del phi_{u1} by del x, dx dy, q_{w1}. We are just evaluating this term. And then we have minus c by 2, c by 2, 0 to l. If we evaluate this term, we will get del phi_{u1} by del x multiplied by B_{11}. So, it is it is not B_{11}, it will be B_{XS} B_{XS} multiplied by twice of del phi_{w1} x y by del x del y, and dx dy q_{w1}. Finally, at the right-hand side we have minus c by 2, c by 2, 0 l del phi_{u1} by del x N_{PX} dx dy.

$$\int_{-c/2}^{c/2}\int_{0}^{l}\left(\frac{\partial \phi_{u1}}{\partial x}\right)A_{XX}\left(\frac{\partial \phi_{u1}}{\partial x}\right)dx\,dy\,q_{u1}+\int_{-c/2}^{c/2}\int_{0}^{l}\left(\frac{\partial \phi_{u1}}{\partial x}\right)B_{XS}\left(2\frac{\partial^2 \phi_{w1}}{\partial x \partial y}\right)dx\,dy\,q_{w1}$$

$$=\int_{-c/2}^{c/2}\int_{0}^{l}\left(\frac{\partial \phi_{u1}}{\partial x}\right)N_{PX}dx\,dy$$

So, there is one equation where q_{u1} and q_{w1} are the unknowns.

(Refer Slide Time: 29:26)



Next, we have to form another equation and, in that equation, would be again with q_{u1} and q_{w1} as the unknowns and that equation would look like this. 0 to L and here we have B_I transpose multiplied by B_W transpose multiplied by the B matrix, dx dy. And this is multiplied with q_{u1} plus minus c by 2, minus c by 2, 0 to l B_W transpose multiplied by D multiplied by B_W dx dy. And then at the right-hand side we have del 2 w1 by del x 2, 0, 2 phi_{w1} del x del y, but there is no actuation because the actuation is purely in plane actuation. Our piezo's are actuated symmetrically with same voltage. So, we can write it here. Pure in plane actuation.

$$\int_{-c/2}^{c/2}\int_{0}^{l}[B_W]^T[B][B_I]dx\,dy\,q_{u1}+\int_{-c/2}^{c/2}\int_{0}^{l}[B_W]^T[D][B_W]dx\,dy\,q_{w1}$$

$$=\int_{-c/2}^{c/2}\int_{0}^{l}\left\{\frac{\partial\phi_{w1}}{\partial x}\quad 0\quad 2\frac{\partial^2\phi_{w1}}{\partial x\partial y}\right\}\begin{Bmatrix}0\\0\\0\end{Bmatrix}dx\,dy$$

Now, if these expressions are simplified, this comes to be this 2 phi$_{w1}$ x y. Then we have B$_{XS}$ phi$_{u1}$ x dx dy q$_{u1}$ and then we have minus c by 2, c by 2, 0 to L and here we have contributions from D matrix and that is phi$_{w1}$ x x D$_{XS}$ phi$_{w1}$ x x plus phi$_{w1}$ x y and then we have D$_{SS}$ again twice phi$_{w1}$ comma x y dx dy. And then at the right-hand side we have 0.

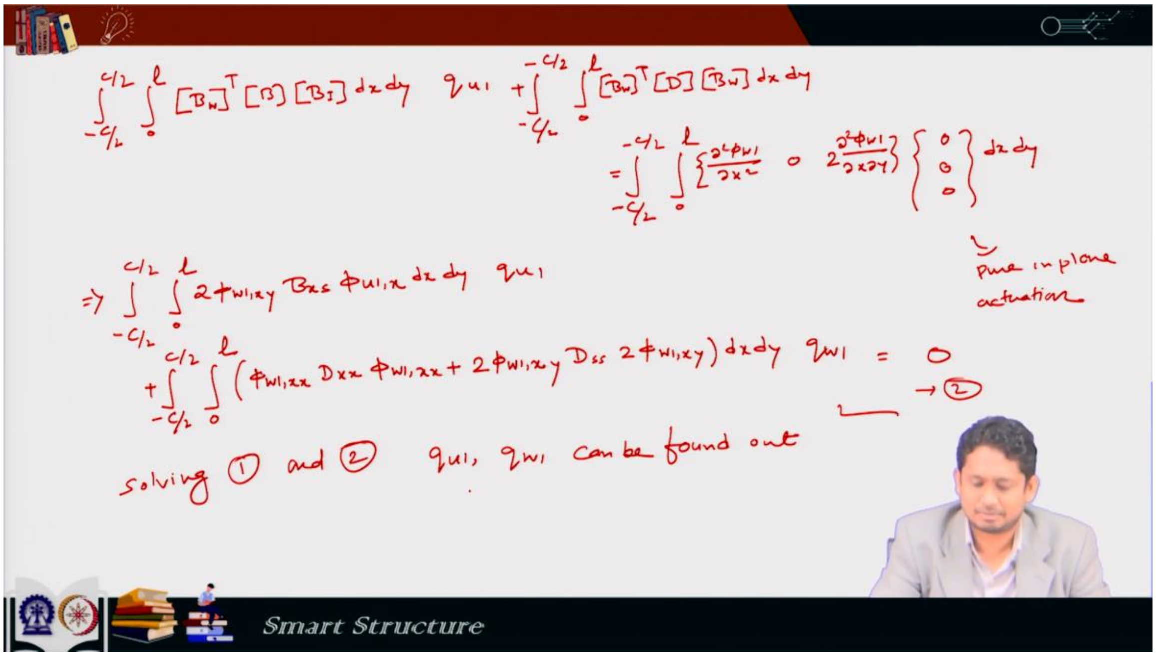
$$\int_{-c/2}^{c/2}\int_{0}^{l}2\phi_{w1,xy}B_{XS}\phi_{u1,x}dx\,dy\,q_{u1}$$

$$+\int_{-c/2}^{c/2}\int_{0}^{l}\left(\phi_{w1,xx}D_{xx}\phi_{w1,xx}+2\phi_{w1,xy}D_{ss}2\phi_{w1,xy}\right)dx\,dy\,q_{w1}=0$$

So, here the derivatives are written as comma with subscripts and here it was written as the derivative itself, but again they are same. So, we have got 2 equations, the previous equation is this. Here I have unknowns q$_{u1}$, q$_{w1}$. Next equation is this. Here I have unknowns q$_{u1}$, q$_{w1}$. And so, 2 equations 2 unknowns we can solve these 2 equations. After solving these equations, we can find out q$_{u1}$ and q$_{w1}$. So, let us give some number to it maybe we call it 1 and maybe we call it 2. So, solving 1 and 2, q$_{u1}$, q$_{w1}$ can be found out. So, in this problem ah it was an antisymmetric laminate the actuation was a pure in plane actuation. Now although the actuation was actuation was in plane actuation because of the B matrix being non zero we had to solve the 2 coupled equations. So, we got q$_{u1}$ and q$_{w1}$ each of these u$_0$ and w had 1 term in the approximation.

So, after this solution we can find out the constants associated with each of these terms unknown constants and after getting these 2 constants we can again back substitute and find out your u$_0$ and w and that solves the problem.

(Refer Slide Time: 34:20)



So, with this I would conclude this lecture.

Thank you.