Smart Structures

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Week 07
Lecture No: 38
Analysis of composite laminate with piezoelectric patches (continued)
Part 03

So, today we will continue with our discussion on Analysis of Composite Laminates which are Piezoelectric Patches. We started with the Galerkin technique. And then, after the Galerkin technique, to define these terms N_{xp}, N_{yp}, N_{sp} that we did in the previous lecture. Now, just one thing to note, this expression is written considering for one patch. So, this should be doubled, if I cannot consider two patches. And here, while writing the moment term, we have considered two patches. At top and bottom, or I should say two identical patches, identical, considering two patches which are identical and which are actuated in an antisymmetric way, which can cause bending. Now, if I consider two patches here, this just quantities get doubled.

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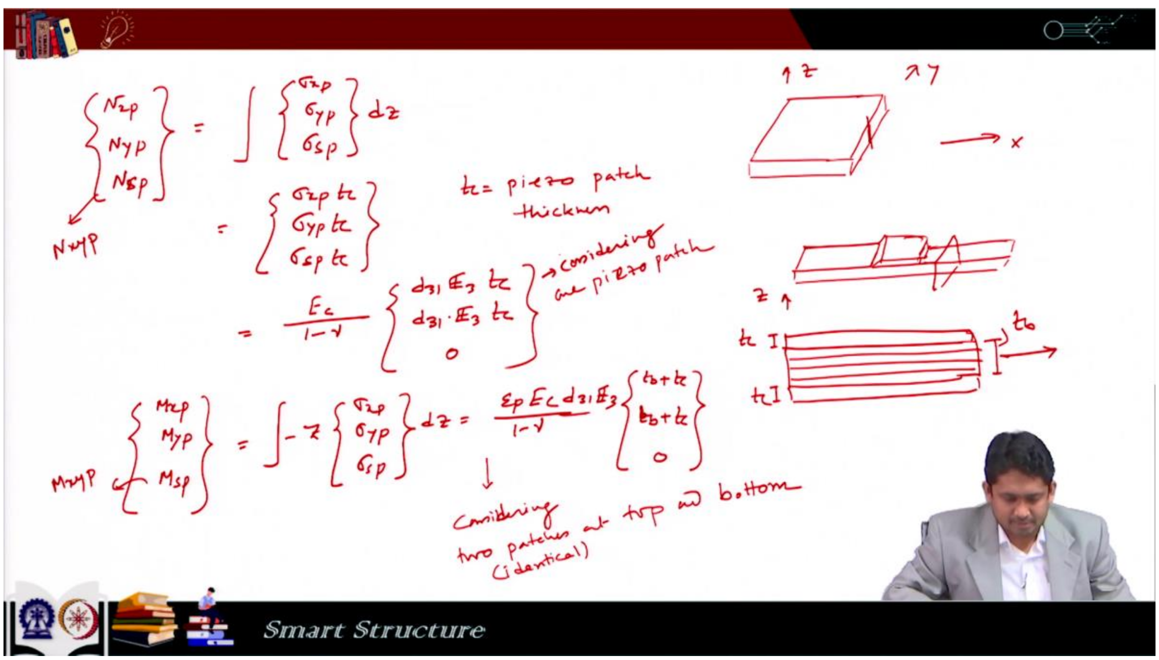

All right. Now then, we looked into the Rayleigh Ritz technique. And in the Rayleigh Ritz technique, this was the variational indicator that we wrote.

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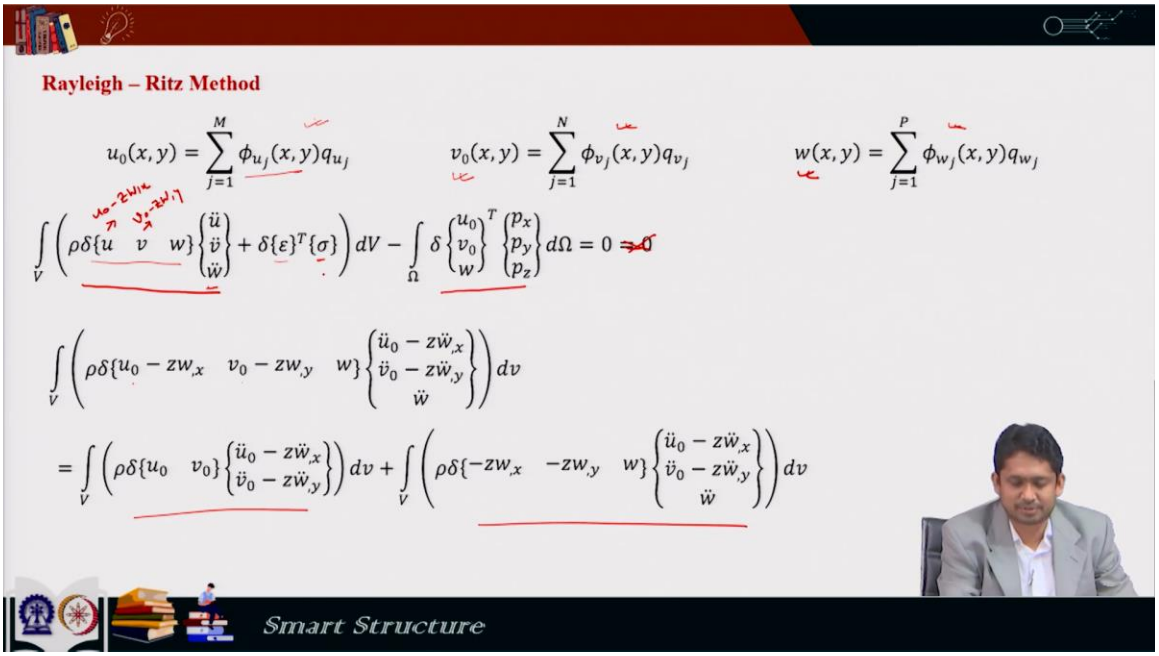

After putting all the expressions, we wrote all the terms, the inertia related terms, the internal virtual work terms, and these external virtual work terms, everything in terms of our approximations. Now, it is time to finally arrive at the final equations.

So, this is what we started with and this we wrote, this we wrote, this we wrote, everything we wrote in terms of other approximations. So finally, if we put all the approximations here, we get a term like this, and then, we will get a term like this, and that is equal to 0.

$$\int_{\Omega} \delta\{q_I\}^T[\cdots]d\Omega + \int_{\Omega} \delta\{q_w\}^T[\cdots]d\Omega = 0$$

And again, we know that, these variations are independent, and these are arbitrary variations. So, this entire quantity can be 0, only when, I have this equal to 0, and this equal to 0. So, this gives us M plus N equations, and this gives us P equations, where M and N are the number of terms used to approximate our u_0 and v_0, and P is the number of terms which is used to approximate our w. So, finally, the equation that would look like is this. Now, we will define these matrices. And, in the righthand side, we have F_I.

$$[M_{II}]\{\ddot{q}_I\} + [M_{IW}]\{\ddot{q}_W\} + [K_{II}]\{q_I\} + [K_{IW}]\{q_W\} = \{F_I\}$$

Similarly, here we have F_W.

$$[M_{WI}]\{\ddot{q}_I\} + [M_{WW}]\{\ddot{q}_W\} + [K_{WI}]\{q_I\} + [K_{WW}]\{q_W\} = \{F_W\}$$

So, this is our M plus N equations, and this is a system of P equations. Now, we have to define these matrices M_{II} is the integral over the surface of N_I transpose, multiplied by m_{II} into N_I. Then, I have M_{IW}, which is N_I m_{IW} multiplied by N_W.

$$[M_{II}] = \int_{\Omega} [N_I]^T [m_{II}][N_I] d\Omega \qquad [M_{IW}] = \int_{\Omega} [N_I]^T [m_{IW}][N_W] d\Omega$$

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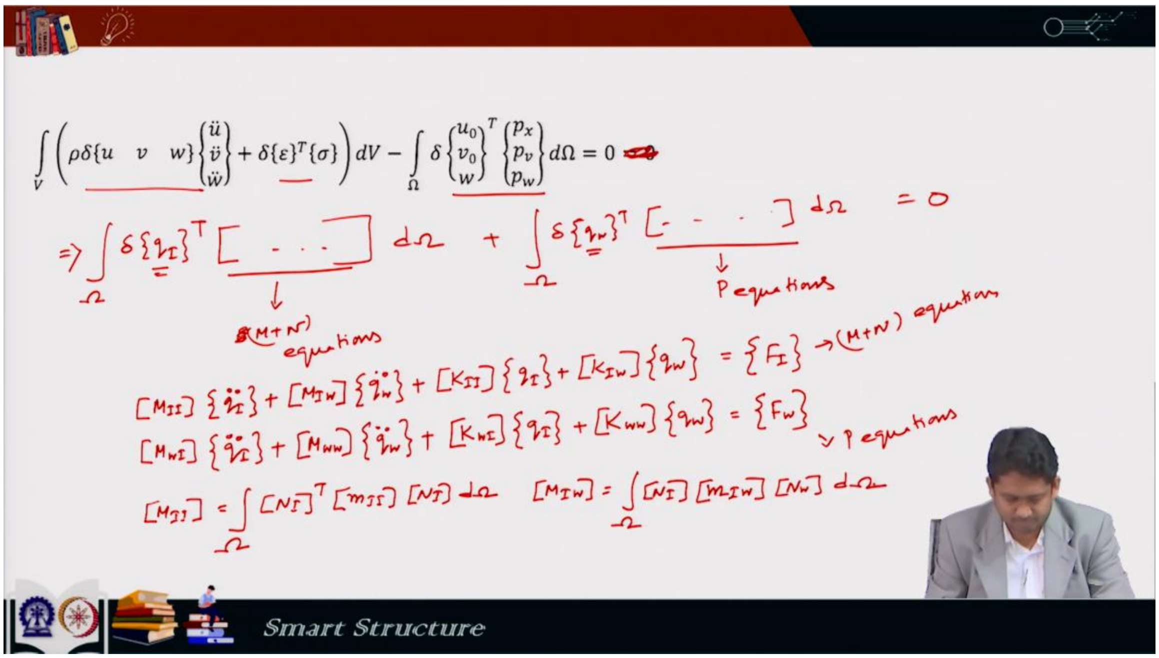

And then, we have, M_{WI} is equal to N_{WT}, multiplied by m_{WI}, multiplied by N_I transpose, d omega. Sorry, this is capital M_W that we are talking about. Similarly, here we have M_{WW}, and we have N_W T, multiplied by m_{WW}, multiplied by N_W, d omega.

$$[M_{WI}] = \int_{\Omega} [N_W]^T [m_{WI}][N_I] d\Omega \qquad [M_{WW}] = \int_{\Omega} [N_W]^T [m_{WW}][N_W] d\Omega$$

Then comes the stiffness terms. So, we have K_{II} which is equal to B_I transpose, multiplied by A, multiplied by B_I, d omega.

Here, we have K_{IW} which is equal to B_I transpose, multiplied by the B matrix, multiplied by B_W transpose.

$$[K_{II}] = \int_{\Omega} [B_I]^T [A][B_I] d\Omega \qquad [K_{IW}] = \int_{\Omega} [B_I]^T [B][B_W]^T d\Omega$$

Then, we have K_{WI} which is equal to B_W transpose, multiplied by the B matrix, multiplied by the B_I matrix. We have K_{WW} which is D matrix, sorry, which is B_W matrix, multiplied by D matrix, multiplied by B_W.

$$[K_{WI}] = \int_{\Omega} [B_W]^T [B][B_I] d\Omega \qquad [K_{WW}] = \int_{\Omega} [B_W]^T [D][B_W] d\Omega$$

And then, if we - so, in the previous - so, we have called it F_I. and then we have the F_I matrix, and which is B_I transpose, multiplied by N_P d omega plus omega, N_I transpose multiplied by q_x q_y.

$$\{F_I\} = \int_{\Omega} [B_W]^T [N_p] d\Omega + \int_{\Omega} [N_p]^T \begin{Bmatrix} q_x \\ q_y \end{Bmatrix} d\Omega$$

And here, we have F_W which is B_W transpose, multiplied by M_P d omega plus N_W transpose, multiplied by q_z.

$$\{F_W\} = \int_{\Omega} [B_W]^T [M_p] d\Omega + \int_{\Omega} [N_W]^T \{q_z\} d\Omega$$

Now, inside F_I, the force matrix, we have contribution from the N_P's and the applied forces. And similarly, in F_W, we have contributions from M_P's and the applied transverse forces. So, this is the entire formulation based on the Rayleigh Ritz technique.

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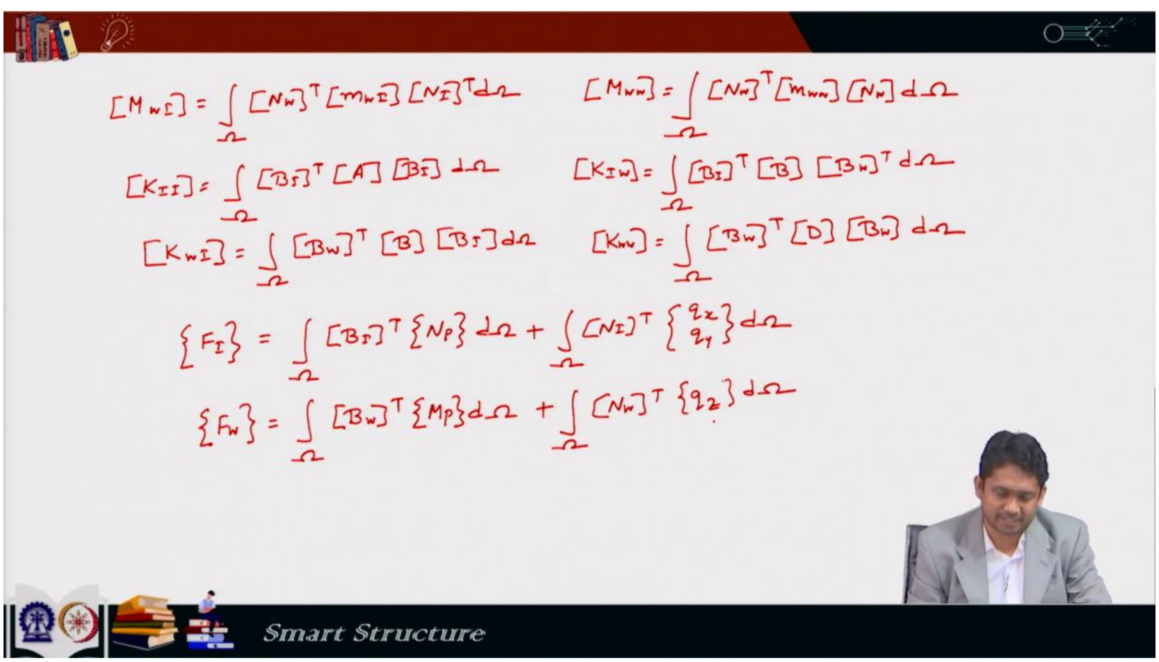

So, here we can see that if the laminate is symmetric, then these two terms vanish; because in a symmetric laminate, B matrix is 0 and this stiffness matrix, and this stiffness matrix, they have B matrices.

So, these two stiffness matrix does a stiffness coupling between the in-plane components and the out-of-plane components. So, if our laminate is symmetric and if we know that the applied force is only along the x y direction, so, we can separately just solve this. We can separately solve M_{II}, q_I double dot, plus K_{II}, q_I is equal to F_I because we know that there is no coupling between the stiffness and in plane and out of plane terms, and the load is purely in plane. So, there is nothing going to be induced in the out of plane direction. In that case, we can solve the decoupled problem. Similarly, if the load is applied purely along the z direction, in that case also we can just solve M_{WW}, q_W double dot, plus K_{WW}, q_W is equal to F_W.

But, if the laminate is not symmetric, in that case B matrix exists. And if the B matrix exists, then this coupling matrices, K_{IW} and K_{WI}, they remain non zero. And when they are non zero, then even if my load is purely in plane, it will induce out of plane deformation, or if the load is purely out of plane, it will induce in plane deformation. In that case, the entire system has to be solved.

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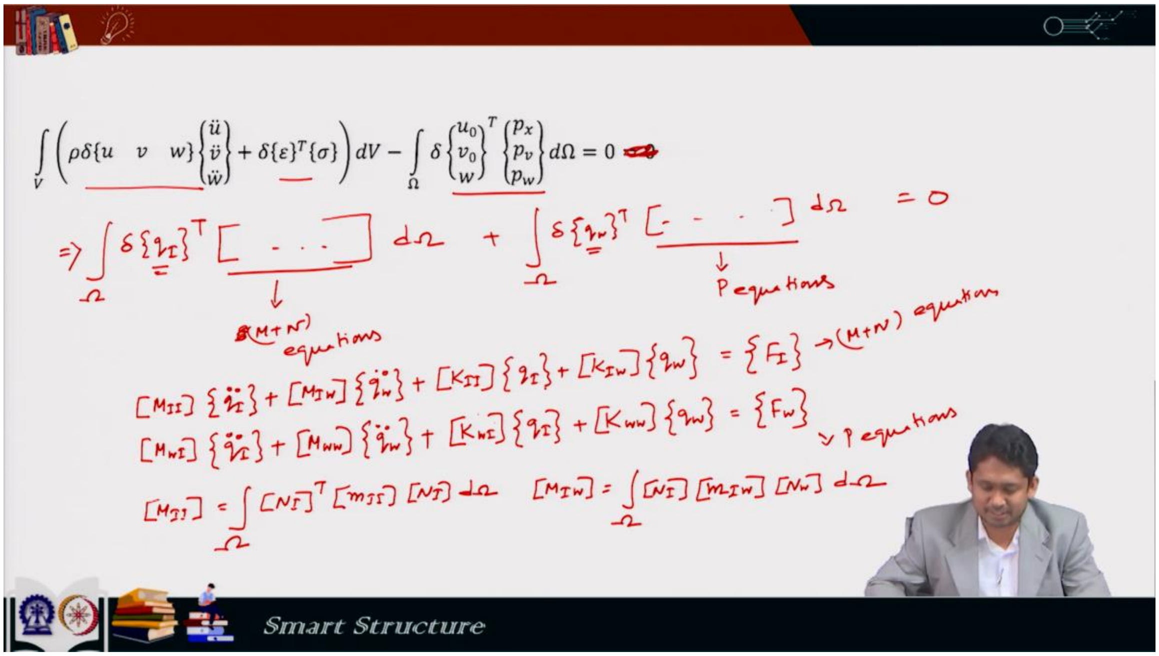

Now, let us look into our example of pure in plane deformation. So, let us imagine that we have a composite laminate with piezoelectric patches which looks like this when we view from top. So, this is our x direction, this is y direction. Along the x direction, the dimension is L, and along the y direction, the dimension is C. So, from here to here, it is C by 2, from here to here it is C by 2. And then, if we look at the same thing in the x z plane, it looks like this.

We can assume that the peizo patch is distributed all over. Even, if it is not distributed all over, our formulation can care of that, that is not a problem. So, this is in the x z plane. So, this part is t_c, and for the sake of simplicity, let us assume that we have two identical piezoelectric patches of same thickness.

So, these are our - this is a piezo patch, and this is composite laminate. And this is another piezo patch. And we have two identical piezo patches actuated symmetrically by voltage V. So, V top is equal to V bottom is equal to voltage V, and because they are actuating symmetrically, it will induce only in plane deformation. And also for the sake of simplicity, while assuming let us imagine that our deformation along y is not accounted for. So, if that is the case, then accordingly, let us make some approximations.

So, assume u_0 which should be as a function of x and y. Let us assume, it to be only function of x, and then, there is no variation along the y direction. So, I would say that it is not just the deformation, it is the deformation variation along y direction. So, it is u_0 and let us approximate only one term solution. So, there is no summation. We are approximating only one term solution. So, let us assume that to be phi_{u1}, multiplied by q_{u1}. And let us assume, phi_{u1} to be x by L. It is x by L multiplied by q_{u1}. Now, this satisfies the essential boundary conditions.

So, at x equal to 0, u_0 is 0. And again, because there is no out of plane deformation, our laminate is symmetric. So, for pure in plane deformation, let us assume symmetric laminate. And the actuation is such that it does not induce any moment. It induces only in plane forces. So, we can solve it as a purely axial problem. So, in that case the stiffness is like q_{IW}, and q_{WI} can be neglected, and let us solve it as a static problem. Now, if that is the case, we solve only this: K_{II} multiplied by q_I is equal to F_I.

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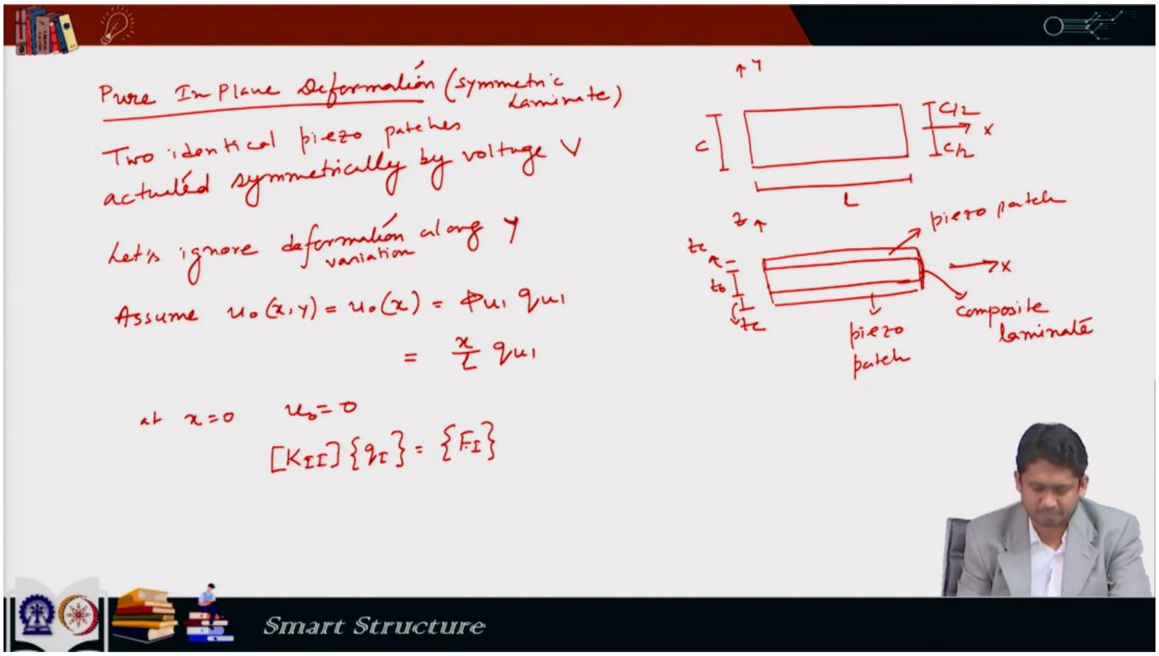

Now, we have K, K consists of only K_{II}, and that is equal to our integral over the surface B_I T, multiplied by A, multiplied by B_I. Now, please understand that because we do not have other components of K's, we have only K_{II}. So, we are just writing it as K, we are not specifying. And this becomes minus c by 2 to c by 2, and we have to put the approximate, I mean, the B matrix in terms of the approximations. So, as per our approximation, epsilon$_{0x}$, epsilon$_{0y}$, and epsilon$_{0s}$, that is in terms of if we write q_{u1}, then it becomes phi$_{u1}$ comma x, phi$_{u1}$ comma y, there is no – phi$_{v1}$, and then we have phi$_{u1}$ comma y. So, not only our variation of the u_0 along y_0, even our v is also 0. So, v_0 is 0. And then this B matrix and then, this entire thing is multiplied with q_{u1}.

$$\begin{Bmatrix} \varepsilon_{0x} \\ \varepsilon_{0y} \\ \varepsilon_{0s} \end{Bmatrix} = \begin{Bmatrix} \phi_{u1,y} \\ 0 \\ \phi_{u1,y} \end{Bmatrix} q_{u1}$$

So, we can say that our B matrix is essentially phi$_{u1}$ comma x is 1 by L, 0, and phi is not a function of y. So, it is also 0. So, B_I is essentially 1 by L, 0, 0. So, this is 1 by L, 0, 0. A matrix remains as it is. And this is 1 by L, 0, 0. And this entire integration has to be evaluated along y, it is from minus c by 2 to c by 2, and along x, it is from 0 to L. Now, if we do the entire multiplication, this expression comes to be 1 by L square multiplied by A_{xx}. And finally, after integrating this comes to be A_{xx}, multiplied by c by L. So, K has only one component. So, we can call it K_{11} also. And similarly, our F matrix: that is also going to have only one component, we will see that.

$$K_{11} = \int_{\Omega} [B_I]^T [A][B_I] d\Omega = \int_0^l \int_{-c/2}^{c/2} \left\{\frac{1}{l} \quad 0 \quad 0\right\} [A] \begin{Bmatrix} \frac{1}{l} \\ 0 \\ 0 \end{Bmatrix} dy\, dx$$

$$K_{11} = \int_0^l \int_{-c/2}^{c/2} \frac{1}{l^2} A_{xx} dy\, dx = \frac{A_{xx} c}{l}$$

And F matrix is going to be - as per our definition, it is B_I transpose, multiplied by the N_P matrix. So, it is again 1 by L, 0, 0. And we have N_{Px}, and N_{Py}, and 0. This expression after the integration comes to be 0 to L, N_{Px} into L into c.

$$F_1 = \int_0^l \int_{-c/2}^{c/2} \left\{\frac{1}{l} \quad 0 \quad 0\right\} \begin{Bmatrix} N_{px} \\ N_{py} \\ 0 \end{Bmatrix} dy\, dx = N_{px} l c$$

Now, we can always solve, and there is only one components, we can call it F_1. So, we can always solve q_{u1}, multiplied by K_1 is equal to F_1 that would give us q_{u1} as F_1 by K_{11}.

$$K_{11} q_{u1} = F_1$$

$$\Longrightarrow q_{u1} = \frac{F_1}{K_{11}}$$

So, once we get our q_{u1}, we know the solution.

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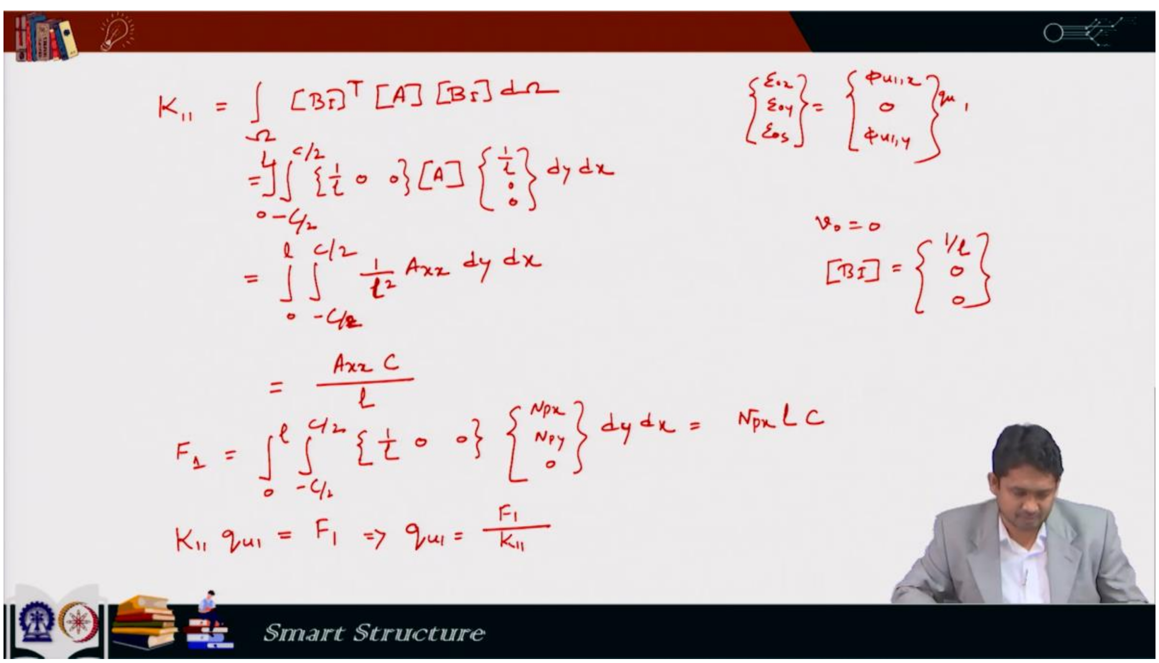

Now, we will solve the same problem using the Galerkin technique.

To solve it using Galerkin technique, let us make a different approximation. Let us approximate that our variation of u_0 along y is not there, and v_0 is not there. So, it is x by L minus half into x by L square, and v_0 is 0. So, there is no v_0. That is a much simplified approximation.

$$u_0(x,y) = u_0(x) = \left[\frac{x}{l} - \frac{1}{2}\left(\frac{x}{l}\right)^2\right] q_{u1}$$

$$v_0(x,y) = 0$$

Now, when we solve it using the Galerkin technique, the phi is this. So, this has to be multiplied with q_{u1} and which we call phi_{u1} multiplied by q_{u1}. So, this phi_{u1} is going to be differentiated twice. So, that is why we had to use a different approximation functions and it is satisfies both the geometric and force boundary conditions. So, our geometric boundary condition is : that this part is clamped. So, u_0 is 0 here and the force boundary conditions is - at the free end, there is no force. So, u_0 is equal to 0 at x is equal to 0, and del u_0 by del x is equal to 0 at x equal to L. And this approximation satisfies those conditions.

$$u_0 = 0 \ \ at \ \ x = 0$$

$$\frac{\partial u_0}{\partial x} = 0 \ \ at \ x = l$$

Now, we know that we already discussed the governing equations. And we know that according to that the differential equation is this.

$$A_{xx}\frac{\partial^2 u_0}{\partial x^2} = \frac{\partial F_{px}}{\partial x}$$

And then, we define an error function - the error function that we define is, or it is a residue. So, the residue that we define is minus of del F_{px} by del x.

$$\epsilon(x) = A_{xx}\frac{\partial^2 u_0}{\partial x^2} - \frac{\partial F_{px}}{\partial x}$$

Now, what we do is that we multiply this residue by the basis functions one by one. In that case, we have only one. So, we just multiply with phi_{u1} and integrate it over the surface and make the residue 0 and that gives us the desired equation solving which we can find out q_{u1}. So, we have phi_{u1}, multiplied by epsilon, d omega integrated over the omega is 0.

$$\int_{\Omega} \phi_{u1} \epsilon \, d\Omega = 0$$

And which also tells us that phi$_{u1}$, multiplied by A_{xx}, multiplied by del 2 phi$_{u1}$ by del x 2, multiplied by q_{u1} minus del F_{px} by del x, d x. It is d omega, and that is equal to 0.

$$\int_{\Omega} \phi_{u1}\left(A_{xx}\frac{\partial^2 \phi_{u1}}{\partial x^2}q_{u1} - \frac{\partial F_{px}}{\partial x}\right)d\Omega = 0$$

Now, here we can see that this phi u had to be differentiated twice. So, that is why we had to make this approximation. The previous approximation that we made while solving using the Rayleigh Ritz technique does not work here.

And then finally, after evaluating all these integrations, and we know that this d omega is equal to 0 to L, minus c by 2 to c by 2 d y d x.

$$\int_{\Omega} d\Omega = \int_{0}^{l}\int_{-c/2}^{c/2} dy\, dx$$

So, after evaluating these integrals finally, q_{u1} comes to be 3 l N_{px} divided by 2 A_{xx}.

$$q_{u1} = \frac{3lN_{px}}{2A_{xx}}$$

Now, here F_{px} is basically our N_{px}. The first term in that is F_{px} vector. So, that is the solution, that is obtained using the Galerkin technique.

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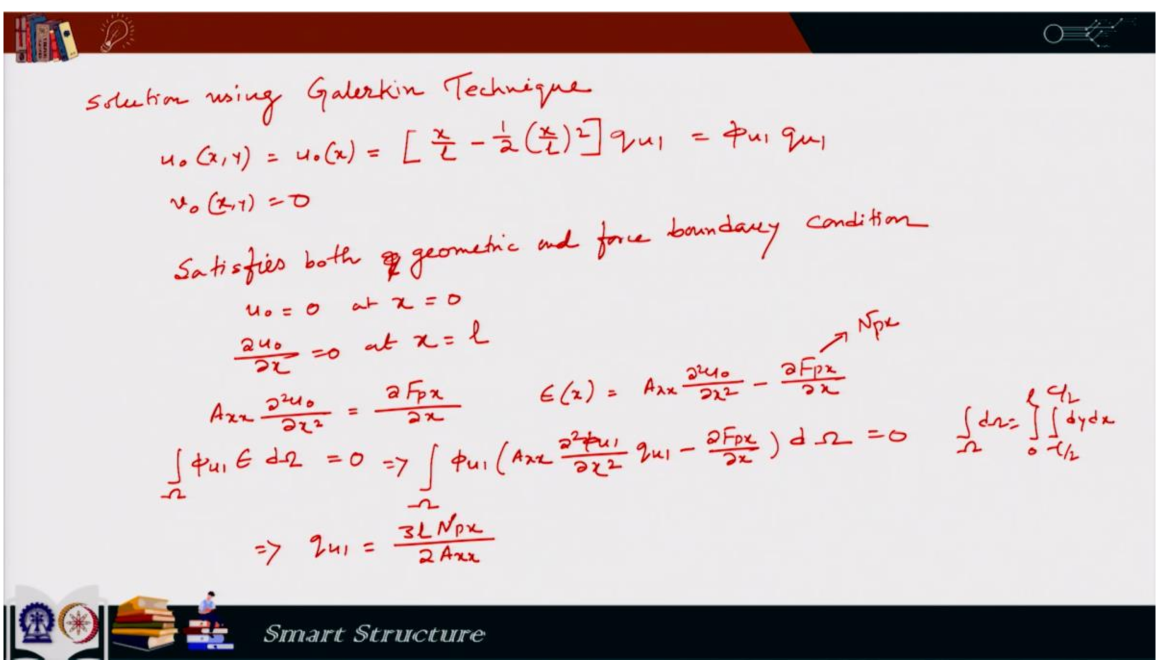

So, we have solved this in plane problem using both the Galerkin technique and the Rayleigh Ritz technique. In the next class, we will solve an out of plane problem, where there will be pure bending.

So, with that let us conclude this lecture.

Thank you